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Bayesian Approach of One Parameter Maxwell Distribution under Two Different Loss Functions

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ABSTRACT

For estimating an unknown parameter for the Maxwell distribution, we obtained some Bayesian estimators under Squared-log error loss function and Precautionary loss function using Non-informative prior, represented by Jefferys prior and Informative priors as Gumbel type –II prior, Inverted Gamma prior, and Inverted Levy. According to Monte-Carlo simulation study, behavior of all estimators, namely Bayes estimators under Squared-Log error loss function and Precautionary loss function, based on different priors, is compared with respect to mean squared errors. The results showed that, the performance of Bayes estimator under Squared–log error loss function with Inverted Gamma prior is the best in performance than other estimators for all cases.

INTRODUCTION

Maxwell distribution plays an important role in Physics and Chemistry. It gives the distribution of speeds of molecules in thermal equilibrium as given by statistical mechanics. For example, this distribution explains many fundamental gas properties in kinetic theory of gases, distribution of energies and moments,…etc.

The Bayesian deduction requires appropriate choice of priors for the parameters. In the last several decades, Bayesian analysis focused on priors that are un-informative. But if we have enough information about the parameter, then it is better to use the informative priors. The parameters of the prior distribution called hyper-parameters.

The Maxwell distribution was first introduced in the literature by J.C. Maxwell (1860) and again described by Boltzmann (1870) with a few assumptions

Tyagi and Bhattacharya (1989a) studied Maxwell distribution as a lifetime model for the first time. They derived some Bayes estimates and minimum variance unbiased estimators of the parameter and reliability function for the Maxwell distribution. Podder and Roy(2003) obtained the Bayes estimation of the parameter of Maxwell distribution under modified linear exponential loss function. In (2005) Bekker and Roux, studied empirical Bayes estimation for this distribution, and they assumed that complete sample information is available. Rasheed and Khalifa studied Semi- Minimax estimation of the parameter of the Maxwell distribution under New loss function with Jeffers prior. In the present study, we are interested in finding the best estimates for the scale parameter of Maxwell distribution under Squared-Log error loss function and Precautionary loss function with different priors, namely, Non-informative prior, represented by Jefferys prior and Informative priors as Gumbel type –II prior, Inverted Gamma prior, and Inverted Levy.

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1. Maxwell Distribution

The probability density function and the cumulative distribution function of Maxwell distribution, with the scale parameter θ are given by (Rasheed and Khalifa, 2016)

$$f(x\,;\theta)=\frac{4}{\sqrt{\pi}}\frac{1}{\theta^{3/2}}x^2 e^{-\frac{x^2}{\theta}} \qquad ; \qquad 0< X,\theta \tag{1}$$

$$F(x)=\frac{1}{\Gamma(\frac{3}{2})}\Gamma(\frac{x^2}{\theta},\frac{3}{2}) \tag{2}$$

Where $\Gamma(x,\alpha)=\int_0^x e^{-u}u^{\alpha-1}\,du$ is the incomplete Gamma function (Krishna and Malik, 2009).

It can also be expressed as follows

$$F(x;\theta)=2erf\left(\frac{x}{\sqrt{\theta}}\right)-\frac{2}{\sqrt{\pi}}\frac{x}{\theta}e^{-\frac{x^2}{\theta}}$$

Where $erf(x)=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-w^2}$ dw ,is the error function.

2.Bayes Estimator Under Squared – Log Error Loss Function:

Farsi pour and Zakerzadeh (2006) studied the estimation of generalized variance under an asymmetric loss function named the squared log error loss function, where is of the form:

$$L(\hat{\theta},\,\theta)=\left(ln\,\hat{\theta}-ln\,\theta\right)^2=\left(ln\frac{\hat{\theta}}{\theta}\right)^2 \tag{3}$$

Which is a balanced with $\lim L(\hat{\theta},\,\theta)\to\infty$ as $\hat{\theta}\to 0$ or ∞

A balanced loss function takes both error of estimation and goodness of fit in to account but the unbalanced loss function considers only error of estimation. This loss function is convex for $\frac{\hat{\theta}}{\theta}\le e$ and concave otherwise, it's risk function has a unique minimum with respect to $\hat{\theta}$ (Dey, S., 2010).

According to the above mentioned loss function, we derive the corresponding Bayes estimators for θ using Risk function $R(\hat{\theta},\theta)$, which minimizes the posterior risk.

$$R(\hat{\theta},\,\theta)=\,E\,[L(\hat{\theta},\theta)] \tag{4}$$

$$=\int_0^\infty\left(ln\,\hat{\theta}-ln\,\theta\right)^2 h(\theta|x_1\ldots\ldots\ldots..x_n)\,d\theta$$

$$\frac{\partial Risk}{\partial\hat{\theta}}=2\left(ln\,\hat{\theta}\right)\frac{1}{\hat{\theta}}-\frac{2}{\hat{\theta}}E\left(ln\,\theta\,|\underline{x}\right) \tag{5}$$

By letting, $\frac{\partial Risk}{\partial\hat{\theta}}=0$, the Bayes estimator for the parameter θ of Maxwell distribution under the Squared-log error loss function will be

$$\hat{\theta}=Exp[E\left(ln\,\theta\,|\underline{x}\right)] \tag{6}$$

3.Bayes Estimation Under Precautionary loss Function:

Precautionary loss function, which is proposed by Norstrom (1996), is one of asymmetric loss function, which can be defined as follows (Rasheed and Aref, 2016) :

$$L(\hat{\theta},\theta)=\frac{(\theta-\hat{\theta})^2}{\hat{\theta}} \tag{7}$$

The Precautionary loss function approach infinitely near the origin to prevent under estimation, thus giving conservative estimators, especially when low failure rates are being estimated. It is very useful when underestimation may lead to serious consequences. For instance, in the case of estimation of a financial charge or size of an order, underestimation has much more serious consequences.

