Australian Journal of Basic and Applied Sciences, 10(1) January 2016, Pages: 550-563

AUSTRALIAN JOURNAL OF BASIC AND APPLIED SCIENCES

ISSN:1991-8178 EISSN: 2309-8414
Journal home page: www.ajbasweb.com

Occurrences of ordered patterns in rectangular space filling curve through homomorphism

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ARTICLE INFO

Article history:
Received 12 February 2016
Accepted 12 March 2016
Available online 20 March 2016

Keywords:
Finite word, Grammar, Ordered Patterns, Rectangular Space Filling Curve, Recurrence relation.

ABSTRACT

Finite Words corresponding to finite approximations of Rectangular Space Filling Curve are formed. By ordering alphabets of the finite words, the numbers of occurrences of certain ordered patterns in these words are investigated. Moreover, a DOL system generating these words was established and occurrences of ordered patterns were discussed throught the homomorphism of the DOL system.

INTRODUCTION

The use of Space Filling Curves (SFC) has found in many applications such as routing system, parallel computing, image processing and data bases. The goal of the research presented in this paper is to extend the concept of Space Filling Curves on square frame to Space Filling Curves on rectangular frame. Just as SFCs are convoluted lines that fill a square, these SFCs are carefully elaborated to fill a rectangle. The author has discussed in (Kitaev, S., 2002) about occurrences of some patterns, subsequences and sub words in sigma-sequence. Combinatorics on words has been analyzed in (De Luca, A., 1999) and (Berstel, J., and D. Perrin, 2007). Masood Ahmed and Shahid Bokhari (Ahmed, M., S. Bokhari, 2007) explained about Space Filling Surfaces. The author has introduced the notion of Hilbert words in (Sưbold, P., 2007). In (Kitaev, S., and T. Mansour, 2002), counting occurrences of some patterns in Peano words was done. Counting ordered patterns in words generated by morphisms was done in (Kitaev, S., *et al.*, 2008). Finite words which correspond to finite approximations of the Rectangular SFC are produced in section 4, after presenting the generation of Rectangular Space Filling Curve by geometrically and through by a grammar. An ordering is given to the alphabets of finite words and discussed about rises and descents in the next section. Finally, the number of occurrences of ordered patterns in the finite words has been executed.

II. Geometric generation of rectangular SFC:

A Space Filling Curve maps a 1-dimensional space onto a higher-dimensional space e.g., the unit interval onto the unit square. A geometric generation principle for the extension of Hilbert curve construction to fill a

To Cite This Article: K.Navaneetham, K. Thiagarajan, S. Jeyabharathi, Ponnammal Natarajan., Occurrences of ordered patterns in rectangular space filling curve through homomorphism. ***Aust. J. Basic & Appl. Sci.,*** *10(1): 550-563, 2016*

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rectangle region is suggested. The method is a work out in recursive thinking and can be summed up in a few lines. Let us assume that the unit interval *I* can be mapped continuously onto the rectangle $\Omega \left[0, \frac{2}{3}\right] x [0,1]$.

2.1. Initial mapping:

If I is partitioned into six congruent subintervals then it should be possible to partition Ω into six congruent sub squares, such that each subinterval will be mapped continuously onto one of the sub squares.

2.2. Iteration mapping:

If the subintervals are partitioned into nine congruent subintervals then it should be possible to partition the sub squares into nine congruent sub squares, such that each subinterval will be mapped continuously onto one of the sub squares. This reasoning can be repeated by again partitioning each subinterval into nine congruent subintervals and doing the same for the respective sub squares. When repeating this procedure make sure that the sub squares are arranged in such a way that adjacent sub squares correspond to adjacent subintervals. Like this the overall continuity of the mapping is preserved. If an interval corresponds to a square, then its subintervals must correspond to the sub squares of that square. This inclusion relationship assures that a mapping of the n^{th} iteration preserves the mapping of the $(n\text{-}1)^{th}$ iteration.

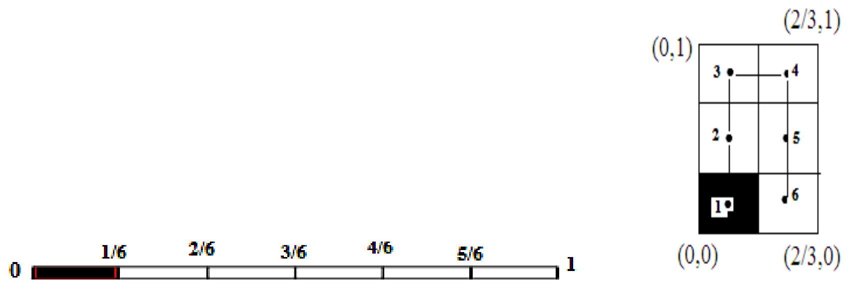
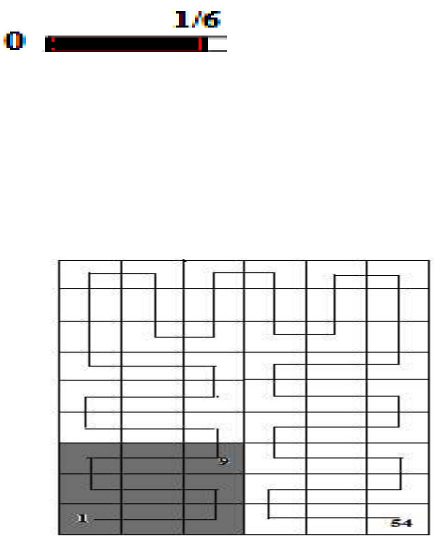


Fig. 1: Initial Iteration



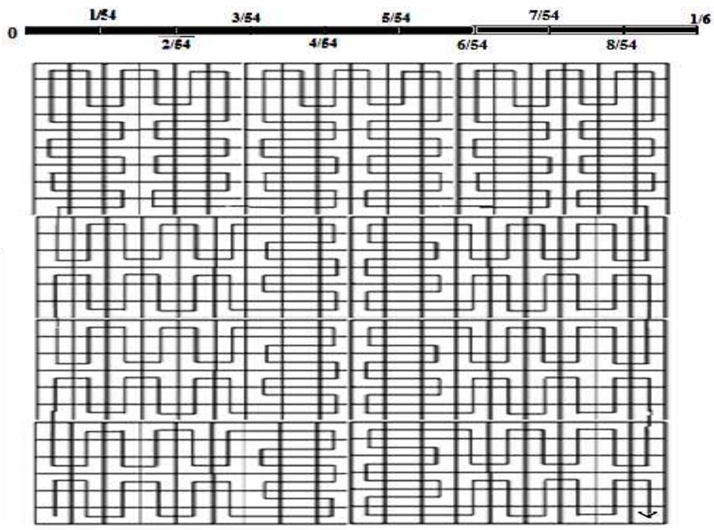


Fig. 2: First iteration **Fig. 3:** Second iteration

III. Representation of the rectangular SFC through a grammar:

In the construction of this Hilbert Rectangular curve, the four templates H, A, B and C illustrated in the figure (1) are used in every iteration. These templates are substituted in every iteration step into a first iteration of this Space Filling Curve. These fixed substitution procedure can be described by a grammar G = (V, T, P, S) where V = $\{H, A, B, C\}$, T = $\{\uparrow, \downarrow, \rightarrow, \leftarrow\}$, S = $\{H\}$ and P is defined by

$$H \rightarrow [C \uparrow C \uparrow C \uparrow H \rightarrow H \rightarrow H \downarrow A \downarrow A \downarrow A] \quad =$$

$$A \rightarrow [B \leftarrow B \leftarrow B \leftarrow A \downarrow A \downarrow A \rightarrow H \rightarrow H \rightarrow H]$$

