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The Approximate Solution for Solving Linear Volterra Weakly Singular Integro-Differential Equations by Using Chebyshev Polynomials of the First Kind

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ABSTRACT

In this paper, we use Chebyshev polynomials method of the first kindof degree n to solve linear Volterra weakly singular integro- differential equations (LVWSIDEs) of the second kind. This techniques transform the linear Volterra weakly singular integro-differential equations to a system of a linear algebraic equations. This application was presented to illustrate the efficiency and accuracy of this method.

INTRODUCTION

Since many physical problems are modeled by integro-differential equations,the numerical solutions of such integro-differential equations have been highly studied by many authors.In recent years some works have been done in order to find the numerical solution of singular integral and integro-differential equations. Therefore many researchers used several numerical methods to solve weakly singular integral equations(WSIEs) includingH. Brunner, A. Pedas, G. Vainikko, (The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations, Math Comp. 68 (1999) 1079–1095).A. Palamora, (Product integration for Volterra integral equations of the second kind with weakly singular kernels,Math. Comp. 65 (215) (1996) 1201–1212)andH. Brunner, (Non polynomial spline collocation for Volterra equations with weakly singular kernels, SIAM J. Numer. Anal., 20, 1106-1119, 1983). As well as several numerical methods were used to solve weakly singular integro-differential equations(WSIDEs) includingSh . Sadigh Behzadia, A.Yildirim, (A Method to Estimate the Solution of a Weakly Singular Non – linear Integro –Differential Equations by Applying the Homotopy Methods, Int. J. Industrial Mathematics Vol.4, No.1, 41 – 51,2012).H. Brunner, A. Pedas and G.Vainikko, (A spline collocation method for linear integro-differential equations with weakly singular kernels", BIT 41, pp 891-900,2001)andMehrdad Lakestani, Behzad Nemati Saray and Mehdi Dehghan,(Numerical solution for the weakly singular Fredholm integro-differential equations using Legendre multiwavelets, J. Comput. Appl. Math.235, pp 3291-3303, 2011).In this paper Chebyshev polynomials of the first kind of degree n are defined in section2.In the section 3 the proposed method forsolving linear

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Volterraweaklysingular integro-differential equations is used. application is given in section 4 for confirming the efficiency of the proposed method .Section 5 contains conclusions of the paper.

2.Chebyshev Polynomials Of the First Kind $T_n(x)$, (Mason, Handscomb, 2003):

The Chebyshev Polynomials of the first kind of degree n is as set of orthogonal polynomials and it is defined by the recurrence relation

$T_0(x) = 1$

$T_1(x) = x$

$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$ for each $n \geq 1$. (1)

2.1 Properties of Chebyshev Polynomials $T_n(x)$:

1. The Chebyshev Polynomials of the first kind $T_n(x), n = 0,1, \ldots$ are aset of orthogonal polynomials over the interval [-1,1] with respect to the weight function$w(x) = (1 - x^2)^{-1/2}$, that is:

$$\int_{-1}^{1} w(x)T_n(x)T_m(x)dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \neq 0 \\ \pi & n = m = 0 \end{cases} \quad (2)$$

2. The Chebyshev polynomials of the first kind can be defined by the trigonometric identity $T_n(\cos(\theta)) = \cos(n\theta)$ for n=0,1,2,3,…

3.$T_n(x)$ has n distinct real roots x_i on the interval [-1,1], these roots are defined by :

$$x_i = \cos\left(\frac{(2i+1)\pi}{2N}\right), i = 0,1,2, \ldots, N-1 \quad (3)$$

are called Chebyshev nodes.$T_n(x)$ assumes its absolute extrema at

$$x_j = \cos\left(\frac{j\pi}{N}\right) \text{ for j=0,1,2,…,N} \quad (4)$$

3. A polynomial of degree N in Chebyshev form is a polynomial

$$p(x) = \sum_{n=0}^{N} a_n T_n(x) \quad (5)$$

Where T_n is the n^{th} Chebyshev form

The first few Chebyshev polynomials of the first kind for N=0,1,2,3,4,5 are given in figure(1)