It can be derive risk function and Bayes estimator for the parameter θ in $R_P(\hat{\theta},\theta)$ and $\hat{\theta}_P$ based on Precautionary loss function as follows:

$$R_P(\hat{\theta},\theta)=E[L(\hat{\theta},\theta)]$$

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$$R_P(\hat{\theta},\theta)=\int_0^{\infty}\frac{(\theta-\hat{\theta})^2}{\hat{\theta}}h(\theta|\underline{x})d\theta$$

$$R_P(\hat{\theta},\theta)=E(\theta^2|\underline{x})\hat{\theta}^{-1}-2E(\theta|\underline{x})+\hat{\theta}$$

Taking the partial derivative for $R_P(\hat{\theta},\theta)$ with respect to $\hat{\theta}$ and setting it equal to zero, we get the Bayes estimator as follows:

$$\hat{\theta}^2=E(\theta^2|\underline{x})$$

Hence, Bayes estimator relative to Precautionary loss function, denoted by $\hat{\theta}_P$ is

$$\hat{\theta}_P=\sqrt{E(\theta^2|\underline{x})} \qquad (8)$$

4.Priors and Posteriors Distributions for (θ):

In order to get better understanding of our Bayesian analysis, we consider informative as well as non-informative priors density for θ, as follows:

4.1. Non- informative prior:

The non-informative prior applied here is represented by Jefferys prior.
Jefferys prior is proposed by Harold Jeffrey in (1946). It is based on Fisher information (Rasheed and Aref. 2016) , such that:

$$g_1 \propto \sqrt{I(\theta)} \qquad (9)$$

Where $I(\theta)=-nE\left(\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right)$ is Fisher's information matrix. Hence,

$$g_1(\theta)=k\sqrt{-nE\left(\frac{\partial^2 \ln f(x;a,\theta)}{\partial\theta^2}\right)} \quad \text{, with k is a constant} \qquad (10)$$

Thus,

$$\frac{\partial^2 \ln f(x\,;\theta)}{\partial\theta^2}=\frac{3}{2\theta^2}-\frac{2x^2}{\theta^3}$$

$$E\left(\frac{\partial^2 \ln f(x\,;\theta)}{\partial\theta^2}\right)=\frac{-3}{2\theta^2} \qquad (11)$$

After substitution into (10), yields

$$g_1(\theta)=\frac{k}{\theta}\sqrt{\frac{3n}{2}}$$

Now, the posterior density function is defined as (Rasheed and Aref, 2016)

$$h(\theta|\underline{x})=\frac{g(\theta)L(\theta;x_1x_2,\dots,x_n)}{\int_0^{\infty}g(\theta)L(\theta;x_1x_2,\dots,x_n)d\theta} \qquad (12)$$

Therefore, the posterior density functions of (θ) with Jefferys prior is given by (Rasheed and Khalifa, 2016)

$$h_1(\theta|\underline{x})=\frac{(\sum_{i=1}^{n}x_i^2)^{\frac{3n}{2}}e^{-\sum_{i=1}^{n}x_i^2/\theta}}{\theta^{(\frac{3n}{2}+1)}\Gamma(\frac{3n}{2})}=\frac{T^{\frac{3n}{2}}e^{-\frac{T}{\theta}}}{\theta^{\left(\frac{3n}{2}+1\right)}\Gamma(\frac{3n}{2})} \qquad (13)$$

It is clear that, $\theta|\underline{x}\sim IG(\frac{3n}{2},T)$, with $E(\theta|\underline{x})=\frac{T}{(\frac{3n}{2}-1)}$, $Var(\theta|\underline{x})=\frac{T^2}{\left(\frac{3n}{2}-1\right)^2\left(\frac{3n}{2}-2\right)}$, $n>1$

Now, to obtain the Bayes estimator of parameter θ under Squared-log error loss function with Jefferys prior information we derived $E(ln\theta|\underline{x})_J$, where

$$E\left(ln\theta|\underline{x}\right)_J = \int_0^\infty ln\,\theta\; h_1\left(\theta|\underline{x}\right)_J d\theta$$

$$E\left(ln\theta|\underline{x}\right)_J = \frac{\left(\sum_{i=1}^n x_i^2\right)^{\frac{3n}{2}}}{\Gamma(\frac{3n}{2})}\int_0^\infty ln\,\theta\,\frac{e^{\frac{-\sum_{i=1}^n x_i^2}{\theta}}}{\theta^{\frac{3n}{2}+1}}\,d\theta$$

Using transformation technique, gives

$$E\left(ln\theta|\underline{x}\right)_J = \frac{ln(\sum_{i=1}^n x_i^2)}{\Gamma(\frac{3n}{2})}\int_0^\infty y^{\frac{3n}{2}-1}e^{-y}dy - \int_0^\infty \frac{ln(y)y^{\frac{3n}{2}-1}e^{-y}}{\Gamma(\frac{3n}{2})}dy$$