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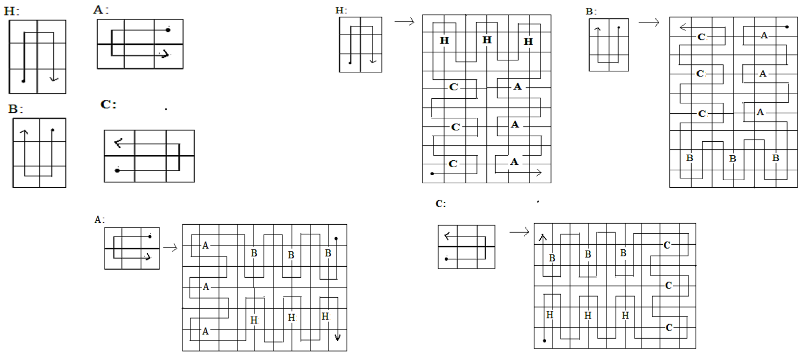
$C \rightarrow [H \rightarrow H \rightarrow H \rightarrow C \uparrow C \uparrow C \leftarrow B \leftarrow B \leftarrow B]$

$B \rightarrow [A \downarrow A \downarrow A \downarrow B \leftarrow B \leftarrow B \uparrow C \uparrow C \uparrow C]$



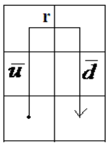
IV. Finite Word H_n for the Rectangular SFC:

The construction of the Hilbert Rectangular curve is obtained by drawing, without removing the pen from the surface of the paper to fill the rectangle of size $3^n x 2^n$, an infinite succession of unit lines and double unit lines left, right, up or down. Thus this succession can be represented by a word over the alphabet $\sum = \{u, d, r, l, \bar{u}, \bar{d}, \bar{r}, \bar{l}\}$ where u stands for unit line up, d stands for unit line down, r stands for unit line right, ℓ stands for unit line left, $\bar{u}$ stands for double units line up, $\bar{d}$ stands for double units line down, $\bar{r}$ stands for double units line right, $\bar{\ell}$ stands for double units line left. Let us explain the algorithm of rectangular Hilbert Curve. The general idea is to divide, at step n, the rectangle into $6(9)^{n-1}$ equal sub squares each of them containing an equal length part of the curve (except the first and the last ones which contain a part of length ½). The curve so obtained is then depicted by a word of length $[4(9)^{n-1} - 1]$ which is called as the n[th] rectangular Hilbert word H_n.

Therefore

$H_1 = \bar{u}\ r\ \bar{d}$

Similarly H_2 $= \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}$



$H_3 = \bar{u}\ r\ \bar{d}\ r$ $\bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}$

u

$\bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ \ell\ \bar{d}\ \ell\ \bar{u}$

$\ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}$

u

$\bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}$

u

$\bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}$

r

$\bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}$

r

$\bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{r}\ u\ \bar{\ell}\ u\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}$

d

$\bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}$

d

$\bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}$

d

$\bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{d}\ \ell\ \bar{u}\ \ell\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ d\ \bar{\ell}\ d\ \bar{r}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}\ r\ \bar{u}\ r\ \bar{d}$

Now let us define on $\sum$ three literal morphisms f, t_ℓ and t_r by

$f(u) = d$, $f(d) = u$, $f(r) = r$, $f(\ell) = \ell$, $f(\bar{u}) = \bar{d}$, $f(\bar{d}) = \bar{u}$, $f(\bar{r}) = \bar{r}$, $f(\bar{\ell}) = \bar{\ell}$

$t_\ell(u) = \ell$, $t_\ell(d) = r$, $t_\ell(r) = u$, $t_\ell(\ell) = d$, $t_\ell(\bar{u}) = \bar{\ell}$, $t_\ell(\bar{d}) = \bar{r}$, $t_\ell(\bar{r}) = \bar{u}$, $t_\ell(\bar{\ell}) = \bar{d}$

$t_r(u) = r$, $t_r(d) = \ell$ $t_r(r) = d$, $t_r(\ell) = u$, $t_r(\bar{u}) = \bar{r}$, $t_r(\bar{d}) = \bar{\ell}$, $t_r(\bar{r}) = \bar{d}$, $t_r(\bar{\ell}) = \bar{u}$

These three morphisms represent a vertical flip, a quarter turn left rotation, and a quarter turn right rotation respectively. Using these morphisms, the recurrence relation for finite word H_n ($n \geq 1$) is generated by

$$\boxed{H_{n+1} = \rho(H_n)u\rho(H_n)u\rho(H_n)uH_nrH_nrH_nd\lambda(H_n)d\lambda(H_n)d\lambda(H_n)}$$

(1) where $\rho = t_\ell \circ f$ and $\lambda = t_r \circ f$

4.1. Theorem:

For any $n \geq 1$,

1. $|H_n| = 4(9)^{n-1} - 1$
2. $|H_n|_{\bar{u}} = |H_n|_{\bar{d}}$ and $|H_n|_{\bar{\ell}} = |H_n|_{\bar{r}} = |H_n|_u = |H_n|_d$
3. $|H_{n+1}|_{\bar{u}} = 3|H_n|_{\bar{u}} + 3|H_n|_{\bar{r}} + 3|H_n|_{\bar{\ell}}$
4. $|H_{n+1}|_{\bar{d}} = 3|H_n|_{\bar{d}} + 3|H_n|_{\bar{\ell}} + 3|H_n|_{\bar{r}}$
5. $|H_{n+1}|_{\bar{r}} = 3|H_n|_{\bar{r}} + 3|H_n|_{\bar{u}} + 3|H_n|_{\bar{d}}$
6. $|H_{n+1}|_{\bar{\ell}} = 3|H_n|_{\bar{\ell}} + 3|H_n|_{\bar{d}} + 3|H_n|_{\bar{u}}$
7. $|H_{n+1}|_r = 3|H_n|_r + 3|H_n|_u + 3|H_n|_d + 2$
8. $|H_{n+1}|_u = 3|H_n|_u + 3|H_n|_r + 3|H_n|_\ell + 3$
9. $|H_{n+1}|_d = 3|H_n|_d + 3|H_n|_\ell + 3|H_n|_r + 3$
10. $|H_{n+1}|_\ell = 3|H_n|_\ell + 3|H_n|_d + 3|H_n|_u$
11. $|H_n|_{\bar{u}} = |H_n|_{\bar{d}} = \begin{cases} \frac{3^{n-1}(3^{n-1}-1)}{2}, & if \quad n \quad is \quad even \\ \frac{3^{n-1}(3^{n-1}+1)}{2}, & if \quad n \quad is \quad odd \end{cases}$
12. $|H_n|_{\bar{r}} = |H_n|_{\bar{\ell}} = |H_n|_u = |H_n|_d = \begin{cases} \frac{3^{n-1}(3^{n-1}+1)}{2}, & if \quad n \quad is \quad even \\ \frac{3^{n-1}(3^{n-1}-1)}{2}, & if \quad n \quad is \quad odd \end{cases}$

$|H_n|_u = 0$ means that there is zero occurrence of u in H_n.

13. $|H_n|_r = \begin{cases} \frac{3^n(3^{n-2}+1)}{2} - 1, & if \quad n \quad is \quad odd \\ \frac{3^{n-1}(3^{n-1}+1)}{2} - 1, & if \quad n \quad is \quad even \end{cases}$
14. $|H_n|_\ell = \begin{cases} \frac{3^{n-1}(3^{n-1}-1)}{2}, & if \quad n \quad is \quad odd \\ \frac{3^n(3^{n-2}-1)}{2}, & if \quad n \quad is \quad even \end{cases}$

Proof

The recurrence relation for $|H_n|$ is given by $|H_{n+1}| = 9|H_n| + 8$, $|H_1| = 3$. Solving this equation with initial condition, the equality (1) is obtained. The equalities from (2) to (10) can be obtained from the definition of ρ, λ and the formation (1) of H_n. Other equalities can be proved by induction on n.