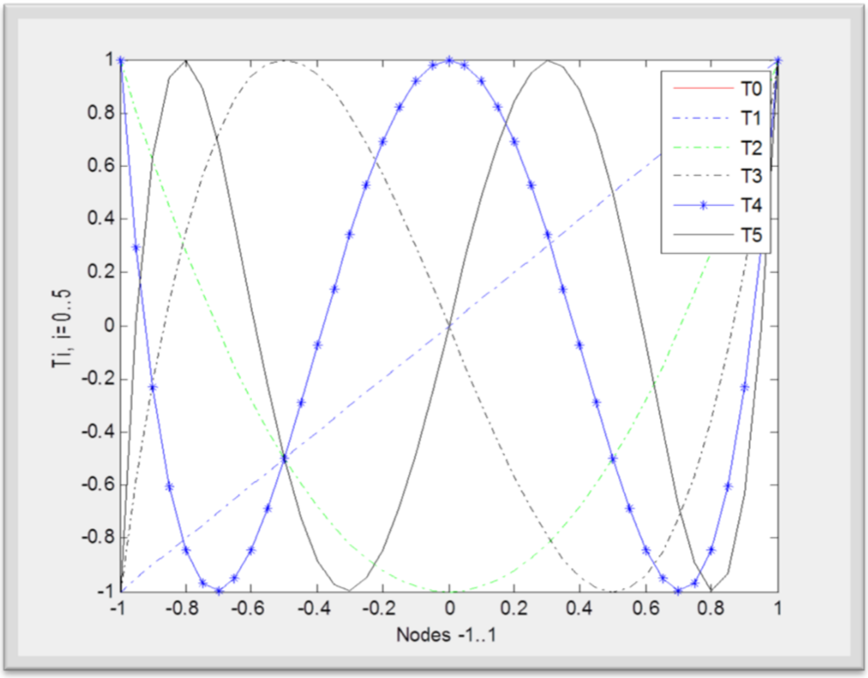

Fig. 1: The first few Chebyshev polynomials of the first kind for N=0,1,2,3,4,5.

2.2 Shifted Chebyshev Polynomials:

Shifted Chebyshev polynomials are also of interest when the range of the independent variable is [0,1] instead of [-1,1].The shifted Chebyshev polynomials of the first kind are defined as

$$T_n^*(x) = T_n(2x - 1), \; 0 \leq x \leq 1 \quad (6)$$

Similarly, one can also build shifted polynomials for a generic interval [a,b] where

$$\tilde{x}_i = \frac{b-a}{2}\bar{x}_i + \frac{b+a}{2} \quad (7)$$

The first few Chebyshev polynomials of the first kind for N=0,1,2,3,4,5 for interval [0,1] are given in figure(2).

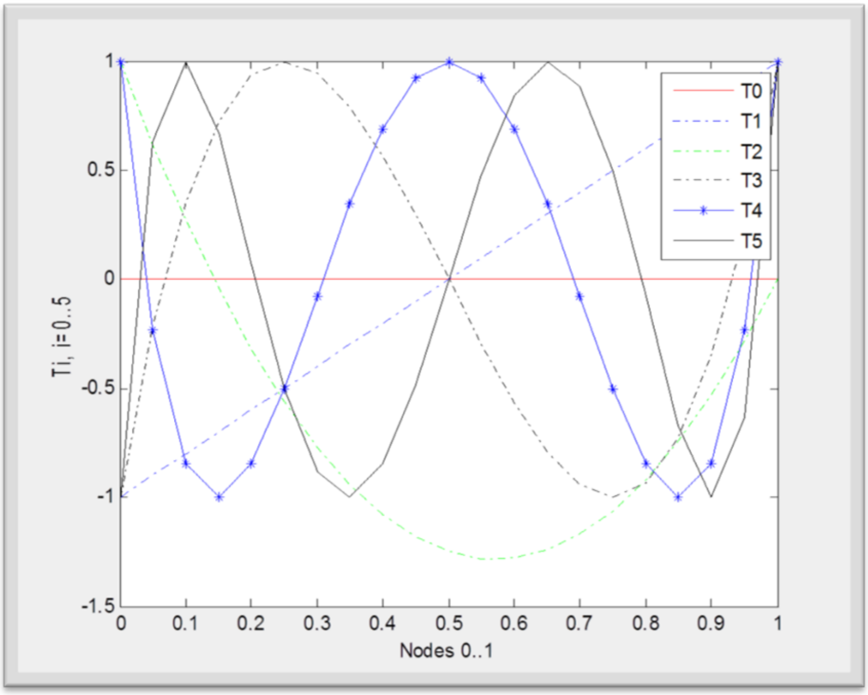

Fig. 2: The first few shifted Chebyshev polynomials of the first kind for N=0,1,2,3,4,5.