Where, $y = \frac{\sum_{i=1}^n x_i^2}{\theta}$

$$E\left(\ln\theta\,|\underline{x}\right)_J = ln(\sum_{i=1}^n x_i^2) - \psi\left(\frac{3n}{2}\right)$$

Such that $\psi\left(\frac{3n}{2}\right) = \frac{\Gamma'\left(\frac{3n}{2}\right)}{\Gamma\left(\frac{3n}{2}\right)}$, where $\psi\left(\frac{3n}{2}\right)$ is a Digamma function

After substitution into (6), yields the Bayes estimator of parameter θ under Squared-log error loss function with Jefferys prior which is denoted by $\hat{\theta}_{SLEJ}$ as:

$$\hat{\theta}_{SLEJ} = \frac{\sum_{i=1}^n x^2}{e^{\psi\left(\frac{3n}{2}\right)}} \tag{14}$$

Now, to obtain the Bayes estimator of parameter θ under Precautionary loss function with Jefferys prior information we derived $E\left(\theta^2|\underline{x}\right)_J$, where

$$E\left(\theta^2|\underline{x}\right)_J = \int_0^\infty \theta^2\,\theta\; h_1\left(\theta|\underline{x}\right)_J d\theta$$

$$E\left(\theta^2|\underline{x}\right)_J = \frac{\left(\sum_{i=1}^n x_i^2\right)^{\frac{3n}{2}}}{\Gamma(\frac{3n}{2})}\int_0^\infty \frac{e^{\frac{-\sum_{i=1}^n x_i^2}{\theta}}}{\theta^{\frac{3n}{2}-1}}\,d\theta$$

Using transformation technique, gives

$$E\left(\theta^2|\underline{x}\right)_J = \frac{\left(\sum_{i=1}^n x_i^2\right)^{\frac{3n}{2}}}{\Gamma(\frac{3n}{2})}\int_0^\infty \frac{e^{-y}\,\frac{\left(\sum_{i=1}^n x_i^2\right)}{y^2}}{\frac{\left(\sum_{i=1}^n x_i^2\right)^{\frac{3n}{2}-1}}{y^{\frac{3n}{2}-1}}}dy$$

$$= \frac{\left(\sum_{i=1}^n x_i^2\right)^2}{\Gamma\left(\frac{3n}{2}\right)}\int_0^\infty y^{\frac{3n}{2}-3}e^{-y}dy$$

$$E\left(\theta^2|\underline{x}\right)_J = \frac{\left(\sum_{i=1}^n x_i^2\right)^2}{\left(\frac{3n}{2}-1\right)\left(\frac{3n}{2}-2\right)} \tag{15}$$

After substituting in (8), we get the Bayes estimator of parameter **θ** under Precautionary loss function with Jefferys prior information which is denoted by $\hat{\theta}_{PJ}$ as:

$$\hat{\theta}_{PJ} = \frac{\sum_{i=1}^n x_i^2}{\sqrt{\left(\frac{3n}{2}-1\right)\left(\frac{3n}{2}-2\right)}} \tag{16}$$

4.2. Informative Priors:

In this section, we will derive Bayes estimators for θ under Squared- Log error loss function and Precautionary loss function based on three different informative priors, as follows:

(i) Gumbel Type II Prior:

The Gumbel type II prior with hyper parameter (b), is defined as: (Ali, S., *et al*., 2012)'

$$g_2(\theta) = b\left(\frac{1}{\theta}\right)^2 Exp\left[\frac{-b}{\theta}\right] \qquad b,\theta > 0 \tag{17}$$

According to (12), the posterior density function for θ based on Gumbel type II prior is (Rasheed, H.A. and Z.N. Khalifa, 2016):

$$h_2(\theta|\underline{x}) = \frac{(T+b)^{\frac{3n}{2}+1} e^{-\frac{(T+b)}{\theta}}}{\theta^{(\frac{3n}{2}+2)}\,\Gamma\left(\frac{3n}{2}+1\right)} \tag{18}$$

Note that: $\theta|\underline{x} \sim IG(\frac{3n}{2}+1\ ,\ T+b)$, such that, $E(\theta|\underline{x}) = \frac{T+b}{\frac{3n}{2}}$, $Var(\theta|\underline{x}) = \frac{(T+b)^2}{\left(\frac{3n}{2}\right)^2\left(\frac{3n}{2}-1\right)}$, $n > 1$

Based on (6), Bayes estimator for θ of Maxwell distribution under Squared-Log error loss function with Gumbel type-II prior can be obtained as follows:

$$E(\ln\theta\,|\underline{x})_{GL} = \int_0^\infty \ln\theta\, h_2(\theta|\underline{x})_{GL}\,d\theta$$

$$E(\ln\theta\,|\underline{x})_{GL} = \frac{(\sum_{i=1}^n x_i^2+b)^{\frac{3n}{2}+1}}{\Gamma(\frac{3n}{2}+1)}\int_0^\infty \ln\theta\,\frac{e^{\frac{-(\sum_{i=1}^n x_i^2+b)}{\theta}}}{\theta^{\frac{3n}{2}+2}}\,d\theta$$

By the same previous way, using transformation technique, where, $y = \frac{(\sum_{i=1}^n x_i^2+b)}{\theta}$, gives

$$E(\ln\theta\,|\underline{x})_{GL} = \ln(\sum_{i=1}^n x_i^2 + b) - \psi\left(\frac{3n}{2}+1\right)$$