V. Rises and descents in H_n:

Let us order the alphabets of as $\bar{u} < u < r < \bar{r} < \bar{d} < d < \ell < \bar{\ell}$

5.1. Proposition:

Let $x_2, \ldots, x_n$ be non-empty words over $\{\bar{u}, u, r, \bar{r}\}$, $y_1, y_2, \ldots, y_{k-1}$ non-empty words over $\{\bar{d}, d, \ell, \bar{\ell}\}$, x_1 is a word over $\{\bar{u}, u, r, \bar{r}\}$ and y_k is a word over $\{\bar{d}, d, \ell, \bar{\ell}\}$ (may be $x_1 = \epsilon$. Or $y_k = \epsilon$, or both). Let $w = x_1y_1x_2y_2 \ldots x_ky_k$, then

R (ρ(w)) = $\begin{cases} D(w)-1, & if \ \ x_1 = y_k = \varepsilon \\ D(w)+1, & if \ x_1 \neq \varepsilon \ and \ y_k \neq \varepsilon \\ D(w), & otherwise \end{cases}$

D(ρ(w)) = $\begin{cases} R(w)+1, & if \ \ x_1 = y_k = \varepsilon \\ D(w)-1, & if \ x_1 \neq \varepsilon \ and \ y_k \neq \varepsilon \\ R(w), & otherwise \end{cases}$

R(λ(w)) = D(w) and D(λ(w)) = R(w) where R (w) and D (w) denote the number of rises and descents respectively in a word w.

Proof

Clearly D (w) = $\sum_{i=1}^{k}\left[D(x_i)+D(y_i)\right]+k-1$

and R (w) = $\sum_{i=1}^{k}\left[R(x_i)+R(y_i)\right]+\begin{cases} k-2, & if \ \ x_1 = y_k = \varepsilon \\ k, & if \ x_1 \neq \varepsilon \ and \ y_k \neq \varepsilon \\ k-1, & otherwise \end{cases}$

Let $\rho(w) = x_1' y_1' x_2' y_2' ......x_k' y_k'$

From the definition of ρ, R (x_i') = D (x_i), D (x_i') = R (x_i), R (y_i') = D (y_i), D (y_i') = R (y_i),

R (ρ (w)) = $\sum_{i=1}^{k}\left[R(x_i')+R(y_i')\right]+\begin{cases} k-2, & if \ \ x_1' = y_k' = \varepsilon \\ k, & if \ x_1' \neq \varepsilon \ and \ y_k' \neq \varepsilon \\ k-1, & otherwise \end{cases}$

= $\sum_{i=1}^{k}\left[D(x_i)+D(y_i)\right]+\begin{cases} k-2, & if \ \ x_1 = y_k = \varepsilon \\ k, & if \ x_1 \neq \varepsilon \ and \ y_k \neq \varepsilon \\ k-1, & otherwise \end{cases}$

This implies that R (ρ (w)) = $\begin{cases} D(w)-1, & if \ \ x_1 = y_k = \varepsilon \\ D(w)+1, & if \ x_1 \neq \varepsilon \ and \ y_k \neq \varepsilon \\ D(w), & otherwise \end{cases}$

Since λ is the literal morphism which inverts the order of the letters, we get R (λ (w)) = D (w) and
D (λ (w)) = R (w)

5.2. Theorem:

For any n ∈ N,

$D(H_{2n+2}) = \{(81)^n 18\} - 1$

$D(H_{2n+1}) = 2\{(81)^n - 1\}$

$R(H_{2n+2}) = \{(81)^n 18\} - 1$

$R(H_{2n+1}) = 2(81)^n$.

Proof

From the construction of H_n,

$D(H_{2n+2}) = 6R(H_{2n+1}) + 3D(H_{2n+1}) + 5$

$D(H_{2n+1}) = 6R(H_{2n}) + 3D(H_{2n}) + 7$

$R(H_{2n+1}) = 3R(H_{2n}) + 6D(H_{2n}) + 9$

$R(H_{2n+2}) = 6D(H_{2n+1}) + 3R(H_{2n+1}) + 11$.

And using the above proposition 5.1, the theorem can be proved.

VI. Occurrences of ordered patterns without internal hashes in H_n:

The finite words H_n can be defined by the following inductive scheme.

$H_1=\overline{u}\,r\,\overline{d}$, $B_1=\overline{r}\,u\,\overline{\ell}$, $C_1=\overline{d}\,\ell\,\overline{u}$, $D_1=\overline{\ell}\,d\,\overline{r}$

Thus $H_n = B_{n-1}uB_{n-1}uB_{n-1}uH_{n-1}rH_{n-1}rH_{n-1}dD_{n-1}dD_{n-1}dD_{n-1}$

where $B_n = H_{n-1}rH_{n-1}rH_{n-1}rB_{n-1}uB_{n-1}uB_{n-1}\ell C_{n-1}\ell C_{n-1}\ell C_{n-1}$

$C_n = D_{n-1}dD_{n-1}dD_{n-1}dC_{n-1}\ell C_{n-1}\ell C_{n-1}uB_{n-1}uB_{n-1}uB_{n-1}$

$D_n = C_{n-1}\ell C_{n-1}\ell C_{n-1}\ell D_{n-1}dD_{n-1}dD_{n-1}rH_{n-1}rH_{n-1}rH_{n-1}$

Let X_n^{τ} denote the number of occurrences of the pattern τ in X_n where $X \in \{H, A, B, C\}$

6.1. Definition:

Suppose a word W = AaB, where A and B are some words of the same length and a is a letter. The Kernel of order k for the word W to be the subword consisting of the k-1 rightmost letters of A, the letter a, and the k – 1 leftmost letters of B. It is denoted by $\kappa_k(W)$. For example, $\kappa_k(111211221)=12112.$ If $|A|<k-1$, then $\kappa_k(W)=\varepsilon$. Also $m_k(\tau,W)$ denotes the number of occurrences of the pattern τ .

6.2. Theorem:

Let $\tau = \tau_1\tau_2\tau_3....\tau_\ell$ be an arbitrary generalized pattern without internal dashes that consists of numbers from {1,2,…8} such that $|\tau_i|$ =1. Suppose $k = 1+\left\lceil \frac{1}{2}\log_3\left(\frac{\ell}{4}\right)\right\rceil$,

$a_{x1} = m_\ell(\tau, H_i rH_i),\ a_{x2} = m_\ell(\tau, B_i uB_i),\ a_{x3} = m_\ell(\tau, C_i \ell C_i),\ a_{x4} = m_\ell(\tau, D_i dD_i),$

$a_{x5} = m_\ell(\tau, B_i uH_i),\ a_{x6} = m_\ell(\tau, H_i rB_i),\ a_{x7} = m_\ell(\tau, C_i uB_i),\ a_{x8} = m_\ell(\tau, D_i rH_i),$

$a_{x9} = m_\ell(\tau, H_i dD_i),\ a_{x10} = m_\ell(\tau, B_i \ell C_i),\ a_{x11} = m_\ell(\tau, D_i dC_i),\ a_{x12} = m_\ell(\tau, C_i \ell D_i)$

where x can be either (-1) or 1 depends on whether i is Odd or Even and $k \le i \le$ n-1