3. The Approximate Solution Of Linear Volterra Weakly Singular Integro-Differential Equations:

We consider the first-order LVWSIDEs of the following form:

$$p(x)y'(x) - y(x) - \int_a^x k(x,t)y(t)dt = f(x)\ x \in [a,b] \quad (8)$$

with initial condition $y(a) = \beta$ where $\beta$is a constant and k and f are given functions and y is the solution to be determined .Moreover we assume that the kernel $k(x,t) = \frac{H(x,t)}{|x-t|^{\alpha}}$ $\forall\ x,t \in [a,b]$where $0 < \alpha < 1$.As well as we assume that the kernel H is in $L_2[a,b]$ and the unknown y and the right hand side f are in $L_2[a,b]$.Also we suppose that $y(x)$ satisfies in the Lipschitz condition with respect to x,

$$|y(x_1) - y(x_2)| \le L_x|x_1 - x_2| \quad (9)$$

to determine an approximate solution of (8).Firstly if the function $y(x)$defined in $[-1,1]$.We suppose this function may be represented by first kind CPs(Qinghua, 2014):

$$y(\bar{x}) \cong \sum_{i=0}^{\infty} T_i(\bar{x})b_i \quad (10)$$

If we truncated the series (4.3), then we can write (4.3) as follows:

$$y(\bar{x}) \cong \sum_{i=0}^{N} T_i(\bar{x})b_i \cong T(\bar{x})B \quad (11)$$

$$y'(\bar{x}) \cong \left(\sum_{i=0}^{N} T_i(\bar{x})b_i\right)' \cong (T(\bar{x})B)' \quad (12)$$

where $T(\bar{x}) = [T_0(\bar{x}), T_1(\bar{x}), T_2(\bar{x}), \dots, T_N(\bar{x})], B = [b_0, b_1, b_2, \dots, b_N]^T$

clearly T is $1 \times (N+1)$ vectors and B is $(N+1) \times 1$ vectors.then the aim is to find chebyshev coefficients, that is the matrix B. we first substitute the chebyshev nodes, which are defined by:

$\bar{x}_i = \cos\left(\frac{(2i+1)\pi}{2N}\right)$ $, i = 0,1,2, \dots, N-1$into (11) and (12) and then rearrange anew matrix form to determine B:

$$py' - y - k = f \quad (13)$$

In which $k$is the linear integral part of (8) and

$$py' = \begin{pmatrix} p(\bar{x}_0)y'(\bar{x}_0) \\ p(\bar{x}_1)y'(\bar{x}_1) \\ \cdot \\ \cdot \\ \cdot \\ p(\bar{x}_N)y'(\bar{x}_N) \end{pmatrix},\ y = \begin{pmatrix} y(\bar{x}_0) \\ y(\bar{x}_1) \\ \cdot \\ \cdot \\ \cdot \\ y(\bar{x}_N) \end{pmatrix},\ f = \begin{pmatrix} f(\bar{x}_0) \\ f(\bar{x}_1) \\ \cdot \\ \cdot \\ \cdot \\ f(\bar{x}_N) \end{pmatrix}, k = \begin{pmatrix} k(\bar{x}_0) \\ k(\bar{x}_1) \\ \cdot \\ \cdot \\ \cdot \\ k(\bar{x}_N) \end{pmatrix} \quad (14)$$

by substituting(11) and(12) into (13) gives linear algebraicequations in $(N+1)$ unknown coefficients .These equations are solved by using (Matlab R2010b) to obtain the unknown coefficients Bwhich are then substitute into (11) to get the approximate solution of (8).Or if the function $y(x)$defined in $[0,1]$. We use shifted CPsby using the transformation $\tilde{x}=\frac{1}{2}[(b-a)\bar{x}+(a+b)]$ transforms the nodes$\bar{x}_i$in $[-1,1]$into the corresponding nodes $\tilde{x}_i$ in$[0,1]$.