Therefore, after substitution into (6), gives the Bayes estimator for θ of Maxwell distribution under Squared-Log error loss function with Gumbel type-II prior that is denoted by $\hat{\theta}_{SLEGL}$ will be:

$$\hat{\theta}_{SLEGL} = \frac{(\sum_{i=1}^n x^2+b)}{e^{\psi\left(\frac{3n}{2}+2\right)}} \tag{19}$$

Now, based on (8), Bayes estimator for θ of Maxwell distribution under Precautionary loss function with Gumbel type-II prior can be obtained as follows:

$$E(\theta^2|\underline{x})_{GL} = \int_0^\infty \theta^2\ h_2(\theta|\underline{x})_{GL}\,d\theta$$

$$E(\theta^2|\underline{x})_{GL} = \frac{(\sum_{i=1}^n x_i^2+b)^{\frac{3n}{2}+1}}{\Gamma(\frac{3n}{2}+1)}\int_0^\infty \frac{e^{\frac{-(\sum_{i=1}^n x_i^2+b)}{\theta}}}{\theta^{\frac{3n}{2}}}\,d\theta$$

Using transformation technique, gives

$$E(\theta^2|\underline{x})_{GL} = \frac{(\sum_{i=1}^n x_i^2+b)^{\frac{3n}{2}+1}}{\Gamma(\frac{3n}{2}+1)}\int_0^\infty \frac{e^{-y}\,\frac{(\sum_{i=1}^n x_i^2+b)}{y^2}}{\frac{(\sum_{i=1}^n x_i^2+b)^{\frac{3n}{2}}}{y^{\frac{3n}{2}}}}\,dy$$

$$= \frac{(\sum_{i=1}^n x_i^2+b)^2}{\Gamma\left(\frac{3n}{2}+1\right)}\int_0^\infty y^{\frac{3n}{2}-2}e^{-y}\,dy$$

$$= \frac{(\sum_{i=1}^n x_i^2+b)^2}{\left(\frac{3n}{2}\right)\left(\frac{3n}{2}-1\right)} \tag{20}$$

Substituting into (8), yields the Bayes estimator of parameter $\boldsymbol{\theta}$ under Precautionary loss function with Gumbel Type II prior information that is denoted by $\hat{\theta}_{PGL}$ will be:

$$\hat{\theta}_{PGL} = \frac{(\sum_{i=1}^n x_i^2+b)}{\sqrt{\left(\frac{3n}{2}\right)\left(\frac{3n}{2}-1\right)}} \tag{21}$$

(ii) Inverted Gamma Prior:

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This conjugate prior distribution is the distribution of the reciprocal of a variable distributed according to Inverted Gamma distribution that is assumed to be (Rasheed, H.A. and Z.N. Khalifa, 2016):

$$g_3(\theta) = \frac{\alpha^{\beta}}{\Gamma\beta}\,\frac{1}{\theta^{\beta+1}}\,e^{-\alpha/\theta} \qquad ; \quad \alpha, \beta,\ \theta > 0 \tag{22}$$

Where α and β are the scale parameter and the shape parameter of the prior distribution, respectively.
Now, the posterior density function is:

$$h_3(\theta|\underline{x}) = \frac{g_3(\theta)L(\theta; x_1, x_2, \dots, x_n)}{\int_0^{\infty} g_3(\theta)L(\theta; x_1, x_2, \dots, x_n)d\theta}$$

After simplification, we get

$$h_3(\theta|\underline{x}) = \frac{(T+\alpha)^{\frac{3n}{2}+\beta} e^{-\frac{(T+\alpha)}{\theta}}}{\theta^{(\frac{3n}{2}+\beta+1)}\,\Gamma\left(\frac{3n}{2}+\beta\right)} \tag{23}$$

It is clear that, $\theta|\underline{x} \sim IG\left(\frac{3n}{2}+\beta\ ,\ T+\alpha\right)$, such that, $E(\theta|\underline{x}) = \frac{T+\alpha}{\frac{3n}{2}+\beta-1}$, $Var(\theta|\underline{x}) \frac{(T+\alpha)^2}{\left(\frac{3n}{2}+\beta-1\right)^2\left(\frac{3n}{2}+\beta-2\right)}$ $\quad n > 1$

The Bayes estimator for θ of Maxwell distribution under Squared-log error loss function with Inverted Gamma prior can be obtained by apply (6), as follows:

$$E(\ln\theta\,|\underline{x})_{IG} = \int_0^{\infty} \ln\theta\; h_3(\theta|\underline{x})_{IG}\,d\theta$$

$$E(\ln\theta\,|\underline{x})_{IG} = \frac{\left(\sum_{i=1}^{n} x_i^2+\alpha\right)^{\frac{3n}{2}+\beta}}{\Gamma(\frac{3n}{2}+\beta)} \int_0^{\infty} \ln\theta\, \frac{e^{\frac{-(\sum_{i=1}^{n} x_i^2+\alpha)}{\theta}}}{\theta^{\frac{3n}{2}+\beta+1}}\,d\theta$$