Then for n > k + 1,

$$H_n^{\tau} = \frac{3^M}{4}\left((2+(-1)^M+3^M)H_{k+1}^{\tau} + (3^M-(-1)^{M)})B_{k+1} + (3^M+(-1)^M-2)C_{k+1}^{\tau} + (3^M-(-1)^M)D_{k+1}^{\tau}\right) +$$
$$\sum_{i=0}^{M-1}\frac{3^i}{4}\left[(2+(-1)^i+3^i)P_{(-1)^{n-(i+1)}1} + (3^i-(-1)^{i)})P_{(-1)^{n-(i+1)}2} + (3^i+(-1)^i-2)P_{(-1)^{n-(i+1)}3} + (3^i-(-1)^i)P_{(-1)^{n-(i+1)}4}\right]$$

$$B_n^{\tau} = \frac{3^M}{4}\left((2+(-1)^M+3^M)B_{k+1}^{\tau} + (3^M-(-1)^{M)})H_{k+1} + (3^M+(-1)^M-2)D_{k+1}^{\tau} + (3^M-(-1)^M)C_{k+1}^{\tau}\right) +$$
$$\sum_{i=0}^{M-1}\frac{3^i}{4}\left[(2+(-1)^i+3^i)P_{(-1)^{n-(i+1)}2} + (3^i-(-1)^{i)})P_{(-1)^{n-(i+1)}1} + (3^i+(-1)^i-2)P_{(-1)^{n-(i+1)}4} + (3^i-(-1)^i)P_{(-1)^{n-(i+1)}3}\right]$$

$$C_n^{\tau} = \frac{3^M}{4}\left((2+(-1)^M+3^M)C_{k+1}^{\tau} + (3^M-(-1)^{M)})D_{k+1} + (3^M+(-1)^M-2)H_{k+1}^{\tau} + (3^M-(-1)^M)B_{k+1}^{\tau}\right) +$$
$$\sum_{i=0}^{M-1}\frac{3^i}{4}\left[(2+(-1)^i+3^i)P_{(-1)^{n-(i+1)}3} + (3^i-(-1)^{i)})P_{(-1)^{n-(i+1)}4} + (3^i+(-1)^i-2)P_{(-1)^{n-(i+1)}1} + (3^i-(-1)^i)P_{(-1)^{n-(i+1)}2}\right]$$

$$D_n^{\tau} = \frac{3^M}{4}\left((2+(-1)^M+3^M)D_{k+1}^{\tau} + (3^M-(-1)^{M)})H_{k+1} + (3^M+(-1)^M-2)B_{k+1}^{\tau} + (3^M-(-1)^M)C_{k+1}^{\tau}\right) +$$
$$\sum_{\substack{i=0\\ even\,i}}^{M-1}\frac{3^i}{4}\left[(2+(-1)^i+3^i)P_{(-1)^{n-(i+1)}4} + (3^i-(-1)^{i)})P_{(-1)^{n-(i+1)}1} + (3^i+(-1)^i-2)P_{(-1)^{n-(i+1)}2} + (3^i-(-1)^i)P_{(-1)^{n-(i+1)}3}\right]$$

Where M = n - (k + 1) , $p_{x1} = 2(a_{X1} + a_{X2} + a_{X4}) + a_{X5} + a_{X9}$,

$p_{x2} = 2(a_{X1} + a_{X2} + a_{X3}) + a_{X6} + a_{X10}$, $p_{x3} = 2(a_{X2} + a_{X3} + a_{X4}) + a_{X7} + a_{X11}$
$p_{x4} = 2(a_{X1} + a_{X3} + a_{X4}) + a_{X8} + a_{X12}$, $X \in \{-1,1\}$

Proof

Suppose n > k + 1. In this case $H_n = B_{n-1}uB_{n-1}uB_{n-1}uH_{n-1}rH_{n-1}rH_{n-1}dD_{n-1}dD_{n-1}dD_{n-1}$

An occurrence of the pattern τ in H is either in B_{n-1} 's or H_{n-1}'s or C_{n-1}'s or $\kappa_k(B_{n-1}uB_{n-1})$ or $\kappa_k(B_{n-1}uH_{n-1})$ or $\kappa_k(H_{n-1}rH_{n-1})$ or $\kappa_k(H_{n-1}dD_{n-1})$ or $\kappa_k(D_{n-1}dD_{n-1})$. So,

$$H_n^\tau = 3B_{n-1}^\tau + 3H_{n-1}^\tau + 3D_{n-1}^\tau + p_{x1} \quad (1)$$

where X can be 1or (-1) depends on n is Odd or Even.

Similarly, we can have

$$B_n^\tau = 3A_{n-1}^\tau + 3B_{n-1}^\tau + 3C_{n-1}^\tau + p_{x2} \quad (2)$$

$$C_n^\tau = 3D_{n-1}^\tau + 3C_{n-1}^\tau + 3B_{n-1}^\tau + p_{x3} \quad (3)$$

$$D_n^\tau = 3C_{n-1}^\tau + 3D_{n-1}^\tau + 3A_{n-1}^\tau + p_{x4} \quad (4)$$

The above system of equations can be written in matrix form

$$\begin{pmatrix} H_n^\tau \\ B_n^\tau \\ C_n^\tau \\ D_n^\tau \end{pmatrix} = \begin{pmatrix} 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 0 \\ 0 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} H_{n-1}^\tau \\ B_{n-1}^\tau \\ C_{n-1}^\tau \\ D_{n-1}^\tau \end{pmatrix} + \begin{pmatrix} p_{X1} \\ p_{X2} \\ p_{X3} \\ p_{X4} \end{pmatrix} \quad (5)$$

Thus we need to solve recurrence relations of the form $Y_n^\tau = AY_{n-1}^\tau + P_X$.

This relation can be solved by diagonalization of the matrix, that is, by writing $A = VDV^{-1}$, where D is a diagonal matrix consisting of eigenvalues of A, and the columns of V are eigen vectors of A.

Hence the proof of the theorem follows from the solution of (5).

6.2.1. Example

Let $\ell = 1$. Thus k = 1, $H_{k+1}^\tau = B_{k+1}^\tau = C_{k+1}^\tau = D_{k+1}^\tau = 35$,

$p_{X1} = p_{X2} = p_{X3} = p_{X4} = 8$, for $X \in \{-1,1\}$. Thus $H_n^\tau = |H_n| = 4(9)^{n-1} - 1$.

Similarly the cases for B, C and D can be done.

6.2.2. Example

Let $\ell = 2$. τ =12. Thus k = 1, $H_{k+1}^\tau = B_{k+1}^\tau = C_{k+1}^\tau = D_{k+1}^\tau = 17$,

$p_{(-1)1} = 8$, $p_{(-1)2} = 5$, $p_{(-1)3} = 11$, $p_{(-1)4} = 8$, $p_{11} = 9$, $p_{12} = 8$, $p_{13} = 8$, $p_{14} = 7$.

Thus $H_{2m+2}^\tau = R(H_{2m+2}) = \{(81)^m 18\} - 1$.

$H_{2m+1}^\tau = R(H_{2m+1}) = 2(81)^m$

The cases for B, C and D are similar.

VII. Occurrences of patterns 1#2 and 2#1:

Let us denote $\bar{u}, u, r, \bar{r}, \bar{d}, d, \ell \text{ and } \bar{\ell}$ as 1,2,3,4,5,6,7 and 8 respectively.

Let $|X_n|_i$ denotes the number of occurrences of the letter corresponding to $i \in \{1,2,3...8\}$ in X_n.

From (11) to (14) of Theorem 1 can be used to find $|H_n|_i$

It is clear that

$|B_n|_4 = |B_n|_8 = |C_n|_5 = |C_n|_1 = |D_n|_8 = |D_n|_4 = |H_n|_1$

$|B_n|_1 = |B_n|_5 = |B_n|_3 = |B_n|_7 = |H_n|_4$

$|C_n|_8 = |C_n|_4 = |C_n|_6 = |C_n|_2 = |H_n|_4$

$\left|D_n\right|_5 = \left|D_n\right|_1 = \left|D_n\right|_7 = \left|D_n\right|_3 = \left|H_n\right|_4$

Thus from (11) to (14) of Theorem 1 can be used again to find $\left|X_n\right|_i$ where $X \in \{B, C, D\}$

Let us denote for i ≥ j, $\left|X_n\right|_{i--j} = \sum_{k=i}^{j} \left|X_n\right|_k$

Let us discuss about the number of occurrence τ =1#2

We have $H_n = B_{n-1}uB_{n-1}uB_{n-1}uH_{n-1}rH_{n-1}rH_{n-1}dD_{n-1}dD_{n-1}dD_{n-1}$

There are six cases.

