Australian Journal of Basic and Applied Sciences, 9(11) May 2015, Pages: 995-998

ISSN:1991-8178

Australian Journal of Basic and Applied Sciences

Journal home page: www.ajbasweb.com

Characterizations of Kumaraswamy Fréchet Distribution

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ARTICLE INFO

Article history:
Received 12 March 2015
Accepted 28 April 2015
Available online 5 May 2015

Keywords:
Kumaraswamy-Fréchet distribution, characterizations, hazard function

ABSTRACT

Background: Several characterizations of a distribution, introduced by Mead *et al*. (2014), are presented. These characterizations are based on: (i) a truncated moment of the random variable; (ii) the hazard function; (iii) a single function of the random variable.

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To Cite This Article: G.G. Hamedani., Characterizations of Kumaraswamy Fréchet Distribution. ***Aust. J. Basic & Appl. Sci.,*** *9(11): 995-998, 2015*

INTRODUCTION

Characterizations of distributions are important to many researchers in the applied fields. An investigator will be vitally interested to know if their model fits the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution.Various characterizations of distributions have been established in many different directions in the literature. In this short note, several characterizations of Kumaraswamy-Fréchet (KF) distribution, introduced by Mead *et al*. (2014), are presented in several directions. These characterizations are based on: (i) a truncated moment of the random variable; (ii) the hazard function and (iii) a single function of the random variable. Our characterizations of KF distribution, (i), will employ a version of an interesting result due to Glänzel (1987) (Theorem 2.1 of Section 2 below). The advantage of these type of characterizations is that, cumulative distribution function, F , need not have a closed form and are given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation.

The cdf (cumulative distribution function) $F(x)$ and pdf (probability density function) $f(x)$ of KF are given by

$$F(x)=F(x;\theta,\beta,a,b)=1-\left[1-e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\right]^{b},\ x\geq 0,\ (1.1)$$

and

$$f(x)=f(x;\theta,\beta,a,b)=ab\beta\theta^{\beta}x^{-\beta-1}e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\left[1-e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\right]^{b-1},\ x>0,\ (1.2)$$

where α,β,a and b are all positive parameters. For further properties and the domain of applicability of KF distribution, we refer the interested reader to Mead *et al*. (2014).

The presentation of the content of this work is as follows: In Section 2, we present our characterization results based on a truncated moment of the random variable . Section 3 is devoted to characterization of KF in terms of the hazard function. Section 4 deals with various characterizations of KF in terms of the conditional expectations of functions of the random variable.

Characterizations based on truncated moment:

In this section we present characterizations of KF distribution in terms of truncated moment. As mentioned in the Introduction, our characterization results presented here will employ a version of an interesting result due to Glänzel (1987) (Theorem 2.1, below). The advantage of the characterizations given here is that, cdf F need not have a closed form and are given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation.

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Theorem 2.1:

Let $(\Omega,\mathcal{F},\mathbf{P})$ be a given probability space and let $H=[a,b]$ be an interval for some $a<b$ ($a=-\infty, b=\infty$ might be as well allowded). Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and let g be a real function defined on H such that

$\mathbf{E}[g(X)|X\geq x]=\eta(x),\quad x\in H$,

is defined with some real function η . Assume that $g\in C^1(H)$, $\eta\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H . Finally, assume that the equation $\eta=g$ has no real solution in the interior of H . Then F is uniquely determined by the functions g and η, particularly

$$F(x)=\int_a^x C\left|\frac{\eta'(u)}{\eta(u)-g(u)}\right|\exp(-s(u))du,$$

where the function s is a solution of the differential equation $s'=\frac{\eta'}{\eta-g}$ and C is a constant, chosen to make $\int_H dF=1$.

Proposition 2.2:

Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $g(x)=\left[1-e^{-a\left(\frac{\theta}{x}\right)^\beta}\right]$ for $x\in(0,\infty)$.