Using the transformation technique, by the same previous way, where, $y = \frac{(\sum_{i=1}^{n} x_i^2+\alpha)}{\theta}$, yields

$$E(\ln\theta\,|\underline{x})_{IG} = \ln\left(\sum_{i=1}^{n} x_i^2+\alpha\right) - \psi\left(\frac{3n}{2}+\beta\right)$$

Therefore, after substitution into (6), gives the Bayes estimator for θ of Maxwell distribution under Squared-Log error loss function with Inverted Gamma prior which is denoted by $\hat{\theta}_{IG}$ as follows:

$$\hat{\theta}_{SLEIG} = \frac{(\sum_{i=1}^{n} x_i^2+\alpha)}{e^{\psi\left(\frac{3n}{2}+\beta\right)}} \tag{24}$$

Now, the Bayes estimator for θ of Maxwell distribution under Precautionary loss function with Inverted Gamma prior can be obtained by apply (8), as follows:

$$E(\theta^2|\underline{x})_{IG} = \int_0^{\infty} \theta^2\,\theta\; h_3(\theta|\underline{x})_{IG}\,d\theta$$

$$E(\theta^2|\underline{x})_{IG} = \frac{\left(\sum_{i=1}^{n} x_i^2+\alpha\right)^{\frac{3n}{2}+\beta}}{\Gamma(\frac{3n}{2}+\beta)} \int_0^{\infty} \frac{e^{\frac{-\left(\sum_{i=1}^{n} x_i^2+\alpha\right)}{\theta}}}{\theta^{\frac{3n}{2}+\beta-1}}\;d\theta$$

Using transformation technique,

$$E(\theta^2|\underline{x})_{IG} = \frac{\left(\sum_{i=1}^{n} x_i^2+\alpha\right)^{\frac{3n}{2}+\beta}}{\Gamma(\frac{3n}{2}+\beta)} \int_{\mathbf{0}}^{\infty} \frac{e^{-y}\,\frac{\left(\sum_{i=1}^{n} x_i^2+\alpha\right)}{y^2}}{\frac{\left(\sum_{i=1}^{n} x_i^2+\alpha\right)^{\frac{3n}{2}+\beta-1}}{y^{\frac{3n}{2}+\beta-1}}}\,dy$$

where, $y = \frac{(\sum_{i=1}^{n} x_i^2+\alpha)}{\theta}$

$$E(\theta^2|\underline{x})_{IG} = \frac{\left(\sum_{i=1}^{n} x_i^2+\alpha\right)^2}{\left(\frac{3n}{2}+\beta-1\right)\left(\frac{3n}{2}+\beta-2\right)} \tag{25}$$

Substituting into (8), gives the Bayes estimator of the scale parameter $\boldsymbol{\theta}$ under Precautionary loss function with Inverted Gamma prior information as:

$$\hat{\theta}_{PIG} = \frac{(\sum_{i=1}^{n} x_i^2+\alpha)}{\sqrt{\left(\frac{3n}{2}+\beta-1\right)\left(\frac{3n}{2}+\beta-2\right)}} \tag{26}$$

(*iii*) *Inverted Levy Prior:*

The inverted Levy prior is assumed to be **(**Sindhu, T., N; Aslam M,2013) **:**

$$g_4(\theta) = \sqrt{\frac{\lambda}{2\pi}}\,\frac{1}{\sqrt{\theta}}\,e^{-\frac{\lambda}{2\theta}}, \qquad \theta, \lambda > 0 \tag{27}$$

Where λ is the hyper parameter.
The posterior density functions of (θ) with Inverted Levy prior is given by (Rasheed, H.A. and Z.N. Khalifa, 2016)

$$h_4(\theta|\underline{x}) = \frac{(T+\frac{\lambda}{2})^{\frac{3n-1}{2}}\, e^{-\frac{(T+\frac{\lambda}{2})}{\theta}}}{\theta^{\frac{3n+1}{2}}\,\Gamma(\frac{3n-1}{2})} \tag{28}$$

Note that: $\theta|\underline{x} \sim IG\,(\frac{3n-1}{2}, T+\frac{\lambda}{2})$, where ,$E(\theta|\underline{x})=\frac{T+\frac{\lambda}{2}}{(\frac{3n-3}{2})}$, $Var.(\theta|\underline{x})=\frac{\left(T+\frac{\lambda}{2}\right)^2}{\left(\frac{3n-3}{2}\right)^2\left(\frac{3n-5}{2}\right)}$

The Bayes estimator for θ of Maxwell distribution under Squared-log error loss function with Inverted Levy prior can be obtained by apply (6), as follows:

$$E(\ln\theta\,|\underline{x})_{IL} = \int_0^{\infty} \ln\theta\; h_4(\theta|\underline{x})_{IL}\, d\theta$$

$$E(\ln\theta\,|\underline{x})_{IL} = \frac{\left(\sum_{i=1}^{n} x_i^2+\frac{\lambda}{2}\right)^{\frac{3n-1}{2}}}{\Gamma(\frac{3n-1}{2})}\int_0^{\infty} \ln\theta\, \frac{e^{\frac{-(\sum_{i=1}^{n} x_i^2+\frac{\lambda}{2})}{\theta}}}{\theta^{\frac{3n+1}{2}}}\, d\theta$$

By the transformation technique, where, $y = \frac{(\sum_{i=1}^{n} x_i^2+\frac{\lambda}{2})}{\theta}$

We can easily prove that,

$$E(\ln\theta\,|\underline{x})_{IL} = \ln\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right) - \psi\left(\frac{3n-1}{2}\right)$$