3.1 The Algorithm for solving (LVWSIDEs) into [-1,1]:

Input : $a, b, \alpha, N, M, y(x), f(x), p(x), \epsilon$.
Output : The approximate solution of the LVWSIDEs.
Step 1: process: Find $T_i(\bar{x}), (T_i(\bar{x}))'$. (CPs)
Step 2: Find roots $\bar{x}_i, i=0,1,\dots,N$. (roots of CPs)
Step 3: Find roots

$$t_{ij}=a+j+k_i\pm\epsilon, k_i=\frac{\bar{x}_i-a}{M}, i=0,1,\dots,N, j=0,1,\dots,M, \bar{x}=t.$$

Step 4: Calculate $R_i=\frac{k_i}{2}[Y_{i0}+2\sum_{k=1}^{M-1}Y_{ik}+Y_{iM}]$ (Trapezoidal rule)Where $Y_{ij}=\frac{t_{ij}(t)}{|\bar{x}_i-t_{ij}|^{\alpha}}$, $i=0,1,\dots,N, j=0,1,\dots,M$.
Step 5: Construct the system:

$$A_{ik}=b_k*\left(p(\bar{x}_i)*\left(T_k(\bar{x}_i)\right)'-T_k(\bar{x}_i)\right)-b_k*R_i, i,k=0,1,\dots,N,$$

(where b_k are unknown values),$B_i=f(\bar{x}_i), i=0,1,\dots,N$.
Step 6: solve the linear system $A_{ik}=B_i$ by using $X_i=inv(A_{ik})*(B_i)^t$and find the unknowns b_k.
Step 7: Calculate the approximate function $y_N(\bar{x})=\sum_{i=0}^{N}T_i(\bar{x})b_i$
Step 8: Calculate absolute error is the comparison between the exact and the approximate solutions.
Step 9: END of the process.

3.2 The Algorithm for solving (LVWSIDEs) into [0,1]:

Input: $a, b, \alpha, N, M, y(x), f(x), p(x), \epsilon$.
Output: The approximate solution of the LVWSIDEs.
Step 1 : process: Find $T_i(\bar{x})$. (CPs)
Step 2: Find $T_i^*(\tilde{x}), (T_i^*(\tilde{x}))'$. (shifted CPs)
Step 3 : Find roots $\bar{x}_i, i=0,\dots,M$. (roots of CPs)
Step 4 : Find roots $\tilde{x}_i$ by using the transformation
$\tilde{x}_i=\frac{1}{2}[(b-a)\bar{x}_i+(a+b)], i=0,\dots,N$ (roots of shifted CPs)
Step5: Find roots

$$t_{ij}{}^*=a+j+k_i\pm\epsilon, k_i=\frac{\tilde{x}_i-a}{M}, i=0,1,\dots,N, j=0,1,\dots,M, \tilde{x}=t.$$

Step 6: Calculate $R_i=\frac{k_i}{2}[Y_{i0}+2\sum_{k=1}^{M-1}Y_{ik}+Y_{iM}]$ (Trapezoidal rule)
Where $Y_{ij}=\frac{\tilde{t}_{ij}{}^*(t)}{|\tilde{x}_i-\tilde{t}_{ij}{}^*|^{\alpha}}$, $i=0,1,\dots,N, j=0,1,\dots,M$.
Step 7: Construct the system:

$$A_{ik}=b_k*(p(\tilde{x}_i)*(T_k^*(\tilde{x}_i))'-T_k^*(\tilde{x}_i))-b_k*R_i, i,k=0,1\dots,N,$$

(where b_k are unknown values), $B_i=f(\tilde{x}_i), i=0,1,\dots,N$.
Step 8: solve the linear system $A_{ik}=B_i$ by using $X_i=inv(A_{ik})*(B_i)^t$
and find the unknowns b_k.
Step 9: Calculate the approximate function $y_N(\tilde{x})=\sum_{i=0}^{N}T_i^*(\tilde{x})b_i$.
Step 10: Calculate absolute error is the comparison between the exact and the approximate solutions.
Step 11: END of the process.

4.Applications:

In this section, numerical example is given to clarify the applicability and activity of the proposed method.The computations have been performed by using MatlabR2010b. Consider the following LVWSIDEs:

$$p(\tilde{x})y'(\tilde{x})-y(\tilde{x})-\int_0^{\tilde{x}}\frac{y(\tilde{t})}{|\tilde{x}-\tilde{t}|^{1/2}}dt=f(\tilde{x}), 0<\alpha<1, \tilde{x}\in[0,1] \quad (15)$$

with initial condition $y(0)=0$ where $a=0$,here the forcing function f is selected such that the exact solution is $y(\tilde{x})=\tilde{x}^3, p(\tilde{x})=-\tilde{x}$.