Then, pdf of X is (1.2) if and only if the function η defined in Theorem 2.1 has the form

$$\eta(x)=\frac{b}{b+1}\left[1-e^{-a\left(\frac{\theta}{x}\right)^\beta}\right],\ x>0.\quad(2.1)$$

Proof. Let X have pdf (1.2) , then

$$E[g(X)|X\geq x]=\frac{b}{b+1}\left\{\frac{\left[1-e^{-a\left(\frac{\theta}{x}\right)^\beta}\right]^{b+1}}{\left[1-e^{-a\left(\frac{\theta}{x}\right)^\beta}\right]^{b}}\right\}$$

$$=\frac{b}{b+1}\left[1-e^{-a\left(\frac{\theta}{x}\right)^\beta}\right],\ x>0,$$

and

$$\eta(x)-g(x)=-\frac{1}{b+1}\left[1-e^{-a\left(\frac{\theta}{x}\right)^\beta}\right]<0\ for\ x>0.$$

Conversely, if $\eta(x)$ is given by (2.1) , then

$$s'(x)=\frac{\eta'(x)}{\eta(x)-g(x)}=\frac{a\beta b\theta^\beta x^{-\beta-1}e^{-a\left(\frac{\theta}{x}\right)^\beta}}{\left[1-e^{-a\left(\frac{\theta}{x}\right)^\beta}\right]},$$

from which we obtain

$$s(x)=-\ln\left\{\left[1-e^{-a\left(\frac{\theta}{x}\right)^\beta}\right]^b\right\},\ x>0.$$

Now, in view of Theorem 2.1, X has cdf (1.) and pdf (1.2) .

Corollary 2.3:

Let $X:\Omega\to(0,\infty)$ be a continuous random variable. Then, pdf of X is (1.2) if and only if there exist functions g and η defined in Theorem 2.1 satisfying the following differential equation

$$\frac{\eta'(x)}{\eta(x)-g(x)}=\frac{a\beta b\theta^\beta x^{-\beta-1}e^{-a\left(\frac{\theta}{x}\right)^\beta}}{\left[1-e^{-a\left(\frac{\theta}{x}\right)^\beta}\right]},\ x>0.\quad(2.2)$$

Remarks 2.4:

(a) The general solution of the differential equation (2.2) is

$$\eta(x)=\left[1-e^{-a\left(\frac{\theta}{x}\right)^\beta}\right]^{-b}\times$$

$$\left\{-\int g(x)\left(a\beta b\theta^\beta x^{-\beta-1}e^{-a\left(\frac{\theta}{x}\right)^\beta}\right)\left[1-e^{-a\left(\frac{\theta}{x}\right)^\beta}\right]^{b-1}+D\right\},$$

for $x>0$, where D is a constant. One set of function (g,η) satisfying the above equation is given in Proposition 2.2 for $D=0$.

(b) Clearly there are other pairs of functions (g,η) satisfying the conditions of Theorem 2.1. We presented one such pair in Proposition 2.2.

3 Characterization based on hazard function:

It is obvious that the hazard function, h_F , of a twice differentiable distribution function, F , satisfies the first order differential equation

$$\frac{h_F'(x)}{h_F(x)}-h_F(x)=q(x),$$

where $q(x)$ is an appropriate integrable function. Although this differential equation has an obvious form since

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$$\frac{h_F^{'}(x)}{h_F(x)} - h_F(x) = \frac{f^{'}(x)}{f(x)}, \qquad (3.1)$$

for many univariate continuous distributions (3.1) seems to be the only differential equation in terms of the hazard function. The goal of the characterization based on hazard function is to establish a differential equation in terms of hazard function, which has as simple form as possible and is not of the trivial form (3.1). Here, we present a characterization of the KF model based on a nontrivial differential equation in terms of the hazard function.