Case (i)

τ can appear either in any one of B_{n-1}'s or H_{n-1}'s or D_{n-1}'s.

Thus $H_n^\tau = 3B_{n-1}^\tau + 3H_{n-1}^\tau + 3D_{n-1}^\tau$

Case (ii)

The letter corresponding to the 1 in τ can be a "u" or a "r" or a "d".

So there are 21 combinations for the occurrences of τ if the letter corresponding to 2 is other u's or r's or d's.

Moreover the letter corresponding to 2 can appear either in B_{n-1} or H_{n-1} or D_{n-1}.

In this case $H_n^\tau = 3\left|B\right|_{3--8} + 9\left|A\right|_{3--8} + 9\left|D\right|_{3--8} + 3\left|A\right|_{4--8} + 6\left|D\right|_{4--8} + 6\left|D\right|_{7--8}$

Case (iii)

The letter corresponding to the 1 in τ can appear any one of B_{n-1}'s .

In this case $H_n^\tau = \sum_{i=0}^{2}\sum_{j=1}^{8} \left|B_{n-1}\right|_j \left[(2-i)\left|B_{n-1}\right|_{(i+1)--8} + 3\left|A_{n-1}\right|_{(i+1)--8} + 3\left|D_{n-1}\right|_{(i+1)--8}\right]$

A similar argument can be applied to other two cases.

Combining all these cases, we have the following theorem.

7.1. Theorem:

Let τ =1#2. (Similarly if τ =2#1) for For n > 1, $H_n^\tau, B_n^\tau, C_n^\tau$ and D_n^τ are given by the following recurrence.

$$\begin{pmatrix} H_n^\tau \\ B_n^\tau \\ C_n^\tau \\ D_n^\tau \end{pmatrix} = \begin{pmatrix} 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 0 \\ 0 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} H_{n-1}^\tau \\ B_{n-1}^\tau \\ C_{n-1}^\tau \\ D_{n-1}^\tau \end{pmatrix} + \begin{pmatrix} h_n \\ b_n \\ c_n \\ d_n \end{pmatrix}$$

where

$$h_n = 3B_{n-1}^\tau + 3H_{n-1}^\tau + 3D_{n-1}^\tau + \sum_{i=0}^{2}\sum_{j=1}^{7} \left|B_{n-1}\right|_j \left[(2-i)\left|B_{n-1}\right|_{(j+1)--8} + 3\left|A_{n-1}\right|_{(j+1)--8} + 3\left|D_{n-1}\right|_{(j+1)--8}\right] +$$
$$\sum_{i=0}^{2}\sum_{j=1}^{7} \left|A_{n-1}\right|_j \left[(2-i)\left|A_{n-1}\right|_{(j+1)--8} + 3\left|D_{n-1}\right|_{(j+1)--8}\right] + \sum_{i=0}^{1}\sum_{j=1}^{7} \left|D_{n-1}\right|_j \left[(2-i)\left|D_{n-1}\right|_{(j+1)--8}\right] +$$
$$3\left|B\right|_{3--8} + 9\left|H\right|_{3--8} + 9\left|D\right|_{3--8} + 3\left|H\right|_{4--8} + 6\left|D\right|_{4--8} + 6\left|D\right|_{7--8} + 21$$

$$b_n = 3B_{n-1}^\tau + 3H_{n-1}^\tau + 3C_{n-1}^\tau + \sum_{i=0}^{2}\sum_{j=1}^{7} \left|A_{n-1}\right|_j \left[(2-i)\left|A_{n-1}\right|_{(j+1)--8} + 3\left|B_{n-1}\right|_{(j+1)--8} + 3\left|C_{n-1}\right|_{(j+1)--8}\right] +$$
$$\sum_{i=0}^{2}\sum_{j=1}^{7} \left|B_{n-1}\right|_j \left[(2-i)\left|B_{n-1}\right|_{(i+1)--8} + 3\left|C_{n-1}\right|_{(i+1)--8}\right] + \sum_{i=0}^{1}\sum_{j=1}^{7} \left|C_{n-1}\right|_j \left[(2-i)\left|C_{n-1}\right|_{(j+1)--8}\right] +$$
$$3\left|H\right|_{4--8} + 9\left|B\right|_{4--8} + 9\left|C\right|_{4--8} + 3\left|B\right|_{3--8} + 6\left|C\right|_{3--8} + 6\left|C\right|_{8} + 15$$

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$$c_n = 3B_{n-1}^{\tau} + 3C_{n-1}^{\tau} + 3D_{n-1}^{\tau} + \sum_{i=0}^{2}\sum_{j=1}^{7}|D_{n-1}|_j\left[(2-i)|D_{n-1}|_{(j+1)--8} + 3|C_{n-1}|_{(j+1)--8} + 3|B_{n-1}|_{(j+1)--8}\right] +$$
$$\sum_{i=0}^{2}\sum_{j=1}^{7}|C_{n-1}|_j\left[(2-i)|C_{n-1}|_{(j+1)--8} + 3|B_{n-1}|_{(j+1)--8}\right] + \sum_{i=0}^{1}\sum_{j=1}^{7}|B_{n-1}|_j\left[(2-i)|B_{n-1}|_{(j+1)--8}\right] +$$
$$3|D|_{7--8} + 9|C|_{7--8} + 9|B|_{7--8} + 3|C|_8 + 6|B|_8 + 6|B|_{3--8} + 6$$

$$d_n = 3C_{n-1}^{\tau} + 3H_{n-1}^{\tau} + 3D_{n-1}^{\tau} + \sum_{i=0}^{2}\sum_{j=1}^{7}|C_{n-1}|_j\left[(2-i)|C_{n-1}|_{(j+1)--8} + 3|D_{n-1}|_{(j+1)--8} + 3|A_{n-1}|_{(j+1)--8}\right] +$$
$$\sum_{i=0}^{2}\sum_{j=1}^{7}|D_{n-1}|_j\left[(2-i)|D_{n-1}|_{(j+1)--8} + 3|A_{n-1}|_{(j+1)--8}\right] + \sum_{i=0}^{1}\sum_{j=1}^{7}|A_{n-1}|_j\left[(2-i)|A_{n-1}|_{(j+1)--8}\right] +$$
$$3|C|_8 + 9|D|_8 + 9|H|_8 + 3|D|_{7--8} + 6|H|_{4--8} + 6|A|_{4--8}$$