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Table1: illustrate the comparison between the exact and the approximate solution depending on Mean Square Error(MSE) and Elapsed Time (ET).

x̃-value	Exact solution	Approximate solution	Absolute error
0.0000	0.000000	-0.000000	0.000000
0.0769	0.000455	0.000428	0.000027
0.1538	0.003641	0.003137	0.000504
0.2308	0.012289	0.009733	0.002557
0.3077	0.029131	0.021310	0.007821
0.3846	0.056896	0.038633	0.018263
0.4615	0.098316	0.062256	0.036060
0.5385	0.156122	0.092592	0.063530
0.6154	0.233045	0.129965	0.103080
0.6923	0.331816	0.174633	0.157183
0.7692	0.455166	0.226812	0.228354
0.8462	0.605826	0.286690	0.319136
0.9231	0.786527	0.354428	0.432099
1.0000	1.000000	0.430175	0.569825
MSE	5.046e-002	ET	450.508284Sec.

Results obtained and errors for example (1):N=13.

Table 2: illustrate the comparison between the exact and the approximate solution depending on Mean Square Error(MSE) and Elapsed Time (ET).

x̃-value	Exact solution	Approximate solution	Absolute error
0.0000	0.000000	-0.000000	0.000000
0.0667	0.000296	0.000282	0.000014
0.1333	0.002370	0.002090	0.000280
0.2000	0.008000	0.006549	0.001451
0.2667	0.018963	0.014463	0.004500
0.3333	0.037037	0.026415	0.010622
0.4000	0.064000	0.042839	0.021161
0.4667	0.101630	0.064065	0.037564
0.5333	0.151704	0.090354	0.061349
0.6000	0.216000	0.121914	0.094086
0.6667	0.296296	0.158919	0.137378
0.7333	0.394370	0.201515	0.192856
0.8000	0.512000	0.249830	0.262170
0.8667	0.650963	0.303978	0.346985
0.9333	0.813037	0.364062	0.448975
1.0000	1.000000	0.430175	0.569825
MSE	4.913e-002	ET	1721.679419Sec.

Results obtained and errors for example (1):N=15.

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Table 3: illustrate the comparison between the exact and the approximate solution depending on Mean Square Error(MSE) and Elapsed Time (ET).

x̃-value	Exact solution	Approximate solution	Absolute error
0.0000	0.000000	-0.000000	0.000000
0.0588	0.000204	0.000195	0.000008
0.1176	0.001628	0.001462	0.000167
0.1765	0.005496	0.004617	0.000879
0.2353	0.013027	0.010267	0.002760
0.2941	0.025443	0.018868	0.006575
0.3529	0.043965	0.030765	0.013200
0.4118	0.069815	0.046227	0.023587
0.4706	0.104213	0.065469	0.038744
0.5294	0.148382	0.088664	0.059718
0.5882	0.203542	0.115954	0.087588
0.6471	0.270914	0.147461	0.123453
0.7059	0.351720	0.183289	0.168431
0.7647	0.447181	0.223532	0.223649
0.8235	0.558518	0.268270	0.290249
0.8824	0.686953	0.317578	0.369375
0.9412	0.833706	0.371525	0.462181
1.0000	1.000000	0.430175	0.569825
MSE	4.812e-002	ET	10916.580284Sec.

Results obtained and errors for example (1):N=17.

