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A New Method to Find All Alternative Extreme Optimal Points for Linear Programming Problem

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Abstract: Anew method to find all alternative extreme optimal points for linear programming problems is given in this paper, this feasible direction method depends on the conjugate gradient projection method, starting with an initial feasible point we generate a sequence of feasible directions towards all alternative extremes. Asimple example is given to clarify this method

Key words: linear program-conjugate projection

1. INTRODUCTION

The problem of linear programming (LP) is one of the earliest formulated problems in mathematical programming where a linear function has to be maximized (minimized) over a convex constraint polyhedron X. The simplex algorithm was early suggested for solving this problem by moving toward a solution on the exterior of the constraint polyhedron X. In 1984, the area of linear programming underwent a considerable change of orientation when Karmarker (1984) introduced an algorithm for solving (LP) problems which moves through the interior of the polyhedron. This algorithm of Karmarker's and subsequent additional variants (Adler *et al.*, 1989; Karmarkar, 1984) established a new class of algorithms for solving linear programming problems known as the interior point methods . In the case of some linear programs sometimes the solution is not unique and decisions may be taken based on thesealternatives. In this paper we present a feasible direction method to find all alternative optimal extreme points for the linear programming problem. This method is based on the conjugate gradient projection method for solving non-linear programming problem with linear constraints (Goldfarb, 1969; Goldfarb and Lapiduo, 1968). In section 2 we give a full description of the problem together with our main results while section 3 contains the steps of our new algorithm. An example to illustrate our algorithm is given in section 4 followed by our conclusion in section 5.

2. Definitions and Theory:

The linear programming problem (LP) arises when a linear function is to be maximized on a convex constraint polyhedron X. this problem can be formulated as follows:

Maximize $F(x) = c^T x$
Subject to
$x \in X = \{x, Ax \leq b\}$ (2-1)

Where c, $x \in R^n$, A is an $(m + n) \times n$ matrix, $b \in R^{m+n}$,we point out that the nonnegative conditions are included in the set of constraints. This problem can also be written in the form:

Maximize $F(x) = c^T x$
Subject to

$$a_i^T x \leq b_i \qquad i = 1, 2... , m + n. \tag{2-2}$$

Here a_i^T represents the i th row of the given matrix A, then we have in the non degenerate case an extreme point (vertex) of X lies on some n linearly independent subset of X. We shall give an iterative method

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for solving this problem and our task is to find all optimal alternatives extreme points for this program, this method starts with an initial feasible point then a sequence of feasible directions toward optimality is generated to find all optimal extremes of this programming, in general if x^{K-1} is a feasible point obtained at iteration k-1 (k = 1, 2 …) then at iteration k our procedure finds a new feasible point x^k given by

$$x^k = x^{k-1} + \alpha_{k-1} d^{k-1} \tag{2-3}$$

Where d^{k-1} is the direction vector along which we move and given by

$$d^{k-1} = H_{k-1} c \tag{2-4}$$

Here H_{k-1} is an n x n symmetric matrix given by

$$H_{k-1} = \begin{cases} I, & \text{for } k = 1 \\ H_{k-1}^{q}, & \text{If } k > 1 \end{cases} \tag{2-5}$$

In (2.5) we have I is an nxn identity matrix and q is the number of active constraints at the current point while H_{k-1}^{q} is defined as follows, for each active constraint s; s = 1, 2,…, q.

$$H_{k-1}^{q} = H_{k-1}^{s-1} - \frac{H_{k-1}^{s-1} a_s a_s^T H_{k-1}^{s-1}}{a_s^T H_{k-1}^{s-1} a_s} \tag{2.6}$$

With $H_{k-1}^{0} = I$. Then H_{k-1} is given by $H_{k-1} = H_{k-1}^{q}$. The step length α_{k-1} is given by $\alpha_{k-1} = \min_{i=1,\ldots,m+n}$

$$\{g_i / g_i = \frac{b_i - a_i^T x^{k-1}}{a_i^T d^{k-1}} \text{ , and } g_i > 0\} \tag{2.7}$$

This relation states that α_{k-1} is always positive. Proposition 2-2 below shows that such a positive value must exist if a feasible point exists. Due to the well known Kuhn-Takucer condition (Greig, 1980; Hillier and Lieberman, 1990) for the point x^k to be an optimal solution of the linear program (2-1) there must exist $u \geq 0$ such that

$$A_r^T u = c, \quad \text{or simply}$$

$$u = (A_r A_r^T)^{-1} A_r c \tag{2-8}$$

Here A_r is a submatrix of the given matrix A containing only the coefficients of the set of active constraints at the current point x^k .This fact will act as a stopping rule of our proposed algorithm, also we have to point out that the matrix $(H_{k-1})^2 = H_{k-1}$ through the following proposition.