VIII. Counting occurrences of ordered patterns through homomorphism:

The generation of RSFC through by a grammar has been explained in section 3. If the templeates H,A,B,Cand arrows $\rightarrow,\leftarrow,\uparrow,\downarrow$ used in the construction of RSFC are named respectively by the letters $a_{\sigma(i)}$, i = 1,2,…8, where $\sigma \in S_8$, then the RSFC can be generated by the DOL system S = $(X, f, a_{\sigma(1)})$ where

$$f(a_{\sigma(1)}) \rightarrow a_{\sigma(4)}a_{\sigma(7)}a_{\sigma(4)}a_{\sigma(7)}a_{\sigma(4)}a_{\sigma(7)}a_{\sigma(1)}a_{\sigma(5)}a_{\sigma(1)}a_{\sigma(5)}a_{\sigma(1)}a_{\sigma(8)}a_{\sigma(2)}a_{\sigma(8)}a_{\sigma(2)}a_{\sigma(8)}a_{\sigma(2)}$$
$$f(a_{\sigma(2)}) \rightarrow a_{\sigma(3)}a_{\sigma(6)}a_{\sigma(3)}a_{\sigma(6)}a_{\sigma(3)}a_{\sigma(6)}a_{\sigma(2)}a_{\sigma(8)}a_{\sigma(2)}a_{\sigma(8)}a_{\sigma(2)}a_{\sigma(5)}a_{\sigma(1)}a_{\sigma(5)}a_{\sigma(1)}a_{\sigma(5)}a_{\sigma(1)}$$
$$f(a_{\sigma(3)}) \rightarrow a_{\sigma(1)}a_{\sigma(5)}a_{\sigma(1)}a_{\sigma(5)}a_{\sigma(1)}a_{\sigma(5)}a_{\sigma(4)}a_{\sigma(7)}a_{\sigma(4)}a_{\sigma(7)}a_{\sigma(4)}a_{\sigma(6)}a_{\sigma(3)}a_{\sigma(6)}a_{\sigma(3)}a_{\sigma(6)}a_{\sigma(3)}$$
$$f(a_{\sigma(4)}) \rightarrow a_{\sigma(2)}a_{\sigma(8)}a_{\sigma(2)}a_{\sigma(8)}a_{\sigma(2)}a_{\sigma(8)}a_{\sigma(3)}a_{\sigma(6)}a_{\sigma(3)}a_{\sigma(6)}a_{\sigma(3)}a_{\sigma(7)}a_{\sigma(4)}a_{\sigma(7)}a_{\sigma(4)}a_{\sigma(7)}a_{\sigma(4)}$$
$$f(a_{\sigma(5)}) \rightarrow a_{\sigma(5)}$$
$$f(a_{\sigma(6)}) \rightarrow a_{\sigma(6)}$$
$$f(a_{\sigma(7)}) \rightarrow a_{\sigma(7)}$$
$$f(a_{\sigma(8)}) \rightarrow a_{\sigma(8)}$$

8.1. Inversions and non-inversions of f^n:

Let n be a non-negative integer.

The vector of non-inversions of f^n is the 8-vector (8x1 column matrix) whose ith entry is the number of occurrences of the ordered pattern 1#2 in the word $f^n(a_{\sigma(i)})$, i.e.,

$$RG(f^n) = \left(\left|f^n(a_{\sigma(i)})\right|_{1\#2}\right)_{1\le i\le 8}$$

The vector of inversions of f^n is the 8-vector (8x1 column matrix) whose ith entry is the number of occurrences of the ordered pattern 2#1 in the word $f^n(a_{\sigma(i)})$, i.e.,

$$DG(f^n) = \left(\left|f^n(a_{\sigma(i)})\right|_{2\#1}\right)_{1\le i\le 8}$$

8.2. Repetitions of one letter:

Let n be a non-negative integer and p a positive integer. The vector of p-repetitions with gaps of one letter of f^n is the 8-vector whose i-th entry is the number of occurrences of the ordered pattern $(1\#)^p$ in the word $f^n(a_{\sigma(i)})$. i.e., $R_pG(f^n) = \left(\left|f^n(a_{\sigma(i)})\right|_{(1\#)^p}\right)_{1\le i\le 8}$

8.3. Ordered patterns with no gaps and morphisms:

8.3.1. Rises, Descents and cubes of f^n:

The vector of rises of f^n is the 8-vector whose ith entry is the number of occurrences of the ordered pattern 12 in the word $f^n(a_{\sigma(i)})$. i.e., $R(f^n) = \left(\left|f^n(a_{\sigma(i)})\right|_{12}\right)_{1\le i\le 8}$

The vector of descents of f^n is the 8-vector whose ith entry is the number of occurrences of the ordered pattern 21 in the word $f^n(a_{\sigma(i)})$. i.e., $D(f^n) = \left(\left|f^n(a_{\sigma(i)})\right|_{21}\right)_{1\le i\le 8}$

The vector of cubes of f^n is the 8-vector whose ith entry is the number of occurrences of the ordered pattern 111 in the word $f^n(a_{\sigma(i)})$. i.e., $R_3(f^n) = \left(\left|f^n(a_{\sigma(i)})\right|_{111}\right)_{1\le i\le 8}$

8.3.2. Incidence Matrix of f^n:

Let n be a non-negative integer.. The incidence matrix of f^n is the 8 x 8 matrix

$M(f^n) = (m_{n,\sigma(j),\sigma(i)})_{1\le i,j\le 8}$ where $m_{n,\sigma(j),\sigma(i)}$ is the number of occurrences of the letter $a_{\sigma(i)}$ in the word $f^n(a_{\sigma(j)})$ i.e., $m_{n,i,j} = \left|f^n(a_{\sigma(j)})\right|_{a_{\sigma(i)}}$. So we can denote $\left|f^n(a_{\sigma(j)})\right|_{a_1}$ as $m_{n,\sigma(j),1}$

Let us denote $\left|f^n(a_{\sigma(j)})\right|_{a_i--a_k}$ as $m_{n,\sigma(j),i--k}$.

Clearly it is understandable that $\left|f^n(a_{\sigma(j)})\right|_{a_1} = m_{n,\sigma(j),i--i}$

Using these notations and generalizing the theorem 6, we have the following theorem.

8.4. Theorem:

$$\left|f^{n+1}(a_{\sigma(1)})\right|_{1\#2} =$$

$$3\left|f^n(a_{\sigma(4)})\right|_{1\#2} + 3\left|f^n(a_{\sigma(1)})\right|_{1\#2} + 3\left|f^n(a_{\sigma(2)})\right|_{1\#2} + 3\left|f^n(a_{\sigma(7)})\right|_{1\#2} + 3\left|f^n(a_{\sigma(8)})\right|_{1\#2} + 2\left|f^n(a_{\sigma(5)})\right|_{1\#2} +$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(4),j}\begin{pmatrix}(3-i)m_{n,\sigma(7),(j+1)--8} + (2-i)m_{n,\sigma(4),(j+1)--8} + 3m_{n,\sigma(1),(j+1)--8}\\ +2m_{n,\sigma(5),(j+1)--8} + 3m_{n,\sigma(8),(j+1)--8} + 3m_{n,\sigma(2),(j+1)--8}\end{pmatrix} +$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(7),j}\begin{pmatrix}(2-i)m_{n,\sigma(7),(j+1)--8} + (2-i)m_{n,\sigma(4),(j+1)--8} + 3m_{n,\sigma(1),(j+1)--8}\\ +2m_{n,\sigma(5),(j+1)--8} + 3m_{n,\sigma(8),(j+1)--8} + 3m_{n,\sigma(2),(j+1)--8}\end{pmatrix} +$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(1),j}\left((2-i)m_{n,\sigma(1),(j+1)--8} + (2-i)m_{n,\sigma(5),(j+1)--8} + 3m_{n,\sigma(8),(j+1)--8} + 3m_{n,\sigma(2),(j+1)--8}\right) +$$

$$\sum_{i=0}^{1}\sum_{j=1}^{7} m_{n,\sigma(5),j}\left((2-i)m_{n,\sigma(1),(j+1)--8} + (1-i)m_{n,\sigma(5),(j+1)--8} + 3m_{n,\sigma(8),(j+1)--8} + 3m_{n,\sigma(2),(j+1)--8}\right) +$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(8),j}\left((2-i)m_{n,\sigma(8),(j+1)--8} + (3-i)m_{n,\sigma(2),(j+1)--8}\right) +$$

$$\sum_{i=0}^{1}\sum_{j=1}^{7} m_{n,\sigma(2),j}\left((2-i)m_{n,\sigma(8),(j+1)--8} + (2-i)m_{n,\sigma(2),(j+1)--8}\right)$$

$$\left|f^{n+1}(a_{\sigma(2)})\right|_{1\#2} =$$

$$3\left|f^n(a_{\sigma(3)})\right|_{1\#2} + 3\left|f^n(a_{\sigma(2)})\right|_{1\#2} + 3\left|f^n(a_{\sigma(1)})\right|_{1\#2} + 3\left|f^n(a_{\sigma(6)})\right|_{1\#2} + 3\left|f^n(a_{\sigma(5)})\right|_{1\#2} + 2\left|f^n(a_{\sigma(8)})\right|_{1\#2}$$