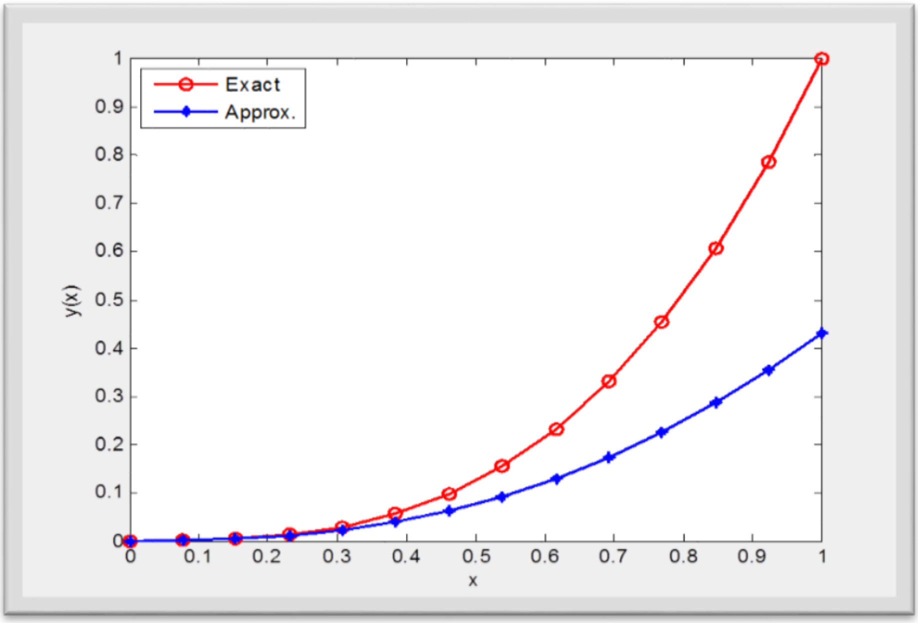

Fig. 3:A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of the first kind of application for N=13.

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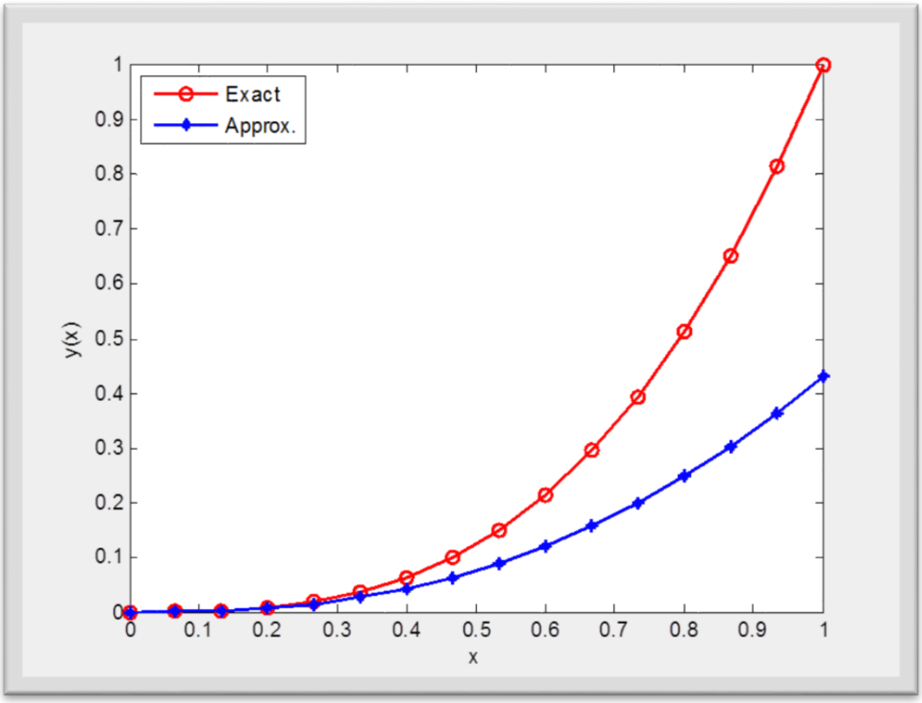

Fig. 4:A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of the first kind of application forN=15.