Thus, the Bayes estimator for θ of Maxwell distribution under Squared-Log error loss function with Inverted Levy prior that is denoted by $\hat{\theta}_{SLEIL}$ will be:

$$\hat{\theta}_{SLEIL} = \frac{\left(\sum_{i=1}^{n} x_i^2+\frac{\lambda}{2}\right)}{e^{\psi\left(\frac{3n-1}{2}\right)}} \tag{29}$$

The Bayes estimator for θ of Maxwell distribution under Precautionary loss function with Inverted Levy prior can be obtained by apply (8), as follows:

$$E(\theta^2|\underline{x})_{IL} = \int_{\mathbf{0}}^{\infty} \theta^2\,\theta\; h_4(\theta|\underline{x})_{IL}\, d\theta$$

$$E(\theta^2|\underline{x})_{IL} = \frac{\left(\sum_{i=1}^{n} x_i^2+\frac{\lambda}{2}\right)^{\frac{3n-1}{2}}}{\Gamma(\frac{3n-1}{2})}\int_0^{\infty} \frac{e^{\frac{-\left(\sum_{i=1}^{n} x_i^2+\frac{\lambda}{2}\right)}{\theta}}}{\theta^{\frac{3n-3}{2}}}\, d\theta$$

By the transformation technique, where, $y = \frac{(\sum_{i=1}^{n} x_i^2+\frac{\lambda}{2})}{\theta}$

$$E(\theta^2|\underline{x})_{IL} = \frac{\left(\sum_{i=1}^{n} x_i^2+\frac{\lambda}{2}\right)^2}{\left(\frac{3n-1}{2}-1\right)\left(\frac{3n-1}{2}-2\right)} \tag{30}$$

Substituting into (8), we get the Bayes estimator of parameter $\boldsymbol{\theta}$ under Precautionary loss function with Inverted Gamma prior information as:

$$\hat{\theta}_{PIL} = \frac{\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right)}{\sqrt{\left(\frac{3n-1}{2}-1\right)\left(\frac{3n-1}{2}-2\right)}} \quad (31)$$

5.Simulation Study:

Mean squared errors (MSE's), are considered to compare the behavior of different estimates of unknown scale parameter θ for Maxwell distribution, that obtained by the method of Bayes estimators under Squared-log error loss and Precautionary function with different priors. In current simulation study, the number of replication used, was I = 5000 samples of sizes n = 5,10, 20, 50 and 100 from the Maxwell distribution with different values of θ where, θ = 0.5, 1.5, 3, hyper parameter of Gumbel type–II prior (b=0.9, 3), hyper parameters of Inverted Gamma prior were ($\alpha = 2, 3,\ \beta$= 0.9, 3). Finally, hyper-parameter of Inverted Levy prior was (λ=0.8).

In this section, Monte-Carlo simulation has been employed to compare the performed of all estimators which were obtained, using mean squared errors (MSE's) as one of an important criteria for comparing the efficiency of the estimators, where:

$$\text{MSE}(\theta) = \frac{\sum_{i=1}^{I}\left(\hat{\theta}_i - \theta\right)^2}{I} \quad (32)$$

The results have been summarized and tabulated in the tables 1, 2, 3 for each estimates and for all sample sizes. The entries within parenthesis indicate the MSE.

Discussion:

The main contribution of this paper is to obtain the best parameter estimates for the Maxwell distribution, using Bayesian estimation under different priors compared between Bayes estimation under Squared- Log error loss function and Precautionary loss function .

Results of a simulation study are presented in tables (1, 2, 3) which contain the expected values and MSE's, we have observed that:

the performance of Bayes estimator under Squared-Log error loss function with Inverted Gamma prior (Provided that chosen the value of α is nearly, equal to the value of β) is the best, comparing to other estimators for all cases.

Table 1: Expected values and MSE's of the different estimators for Maxwell distribution when θ=0.5