Proposition 3.1:

Let $X:\Omega\to(0,\infty)$ be a continuous random variable. Then, X has pdf (1.2) if and only if its hazard function h_F satisfies the differential equation

$$h_F^{'}(x)+(\beta+1)x^{-1}h_F(x) = a^2\beta^2 b\theta^{2\beta}x^{-2\beta-2}e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\left[1-e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\right]^{-2}, x>0. \quad (3.2)$$

Proof:

If X has pdf (1.2), then clearly (3.2) holds. Now, if (3.2) holds, then after multiplying both sides of (3.2) by $x^{\beta+1}$, we arrive at

$$\frac{d}{dx}\left\{x^{\beta+1}h_F(x)\right\} = a\beta b\theta^{\beta}\frac{d}{dx}\left\{\left[1-e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\right]^{-1}\right\}.$$

Integrating both sides of the above equation from 0 to x, we have

$$x^{\beta+1}h_F(x) = a\beta b\theta^{\beta}\left[1-e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\right]^{-1} - a\beta b\theta^{\beta},$$

or

$$h_F(x) = \frac{f(x)}{1-F(x)} = a\beta b\theta^{\beta}x^{-\beta-1}\left[1-e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\right]^{-1}$$

$$-a\beta b\theta^{\beta}x^{-\beta-1} = \frac{a\beta b\theta^{\beta}x^{-\beta-1}e^{-a\left(\frac{\theta}{x}\right)^{\beta}}}{\left[1-e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\right]}. \quad (3.3)$$

Integrating both sides of (3.3) from 0 to x, we arrive at

$$-\ln\left(1-F(x)\right) = -\ln\left\{\left[1-e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\right]^{b}\right\}.$$

from which we obtain

$$F(x) = 1-\left[1-e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\right]^{b}, \quad x\geq 0.$$

4 Characterizations based on single function of the random variable:

In this section we employ a single function ψ of X and state characterization results in terms of $\psi(X)$. The following proposition has already appeared in a Technical Report (Hamedani, 2013), so we will just state it here as an application for KF distribution.

Proposition 4.1:

Let $X:\Omega\to(a,b)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (a,b) with $\lim_{x\to a^+}\psi(x)=1$. Then for $\delta\neq 1$,

$E\left[\psi(X)\,|\,X\geq x\right] = \delta\psi(x), \; x\in(a,b),$

if and only if

$$\psi(x) = \left(1-F(x)\right)^{\frac{1}{\delta}-1}, \quad x\in(a,b)$$

Remark 4.2:

Taking, e.g., $\psi(x) = \left[1-e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\right]^{\frac{b(1-\delta)}{\delta}}$, $(a,b)=(0,\infty)$, Proposition 4.1 gives a characterization of KF distribution.

The following proposition has appeared in our previous work and we state here as an application for KF distribution.

Proposition 4.3:

Let $X:\Omega\to(a,b)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (a,b) with $\lim_{x\to a^+}\psi(x)=0$. Then for $0<\delta<1$,

$E\left[\psi(X)\,|\,X\geq x\right] = \delta+(1-\delta)\psi(x), \quad x\in(a,b),$

if and only if

$$\psi(x) = 1-\left(1-F(x)\right)^{\frac{\delta}{1-\delta}}, \quad x\in(a,b)$$

Remark 4.4:

Taking, e.g.,

$$\psi(x) = 1-\left[1-e^{-a\left(\frac{\theta}{x}\right)^{\beta}}\right]^{\frac{b\delta}{1-\delta}}, (a,b)=(0,\infty),$$ Proposition 4.3 gives a characterization of KF distribution

REFERENCES

Glänzel, W., 1987. A characterization theorem based on truncated moments and its application to some distribution families, Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, 75-84.

Hamedani, G.G., 2013. On certain generalized gamma convolution distributions **II**, Technical Report, No. 484, MSCS, Marquette University.

Mead, M.E. and A.R. Abd-Eltawab, 2014. A note on Kumaraswamy Fréchet distribution, Australian J. of Basic and Applied Sciences, 8: 294-300.