Proposition 2-1:

For H_{k-1} defined by relation (2.5) above we have $(H_{k-1})^2 = H_{k-1}$.

Proof:

This can be proved by induction, define a matrix $Q_1 = \frac{a_1 a_1^T}{a_1^T a_1}$ and since $H_{k-1}^{1} = (I - \frac{a_1 a_1^T}{a_1^T a_1})$ then

$H_{k-1}^{1} Q_1 = 0$, $Q_1^2 = Q_1$, $(H_{k-1})^2 = H_{k-1}^{1}$ and H_{k-1}^{1} is an orthongal projective matrix. Also, if we

define $Q_2 = \frac{a_2 a_2^T H_{k-1}^1}{a_2^T H_{k-1}^1 a_2}$ and $H_{k-1}^{*} = (I - \frac{a_2 a_2^T H_{k-1}^1}{a_2^T H_{k-1}^1 a_2})$ then since $H_{k-1}^2 = H_{k-1}^1 (I - \frac{a_2 a_2^T H_{k-1}^1}{a_2^T H_{k-1}^1 a_2})$,

we have $H_{k-1}^{*} Q_2 = 0$, $Q_2^2 = Q_2$ and $(H_{k-1}^{*})^2 = H_{k-1}^{*}$. Now, since $H_{k-1}^2 = H_{k-1}^1 H_{k-1}^{*}$ and both matrices H_{k-1}^1 and H_{k-1}^{*} are orthogonal projective, then H_{k-1}^2 is orthogonal projective matrix and we have $(H_{k-1}^2)^2 = H_{k-1}^2$. Applying the same argument, we conclude that $H_{k-1} = H_{k-1}^{q}$ is an orthogonal projective matrix such that $(H_{k-1})^2 = H_{k-1}$.

Proposition 2-2:

Any solution x^k given by equation (2-3) is feasible and increases the objective function value.

Proof:

$$\begin{aligned} F(x^k) - F(x^{k-1}) &= c^T x^k - c^T x^{k-1} \\ &= c^T \alpha_{k-1} d^{k-1} \\ &= \alpha_{k-1} c^T H_{k-1} c \\ &= \alpha_{k-1} c^T H_{k-1}^2 c \\ &= \alpha_{k-1} \| H_{k-1} c \| > 0 \end{aligned}$$

This proves that x^k increases the objective function. Next, we shall prove that x^k is a feasible point. For x^k to be a feasible point it must satisfy all constraints of problem (2-1), then

$$a_i^T (x^{k-1} + \alpha_{k-1} d^{k-1}) \leq b_i$$

Must hold for all i Є {1, 2… m+n} which can be written

$$a_i^T \alpha_{k-1} d^{k-1} \leq b_i - a_i^T x^{k-1}, \quad i = 1, 2\ldots, m+n$$

And this is valid for any i since if there is p Є {1, 2… m+n} such that

$$a_p^T d^{k-1} > 0 \text{ and } a_p^T d^{k-1} > b_p - a_p^T x^{k-1}, \text{ then } \frac{b_p - a_{pi}^T x^{k-1}}{a_p^T d^{k-1}} < \alpha_{k-1}$$

That will contradict our definition of α_{k-1}. Next, we shall give a result that guarantees the existence of α_{k-1} defined by relation (2-7) above.

Proposition 2-3:

At any iteration k if a feasible point that will increase the objective function exists then α_{k-1} as defined by relation (2-7) must exist.

Proof:

To prove this result it is enough to prove that

$$a_i^T d^{k-1} \leq 0 \qquad (2\text{-}9)$$

Can not be true for all i $\in$ {1,2,…, m+n}.Now suppose that relation (2-9) is true for i $\in$ {1,2…, m+n} then writing (2-9) in matrix form and multiplying both sides by $u^T \geq 0$,

we get

$u^T A d^{k-1} \leq 0$

i.e., $u^T A H_{k-1} c \leq 0$ (2-10)

Since the constraints of the dual problem for the linear programming problem (2-1) can be written in the form $u^T A = c^T$, $u \geq 0$, then (2-10) can be written as:

$c^T H_{k-1}^2 c \leq 0$, since $H_{k-1} = H_{k-1}^2$

i.e., $\| H_{k-1} c \| \leq 0$

This contradicts the fact that the norm must be positive, which implies that relation (2-7) cannot be true for all, $i \in \{1, 2 \ldots m+n\}$. Thus if a feasible point x^k exists then α_{k-1} as defined by relation (2-7) must exist.

Based on the above results we shall give in the next section a full description of our algorithm for solving the linear programming problem (LP) problems, to find all alternative optimal extremes.