Estimators				n / Criteria	5	10	20	50	100
Square- Log error Loss Function	Jeffery			EXP.	0.533443	0.516863	0.507662	0.503138	0.501778
				MSE	0.039752	0.018423	0.008657	0.003356	0.001670
	b=0.9		Gumbel	EXP.	0.579251	0.541569	0.520496	0.508413	0.504410
				MSE	0.035867	0.017601	0.008463	0.003329	0.001664
	b=3		Gumbel	EXP.	0.841578	0.677031	0.589350	0.536234	0.518370
				MSE	0.146261	0.047213	0.016026	0.004571	0.001982
	α=2	β=0.9	Inverted Gamma	EXP.	0.725747	0.616519	0.558402	0.523666	0.512080
				MSE	0.081303	0.029658	0.011507	0.003827	0.001793
		β=3	Inverted Gamma	EXP.	0.573468	0.542555	0.522319	0.509513	0.505012
				MSE	0.024342	0.014265	0.007582	0.003183	0.001627
	α=3	β=0.9	Inverted Gamma	EXP.	0.852248	0.681446	0.591299	0.536931	0.518733
				MSE	0.154420	0.049004	0.016431	0.004630	0.001998
		β=3	Inverted Gamma	EXP.	0.673425	0.599691	0.553090	0.522420	0.511573
				MSE	0.049021	0.022393	0.009902	0.003595	0.001735
	λ=0.8		Inverted Levy	EXP.	0.635836	0.563895	0.530213	0.511976	0.506122
				MSE	0.063238	0.023540	0.009810	0.003535	0.001715
Precautionary Loss Function	Jeffery			EXP.	0.626959	0.554742	0.526033	0.509918	0.504607
				MSE	0.070058	0.024269	0.009952	0.003529	0.001700
	b=0.9		Gumbel	EXP.	0.665797	0.578542	0.538707	0.515154	0.507253
				MSE	0.067044	0.024605	0.010155	0.003569	0.001709
	b=3		Gumbel	EXP.	0.966566	0.723457	0.609904	0.543343	0.521299
				MSE	0.257238	0.068369	0.020735	0.005218	0.002110
	α=2	β=0.9	Inverted Gamma	EXP.	0.835337	0.659001	0.577961	0.530632	0.514954
				MSE	0.153167	0.043974	0.014793	0.004287	0.001882
		β=3	Inverted Gamma	EXP.	0.639730	0.575043	0.539421	0.516065	0.507817
				MSE	0.043404	0.019865	0.009146	0.003425	0.001674
	α=3	β=0.9	Inverted Gamma	EXP.	0.980644	0.728485	0.611979	0.544074	0.521648
				MSE	0.271737	0.070899	0.021255	0.005291	0.002127
		β=3	Inverted Gamma	EXP.	0.751013	0.635679	0.571170	0.529138	0.514418
				MSE	0.086887	0.032642	0.012657	0.004016	0.001821
	λ=0.8		Inverted Levy	EXP.	0.757441	0.606901	0.549716	0.518891	0.509015
				MSE	0.130553	0.034370	0.012081	0.003835	0.001772

Table 2: Expected values and MSE's of the different estimators for Maxwell distribution when θ=1.5

Estimators				Criteria \ n	5	10	20	50	100
Square- Log error Loss Function	Jeffery			EXP.	1.600330	1.550590	1.522989	1.509410	1.505334
				MSE	0.357765	0.165807	0.077914	0.030201	0.015027
	b=0.9		Gumbel	EXP.	1.512901	1.508595	1.502469	1.501392	1.501263
				MSE	0.266443	0.142934	0.072391	0.029324	0.014801
	b=3		Gumbel	EXP.	1.775227	1.644059	1.571325	1.529213	1.515223
				MSE	0.342025	0.163613	0.077472	0.030176	0.015031
	α=2	β=0.9	Inverted Gamma	EXP.	1.671236	1.589853	1.543621	1.517936	1.509632
				MSE	0.302392	0.152803	0.074767	0.029721	0.014913
		β=3	Inverted Gamma	EXP.	1.320569	1.399117	1.443877	1.476910	1.488796
				MSE	0.202695	0.122264	0.066902	0.028364	0.014539
	α=3	β=0.9	Inverted Gamma	EXP.	1.797736	1.654778	1.576518	1.531204	1.516287
				MSE	0.361718	0.168686	0.078719	0.030372	0.015085
		β=3	Inverted Gamma	EXP.	1.420527	1.456254	1.474647	1.489817	1.495353
				MSE	0.176815	0.114000	0.064395	0.027935	0.014435
	λ=0.8		Inverted Levy	EXP.	1.784552	1.634550	1.563051	1.525116	1.512994
				MSE	0.484048	0.193223	0.084053	0.031156	0.015267
Precautionary Loss Function	Jeffery			EXP.	1.880879	1.664225	1.578094	1.529757	1.513822
				MSE	0.630520	0.218420	0.089569	0.031765	0.015301
	b=0.9		Gumbel	EXP.	1.739588	1.611415	1.555096	1.521302	1.509717
				MSE	0.413401	0.178337	0.080942	0.030510	0.015003
	b=3		Gumbel	EXP.	2.040357	1.756328	1.626293	1.549487	1.523761
				MSE	0.647984	0.231629	0.093856	0.032506	0.015473
	α=2	β=0.9	Inverted Gamma	EXP.	1.924771	1.699055	1.597807	1.538130	1.518093
				MSE	0.546879	0.207862	0.088004	0.031591	0.015256
		β=3	Inverted Gamma	EXP.	1.474060	1.482596	1.491258	1.495908	1.497046
				MSE	0.215594	0.128406	0.068401	0.028522	0.014526
	α=3	β=0.9	Inverted Gamma	EXP.	2.070080	1.768540	1.631829	1.551574	1.524781
				MSE	0.691440	0.240353	0.095816	0.032797	0.015543
		β=3	Inverted Gamma	EXP.	1.585342	1.543234	1.523010	1.508979	1.503647
				MSE	0.222204	0.129972	0.068854	0.028586	0.014531
	λ=0.8		Inverted Levy	EXP.	2.126261	1.759118	1.620573	1.545714	1.521637
				MSE	0.970702	0.273626	0.101017	0.033394	0.015680

Table 3: Expected values and MSE's of the different estimators for Maxwell distribution when θ=3