+

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(3),j}\begin{pmatrix}(3-i)m_{n,\sigma(6),(j+1)--8}+(2-i)m_{n,\sigma(3),(j+1)--8}+3m_{n,\sigma(2),(j+1)--8}\\+2m_{n,\sigma(8),(j+1)--8}+3m_{n,\sigma(5),(j+1)--8}+3m_{n,\sigma(1),(j+1)--8}\end{pmatrix}+$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(6),j}\begin{pmatrix}(2-i)m_{n,\sigma(6),(j+1)--8}+(2-i)m_{n,\sigma(3),(j+1)--8}+3m_{n,\sigma(2),(j+1)--8}\\+2m_{n,\sigma(8),(j+1)--8}+3m_{n,\sigma(5),(j+1)--8}+3m_{n,\sigma(1),(j+1)--8}\end{pmatrix}+$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(2),j}\left((2-i)m_{n,\sigma(2),(j+1)--8}+(2-i)m_{n,\sigma(8),(j+1)--8}+3m_{n,\sigma(5),(j+1)--8}+3m_{n,\sigma(1),(j+1)--8}\right)+$$

$$\sum_{i=0}^{1}\sum_{j=1}^{7} m_{n,\sigma(8),j}\left((2-i)m_{n,\sigma(2),(j+1)--8}+(1-i)m_{n,\sigma(8),(j+1)--8}+3m_{n,\sigma(5),(j+1)--8}+3m_{n,\sigma(1),(j+1)--8}\right)+$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(5),j}\left((2-i)m_{n,\sigma(5),(j+1)--8}+(3-i)m_{n,\sigma(1),(j+1)--8}\right)+$$

$$\sum_{i=0}^{1}\sum_{j=1}^{7} m_{n,\sigma(1),j}\left((2-i)m_{n,\sigma(5),(j+1)--8}+(2-i)m_{n,\sigma(1),(j+1)--8}\right)$$

$$\left|f^{n+1}(a_{\sigma(3)})\right|_{1\#2}=$$

$$3\left|f^{n}(a_{\sigma(1)})\right|_{1\#2}+3\left|f^{n}(a_{\sigma(4)})\right|_{1\#2}+3\left|f^{n}(a_{\sigma(3)})\right|_{1\#2}+3\left|f^{n}(a_{\sigma(5)})\right|_{1\#2}+3\left|f^{n}(a_{\sigma(6)})\right|_{1\#2}+2\left|f^{n}(a_{\sigma(8)})\right|_{1\#2}+$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(1),j}\begin{pmatrix}(3-i)m_{n,\sigma(5),(j+1)--8}+(2-i)m_{n,\sigma(1),(j+1)--8}+3m_{n,\sigma(4),(j+1)--8}\\+2m_{n,\sigma(7),(j+1)--8}+3m_{n,\sigma(6),(j+1)--8}+3m_{n,\sigma(3),(j+1)--8}\end{pmatrix}+$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(5),j}\begin{pmatrix}(2-i)m_{n,\sigma(5),(j+1)--8}+(2-i)m_{n,\sigma(1),(j+1)--8}+3m_{n,\sigma(4),(j+1)--8}\\+2m_{n,\sigma(7),(j+1)--8}+3m_{n,\sigma(6),(j+1)--8}+3m_{n,\sigma(3),(j+1)--8}\end{pmatrix}+$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(4),j}\left((2-i)m_{n,\sigma(4),(j+1)--8}+(2-i)m_{n,\sigma(7),(j+1)--8}+3m_{n,\sigma(6),(j+1)--8}+3m_{n,\sigma(3),(j+1)--8}\right)+$$

$$\sum_{i=0}^{1}\sum_{j=1}^{7} m_{n,\sigma(7),j}\left((2-i)m_{n,\sigma(4),(j+1)--8}+(1-i)m_{n,\sigma(7),(j+1)--8}+3m_{n,\sigma(6),(j+1)--8}+3m_{n,\sigma(3),(j+1)--8}\right)+$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(6),j}\left((2-i)m_{n,\sigma(6),(j+1)--8}+(3-i)m_{n,\sigma(3),(j+1)--8}\right)+$$

$$\sum_{i=0}^{1}\sum_{j=1}^{7} m_{n,\sigma(3),j}\left((2-i)m_{n,\sigma(6),(j+1)--8}+(2-i)m_{n,\sigma(3),(j+1)--8}\right)$$

$$\left|f^{n+1}(a_{\sigma(4)})\right|_{1\#2}=$$

$$3\left|f^{n}(a_{\sigma(2)})\right|_{1\#2}+3\left|f^{n}(a_{\sigma(3)})\right|_{1\#2}+3\left|f^{n}(a_{\sigma(4)})\right|_{1\#2}+3\left|f^{n}(a_{\sigma(8)})\right|_{1\#2}+3\left|f^{n}(a_{\sigma(7)})\right|_{1\#2}+2\left|f^{n}(a_{\sigma(6)})\right|_{1\#2}$$
$+$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(2),j}\begin{pmatrix}(3-i)m_{n,\sigma(8),(j+1)--8}+(2-i)m_{n,\sigma(2),(j+1)--8}+3m_{n,\sigma(3),(j+1)--8}\\+2m_{n,\sigma(6),(j+1)--8}+3m_{n,\sigma(7),(j+1)--8}+3m_{n,\sigma(4),(j+1)--8}\end{pmatrix}+$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(8),j}\begin{pmatrix}(2-i)m_{n,\sigma(8),(j+1)--8}+(2-i)m_{n,\sigma(2),(j+1)--8}+3m_{n,\sigma(3),(j+1)--8}\\+2m_{n,\sigma(6),(j+1)--8}+3m_{n,\sigma(7),(j+1)--8}+3m_{n,\sigma(4),(j+1)--8}\end{pmatrix}+$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(3),j}\left((2-i)m_{n,\sigma(3),(j+1)--8}+(2-i)m_{n,\sigma(6),(j+1)--8}+3m_{n,\sigma(7),(j+1)--8}+3m_{n,\sigma(4),(j+1)--8}\right)+$$

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$$\sum_{i=0}^{1}\sum_{j=1}^{7} m_{n,\sigma(6),j}\left((2-i)m_{n,\sigma(3),(j+1)--8} + (1-i)m_{n,\sigma(6),(j+1)--8} + 3m_{n,\sigma(7),(j+1)--8} + 3m_{n,\sigma(4),(j+1)--8}\right)+$$

$$\sum_{i=0}^{2}\sum_{j=1}^{7} m_{n,\sigma(7),j}\left((2-i)m_{n,\sigma(7),(j+1)--8} + (3-i)m_{n,\sigma(4),(j+1)--8}\right)+$$

$$\sum_{i=0}^{1}\sum_{j=1}^{7} m_{n,\sigma(4),j}\left((2-i)m_{n,\sigma(7),(j+1)--8} + (2-i)m_{n,\sigma(4),(j+1)--8}\right)$$

$$\left|f^{n+1}(a_{\sigma(1)})\right|_{2\#1} =$$

$$3\left|f^{n}(a_{\sigma(4)})\right|_{2\#1} + 3\left|f^{n}(a_{\sigma(1)})\right|_{2\#1} + 3\left|f^{n}(a_{\sigma(2)})\right|_{2\#1} + 3\left|f^{n}(a_{\sigma(7)})\right|_{2\#1} + 3\left|f^{n}(a_{\sigma(8)})\right|_{2\#1} + 2\left|f^{n}(a_{\sigma(5)})\right|_{2\#1}$$