3- New Algorithm for Solving (LP) Problems:

Our algorithm for solving (LP) problems to find all optimal alternatives extreme points consists of the following two phases as follows

Phase I:

Step 0: set k=1, Ho =I, d^0=c, let x^0 be an initial feasible point, and applies relations (2-7) to compute α_0.

Step 1: Apply relation (2-3) to find a new solution x^k.

Step 2: Apply relation (2-8) to compute u, if $u \geq 0$ stop. The current solution x^k is the optimal solution otherwise go to step 3.

Step 3: Set k = k+1, apply relations (2-5), (2-4) and (2-7) to compute H_{k-1}, d^{k-1} and α_{k-1} respectively and go to step 1.

Given an initial feasible point x^0 and a vector c, step 0 computes α_0 in O(m+n) steps. Computing x^k in step 1 requires O(n) steps while testing the optimality of the current solution x^k, in step 2 requires $O(n^3)$ steps. Step 3 of the algorithm requires $O(n^3)$ steps to compute H_{k-1} while computing d^{k-1} , the feasible direction that increase the value of the objective function, requires $O(n^2)$ steps, finally to compute α_{k-1} requires O(m+n) steps. Hence the application of each iteration of our algorithm requires $O(\max\{m+n,n^3\})$ steps. Proposition 3-1 below states that the above algorithm solves the LP problem in at most m+n iteration.

Remark 3-1:

Assuming that q is the number of active constraints at point x^k then if q<n and relation (2-8)is satisfied this indicates that x^k is an optimal non-extreme point, in this case the objective function can not be improved through any feasible direction and we have $H_k c=0$ at this point x^k, we note that although the matrix H_k is singular it does not cause the breakdown of this algorithm but indicate that all subsequent search directions d^{k+1} will be orthogonal to c.

Suppose at iteration k-1, we have x^{k-1} is an extreme non optimal (i.e. q=n, and relation (2-8) is not satisfied), then a move has to be made through a direction d^{k-1} lies in the nullity of a subset of the set of the active constrains at x^{k-1}. Each constraint in this subset satisfies relation (2-7). Also we have to show that if an active constraint at any iteration becomes inactive for the next iteration then it will never be active again at any subsequent iterations this can be shown as follows .Let x^{k-2}, x^{k-1} , x^k represents three successive points generated by relation (2-3) and suppose more that x^{k-2} is an extreme non optimal point , then at x^{k-2} we have

$a_s^T x^{k-2} = b_s$ s=1,2,…,n

$a_j^T x^{k-2} < b$ j=n+1,…, n+m

If the t^{th} active constrain of the form

$a_t^T x^{k-2} = b_t$

is now dropped from the above set of the active constraints (the one with most negative u_i in relation (2-8), and a feasible direction d^{k-2} is defined via the remaining subset of the active constraints, we note that this subset will remains active for the next iteration correspond to the point x^{k-1} together with one constraint from the above non active constraint satisfies relation (2-7) will also be active at x^{k-1}, the t^{th} active constraint at x^{k-2} will be inactive at the point x^{k-1} together with the remaining subset of the non active constraints at x^{k-2}, then if the t^{th} constraint will become active again for the point x^k ,we have

$a_t^T x^k = b_t$, which gives

$a_t^T (\alpha_{k-2} d^{k-2} + \alpha_{k-1} d^{k-1}) = 0$

and hence a direction $d = \alpha_{k-2} d^{k-2} + \alpha_{k-1} d^{k-1}$ with $\alpha_{k-1} \geq 0$ and $\alpha_{k-2} \geq 0$, can be defined such that $a_t^T d = 0$ and this contradicts that the feasible direction d^{k-2} is the only direction of movement defined for all s=1,… ,n and $t \neq s$, hence if an active constraint at a given extreme non optimal becomes inactive for the next iteration it will never be active again at any iteration of the algorithm. Next, we shall prove that the number of iterations that our algorithm requires to solve the (LP) problem is limited by m+n iterations.

Remark 3-1:

Assuming that q is the number of active constraints at point x^k then if q<n and relation (2-8)is satisfied this indicates that x^k is an optimal non-extreme point, in this case the objective function can not be improved through any feasible direction and we have $H_k c=0$ at this point x^k, we note that although the matrix H_k is singular it does not cause the breakdown of this algorithm but indicate that all subsequent search directions d^{k+1} will be orthogonal to c.

Proposition 3-1:

Our algorithm solves the (LP) problem in at most m+n iterations.