Estimators				Criteria \ n	5	10	20	50	100
Square- Log error Loss Function	Jeffery			EXP.	2.998938	2.993549	2.997918	2.998251	2.997346
				MSE	1.234127	0.619490	0.301240	0.118626	0.059243
	b=0.9		Gumbel	EXP.	2.913375	2.959135	2.975425	2.990863	2.996541
				MSE	1.072609	0.573110	0.290144	0.117374	0.059208
	b=3		Gumbel	EXP.	3.175692	3.094600	3.044290	3.018677	3.010507
				MSE	1.095973	0.580389	0.291501	0.117639	0.059306
	α=2	β=0.9	Inverted Gamma	EXP.	3.089465	3.049852	3.021453	3.009343	3.005958
				MSE	1.100289	0.581403	0.291918	0.117682	0.059314
		β=3	Inverted Gamma	EXP.	2.441220	2.683958	2.826210	2.928011	2.964464
				MSE	0.994235	0.548227	0.285211	0.116507	0.058916
	α=3	β=0.9	Inverted Gamma	EXP.	3.215963	3.114776	3.054348	3.022611	3.012613
				MSE	1.138927	0.592092	0.294411	0.118106	0.059438
		β=3	Inverted Gamma	EXP.	2.541179	2.741094	2.856982	2.940917	2.971030
				MSE	0.892517	0.515376	0.275462	0.114815	0.058493
	λ=0.8		Inverted Levy	EXP.	3.507629	3.240533	3.112304	3.044830	3.023303
				MSE	1.870006	0.758331	0.332924	0.124110	0.060935
Precautionary Loss Function	Jeffery			EXP.	3.761759	3.328450	3.156188	3.059513	3.027643
				MSE	2.522081	0.873682	0.358277	0.127062	0.061204
	b=0.9		Gumbel	EXP.	3.350277	3.160724	3.079682	3.030516	3.013412
				MSE	1.546683	0.689528	0.317972	0.121158	0.059813
	b=3		Gumbel	EXP.	3.651045	3.305637	3.150883	3.058711	3.027460
				MSE	1.847851	0.757111	0.334388	0.123673	0.060388
	α=2	β=0.9	Inverted Gamma	EXP.	3.558928	3.259136	3.127582	3.049387	3.022787
				MSE	1.778183	0.740108	0.330025	0.122988	0.060233
		β=3	Inverted Gamma	EXP.	2.725550	2.843927	2.919017	2.965671	2.980891
				MSE	0.935008	0.536771	0.279856	0.115200	0.058434
	α=3	β=0.9	Inverted Gamma	EXP.	3.704238	3.328621	3.161598	3.062824	3.029491
				MSE	1.961733	0.780948	0.339862	0.124496	0.060583
		β=3	Inverted Gamma	EXP.	2.836834	2.904557	2.950773	2.978740	2.987497
				MSE	0.886309	0.521520	0.275722	0.114474	0.058226
	λ=0.8		Inverted Levy	EXP.	4.179502	3.487446	3.226852	3.085950	3.040575
				MSE	3.705193	1.063535	0.397381	0.132605	0.062495

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REFERENCES

Ali, S., M. Aslam, N. Abbas and S.M. Ali Kazmiz, 2012. "Scale Parameter Estimation of the Laplace Modeling Using Different Loss Function ". International Journal of the statistics and Probability, 1(1): 105-127.

Bekker, A. and J.J. Roux, 2005. "Reliability characteristics of the Maxwell distribution: A Bayes estimation study", Comm. Stat. (Theory & Math.), 34(11): 2169-2178.

Dey, S., 2010. "Bayesian Estimation of the Parameter of Generalized Exponential Distribution under Different Loss Function", pak. J. Stat. Opera. Res., 6(2): 163-174.

Farsi pour, N. and H. Zakerzadeh, 2006. "Estimation of Generalized Variance Under an Asymmetric Loss Function Squared Log Error". Publishing Technology, 35(4): 571-581.

Krishna and Malik, 2009. "Reliability estimation in Maxwell distribution with Type-II censored data", Int. Journal of Quality and Reliability management, 26(2): 184-195.

Rasheed, H.A., 2013. "Minimax Estimation of The Parameter of the Maxwell Distribution under Quadratic loss function", Journal of Al-Rafidain University College, ISSN(1681-6870), Issue 31.

Rasheed and Aref, 2016. "Bayesian Approach in Estimation of Scale Parameter of Inverse Rayleigh distribution", Mathematics and Statistics Journal, ISSN-2077-4591, 2(1): 8-13.

Rasheed and Aref, 2016. " Reliability Estimation in Inverse Rayleigh Distribution using Precautionary Loss Function ", Mathematics and Statistics Journal, ISSN-2077-4591, 2(3): 9-15.

Rasheed, H.A. and Z.N. Khalifa, 2016. "Bayes Estimators for the Maxwell Distribution under Quadratic Loss Function Using Different Priors", Mathematics and Statistics Journal, ISSN-2077-4591, 2(3): 16-22.

Rasheed and Khalifa, 2016. "Semi-Minimax Estimators of Maxwell Distribution under New Loss Function", Australian Journal of Basic and Applied Sciences, ISSN-2077-459, 10(6): 97-103.

Sindhu, T., N. Aslam, 2013. "Bayesian Estimation on the Proportional Inverse Weibul Distribution under Different Loss Functions", Advances in Agriculture, Science and Engineering Research, 3(2): 641-655.

Tyagi, R.K. and S.K. Bhattacharya, 1989a. Bayes Estimation of the Maxwell's Velocity Distribution Function. Statistica, 29(4): 563-567.

Podder, C.K. and M.K. Roy, 2003. "Bayesian estimation of the parameter of Maxwell distribution under MLINEX loss function", Journal of Statistical Studies, 23: 11-16.