$+$

$$\sum_{i=0}^{2}\sum_{j=2}^{8} m_{n,\sigma(4),j}\begin{pmatrix}(3-i)m_{n,\sigma(7),1--(j-1)} + (2-i)m_{n,\sigma(4),1--(j-1)} + 3m_{n,\sigma(1),1--(j-1)}\\ + 2m_{n,\sigma(5),1--(j-1)} + 3m_{n,\sigma(8),1--(j-1)} + 3m_{n,\sigma(2),1--(j-1)}\end{pmatrix}+$$

$$\sum_{i=0}^{2}\sum_{j=2}^{8} m_{n,\sigma(7),j}\begin{pmatrix}(2-i)m_{n,\sigma(7),1--(j-1)} + (2-i)m_{n,\sigma(4),1--(j-1)} + 3m_{n,\sigma(1),1--(j-1)}\\ + 2m_{n,\sigma(5),1--(j-1)} + 3m_{n,\sigma(8),1--(j-1)} + 3m_{n,\sigma(2),1--(j-1)}\end{pmatrix}+$$

$$\sum_{i=0}^{2}\sum_{j=2}^{8} m_{n,\sigma(1),j}\left((2-i)m_{n,\sigma(1),1--(j-1)} + (2-i)m_{n,\sigma(5),1--(j-1)} + 3m_{n,\sigma(8),1--(j-1)} + 3m_{n,\sigma(2),1--(j-1)}\right)+$$

$$\sum_{i=0}^{1}\sum_{j=2}^{8} m_{n,\sigma(5),j}\left((2-i)m_{n,\sigma(1),1--(j-1)} + (1-i)m_{n,\sigma(5),1--(j-1)} + 3m_{n,\sigma(8),1--(j-1)} + 3m_{n,\sigma(2),1--(j-1)}\right)+$$

$$\sum_{i=0}^{2}\sum_{j=2}^{8} m_{n,\sigma(8),j}\left((2-i)m_{n,\sigma(8),1--(j-1)} + (3-i)m_{n,\sigma(2),1--(j-1)}\right)+$$

$$\sum_{i=0}^{1}\sum_{j=2}^{8} m_{n,\sigma(2),j}\left((2-i)m_{n,\sigma(8),1--(j-1)} + (2-i)m_{n,\sigma(2),1--(j-1)}\right)$$

The recursive relation for $\left|f^{n+1}(a_{\sigma(2)})\right|_{2\#1}$, $\left|f^{n+1}(a_{\sigma(3)})\right|_{2\#1}$ and $\left|f^{n+1}(a_{\sigma(4)})\right|_{2\#1}$ can be expressed in a similar manner.

8.5. Theorem:

Let $L_{n,i} = \sigma(k)$ if the last letter of $f^{n}(a_{\sigma(i)})$ is $a_{\sigma(k)}$,

$F_{n,i} = \sigma(k)$ if the First letter of $f^{n}(a_{\sigma(i)})$ is $a_{\sigma(k)}$,

$$K_{n,i,j} = \begin{cases} 1, & if \quad L_{n,i} < F_{n,j} \\ -1, & if \quad L_{n,i} > F_{n,j} \\ 0, & if \quad L_{n,i} = F_{n,j} \end{cases}$$

Then $\left|f^{n+1}(a_{\sigma(1)})\right|_{12} =$

$$3\left|f^{n}(a_{\sigma(4)})\right|_{12} + 3\left|f^{n}(a_{\sigma(1)})\right|_{12} + 3\left|f^{n}(a_{\sigma(2)})\right|_{12} + 3\left|f^{n}(a_{\sigma(7)})\right|_{12} + 3\left|f^{n}(a_{\sigma(8)})\right|_{12} + 2\left|f^{n}(a_{\sigma(5)})\right|_{12} +$$

$$3K_{n,4,7} + 2K_{n,7,4} + 2K_{n,1,5} + 2K_{n,5,1} + 3K_{n,8,2} + 2K_{n,2,8} + K_{n,7,1} + K_{n,1,8}$$

$\left| f^{n+1}(a_{\sigma(1)}) \right|_{21} =$

$3\left| f^{n}(a_{\sigma(4)}) \right|_{21} + 3\left| f^{n}(a_{\sigma(1)}) \right|_{21} + 3\left| f^{n}(a_{\sigma(2)}) \right|_{21} + 3\left| f^{n}(a_{\sigma(7)}) \right|_{21} + 3\left| f^{n}(a_{\sigma(8)}) \right|_{21} + 2\left| f^{n}(a_{\sigma(5)}) \right|_{21}$ -

$3\,K_{n,4,7}$ -2 $K_{n,7,4}$ -2 $K_{n,1,5}$ - $2K_{n,5,1} - 3K_{n,8,2} - 2K_{n,2,8} - K_{n,7,1} - K_{n,1,8}$

8.6. Theorem:

The incidence matrix of f^n is

$$\begin{pmatrix} s_3 & s_2 & s_2 & s_2 & s_3-1 & s_2 & s_2 & s_2 \\ s_2 & s_3 & s_2 & s_2 & s_2 & s_2 & s_2 & s_3-1 \\ s_2 & s_2 & s_2 & s_3 & s_2 & s_2 & s_3-1 & s_2 \\ s_2 & s_2 & s_3 & s_2 & s_2 & s_3-1 & s_2 & s_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$, when n is even.

$$\begin{pmatrix} s_1 & s_1 & s_4 & s_1 & s_1-1 & s_4 & s_1 & s_1 \\ s_1 & s_1 & s_1 & s_4 & s_1 & s_1 & s_4 & s_1-1 \\ s_1 & s_4 & s_1 & s_1 & s_1 & s_1 & s_1-1 & s_4 \\ s_4 & s_1 & s_1 & s_1 & s_4 & s_1-1 & s_1 & s_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$, when n is odd.

$s_1 = \frac{3^n(3^n+1)}{4}$, $s_2 = \frac{3^n(3^n-1)}{4}$, $s_3 = \frac{3^{n+1}(3^{n-1}+1)}{4}$, $s_4 = \frac{3^{n+1}(3^{n-1}-1)}{4}$

$R_p G(f^n) = \left(\left| f^n(a_{\sigma(i)}) \right|_{(1\#)^p} \right)_{1\le i\le 8}$ is

$$\begin{pmatrix} (6s_2)C_p + (s_3)C_p + (s_3-1)C_p \\ (6s_2)C_p + (s_3)C_p + (s_3-1)C_p \\ (6s_2)C_p + (s_3)C_p + (s_3-1)C_p \\ (6s_2)C_p + (s_3)C_p + (s_3-1)C_p \\ 1C_p \\ 1C_p \\ 1C_p \\ 1C_p \end{pmatrix}$$, if n is even

$$\begin{pmatrix} s_1 & s_1 & s_4 & s_1 & s_1-1 & s_4 & s_1 & s_1 \\ s_1 & s_1 & s_1 & s_4 & s_1 & s_1 & s_4 & s_1-1 \\ s_1 & s_4 & s_1 & s_1 & s_1 & s_1 & s_1-1 & s_4 \\ s_4 & s_1 & s_1 & s_1 & s_4 & s_1-1 & s_1 & s_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} (5s_1)C_p + (2s_4)C_p + (s_1-1)C_p \\ (5s_1)C_p + (2s_4)C_p + (s_1-1)C_p \\ (5s_1)C_p + (2s_4)C_p + (s_1-1)C_p \\ (5s_1)C_p + (2s_4)C_p + (s_1-1)C_p \\ 1C_p \\ 1C_p \\ 1C_p \\ 1C_p \end{pmatrix}$$

Conclusion:

In this paper, we observed recursive occurrences of rises, descents and ordered patterns in the finite words which represent finite approximations of the Rectangular Space Filling Curve.

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