Proof:

For this algorithm at least one constraint is added at a time starting with $H_0^0 = I$, then an optimal extreme point may be reached in n steps and the algorithm terminate in at most n iterations. On the other hand if at a given iteration we have non optimal extreme point and at least one constraint has to be dropped from the set of active constraints, this constraint can not be active again at any subsequent iterations of the algorithm. Since our allowed directions (given by 2-4) that improve the value of the objective function lies in the nullity of a subset of the given matrix A, then we are moving in the direction parallel to a certain subset of the (m+n) constraints and hence in the worst case the maximum number of iterations required to reach the optimal point is limited by m+n.

In our analysis to find all optimal extreme points in linear program we do this by proceeding from a given optimal point to its adjacent extremes.

By defining a frame for Cone (H) denoted by F, called a minimal spaning system. For a n x n matrix H denote the set of indices of the columns of H by Id_H .Hence if H = $(h^1,\ldots.h^n)$, then $\mathbf{I}\ d_H=\{1,2\ldots n\}$. For a matrix H we define the positive cone spanned by the columns of H
(Called a conical or positive hull by Stoer and Witzgall, 1970)

Cone (H) = Cone $(h^i;\ i \in \mathbf{I}\ d_H)$

$= \{h \in R^n;\ h = \sum_{i \in Id_H} \beta_i h^i,\ \beta_i \geq 0\}$

Then a frame F for Cone (H) is a collection of columns of H such that Cone $(h^i;\ i \in \mathbf{I}\ d_H)$ = Cone (H) and for each $j \in Id_H$
We have Cone $(h^i;\ i \in \mathbf{I}\ d_H \backslash \{j\}) \neq$ Cone (H).

Remark 3-2:

It has to be clear that if a_s , s= 1,2,…q, corresponding to the active constraints, adding the $(q + 1)^{th}$ constraint we have to update the projective matrix given by relation (2.5) to be in the form

$$H_{k+1}^{q+1} = H_k^q - \frac{H_k^q a_{q+1} a_{q+1}^T H_k^q}{a_{q+1}^T H_k^q a_{q+1}}$$

Phase II:

Step 1: Let x^k be an optimal point, compute H_k correspond to this point x^k

Step 2: Constuct a frame F of cone H_k using e.g .the method of Wets and Witzgall, 1967.

Step 3: for each $h^i \in F$ determine α^* obtained by solving the system of linear inequalities of the form $\alpha A h^i \leq b - A x^k$, the boundary points of this interval will define α^*. Then

$x^* = x^k + \alpha^* h^i$

is the optimal extreme point for this (LP) problem and go to step 1.

Remark 3-3:

In the case when q=n and relation (2-8) is satisfied this indicates that x^k is an optimal extreme point, the columns of H_k has to be computed via a subset of these active constraints at x^k such that (2-8) is also satisfied.

4- Example:

Consider the linear programming problem

Maximize $F(x) = x_1 + 2 x_2$

Subject to: $2 x_1 + x_2 \leq 8$

$x_1 + 2x_2 \leq 6$

$x_1 \geq 0, x_2 \geq 0$

To find all altenative optimal extreme points of the above linear programming problem we have to go through the following two phases

Phase I:

Step 0: k=1, $H_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $d^0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Let $x^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be an initial feasible point, then (2-7) gives

$\alpha_0 = 3/5$ and we go to step 1

Step 1: apply relation (2-3) to get $x^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3/5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8/5 \\ 11/5 \end{pmatrix}$

and we go to step 2

Step 2: for this point x^1 the second constraint is the only active constraint and since relation (2-8) is satisfied indicates that the point is optimal non extreme for this linear program and we start phase 2.

Phase II:

We notice that the objective function value can not be improved since

$H_1 c = 0$, where

$H_1 = \begin{pmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{pmatrix}$, solving the system of linear inequalities

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4/5 \\ -2/5 \end{pmatrix} \alpha \leq \begin{pmatrix} 13/5 \\ 0 \\ 8/5 \\ 4/5 \end{pmatrix}$$

Gives $\alpha \in$ [-2 13/6], using these end values of the interval above we get for α^*= -2

$$x_1^* = \begin{pmatrix} 8/5 \\ 11/5 \end{pmatrix} + (-2)\begin{pmatrix} 4/5 \\ -2/5 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

And also for α^* =13/6 we get

$$x_2^* = \begin{pmatrix} 8/5 \\ 11/5 \end{pmatrix} + 13/6 \begin{pmatrix} 4/5 \\ -2/5 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 4/3 \end{pmatrix}$$

Both x_1^* , x_2^* are optimal extreme points for this linear program problem, we notice that the second column of H_1 can be used to get the same result since the columns of H_1 are linearly dependent

Conclusion:

In this paper we gave a feasible direction method to find all optimal alternative extreme points of linear programming problem .since decisions may be taken depending on these alternatives, our method is based on conjugate projection method and doesn't depend on the simplex version of linear program

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