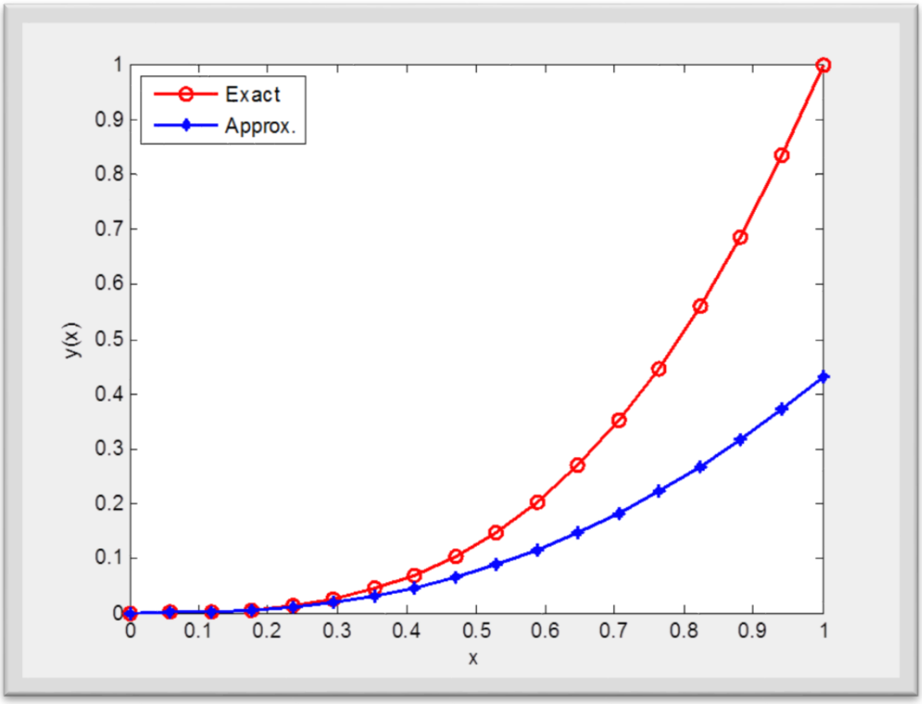

Fig. 5: A Comparison between the exact and the approximate solution using expansion method of Chebyshev polynomials of the first kind of application for N=17.

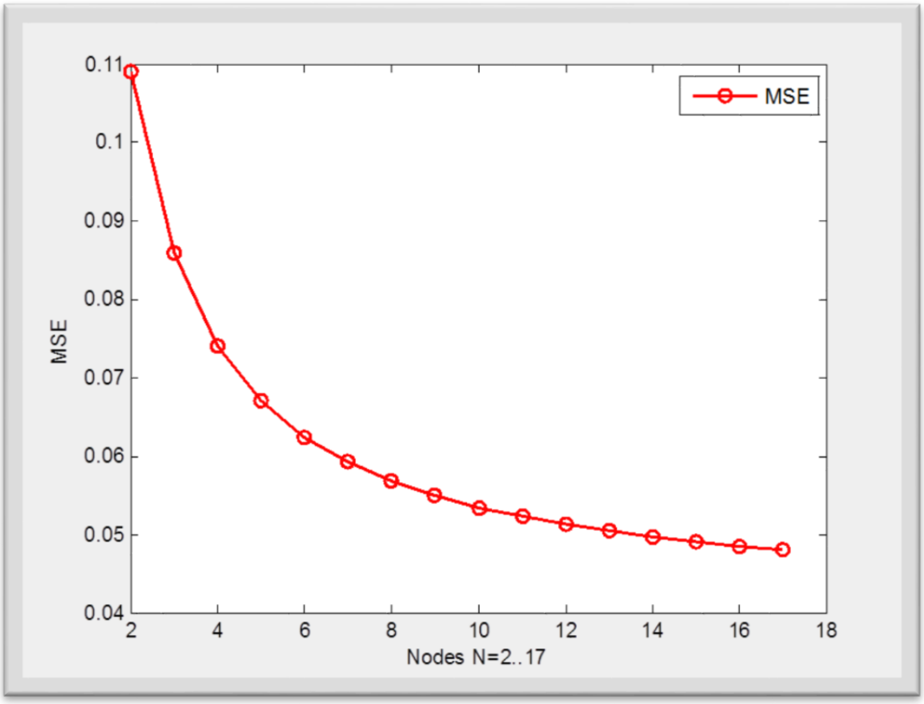

Fig. 6: A Comparison between the Shifted Chebyshev nodes and the MSE of application when N=2….14.

Conclusions:

In this paper, we have submitting expansion method using Chebyshev polynomials of the first kind of degree n as basis function for approximating the solution of one weakly singular integro-differential equation: which is the LVWSIDEs.In the application we have reduced the solution of LVWSIDEs to thesystem of linear equations by removing the singularity using an approximate point t, andwe have the following results:When the degree of expansion method of Chebyshev polynomials of the first kind is increases the error is decreases. Which is shown in Tables (1), (2) and (3).As well asthe proposed method is a delicate and an effective to solve LVWSIDEs.Finally this method can be extended and applied to the system of LVWSIDEs.

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