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Geometry Tensor Conharmonic Curvature Kahler and Nearly Kahler Manifolds

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ABSTRACT

In this paper we study the relationship between tensor is an algebraic curvature tensor, i.e. it has a classical symmetry properties of the Riemann curvature tensor. Replenishing Riemannian structure to the Kaher (K) and nearly Kahler (NK) structures allows to allocate structures some more conharmonic invariants - elements of a spectrum of the tensor conharmonic curvature, and also additional properties of symmetry of the tensor conharmonic curvature invariants and additional properties of symmetry tensor conharmonic curvature, as well as geometrical sense of the reference (manipulation) in zero of these invariants are studied.

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INTRODUCTION

Conformaltransformations of Riemannian structures are the important object of studying in differential geometry. Significant interest represents special type of conformal transformations - conharmonic transformations, i.e. conformal transformations which keeping the property of smooth harmonic functions. This type of transformations was introduced in consideration of Ishii in 1957 and now is studied from the various points of view. It is known, that such transformations have tensor invariant- so-called tensor conharmoniccurvature.

Let M - smooth manifold of dimension $2n$; $C^{\infty}(M)$ - algebra of smooth functions on M; $X(M)$- the module of smooth vector fields on manifoldof M; $g = \langle \cdot, \rangle$ - Riemannian metrics; ∇ – Riemannian connection of the metrics g on M; d - the operator of exterior differentiation. In the further all manifold, Tensor fields, etc. objects are assumed smooth a class C^{∞}.

Definition 1:

Let (M, J, g) - *NK- manifold*. We shall remind, that curvature tensor conharmonic was introduced Ishii (1957) as a tensor of type (4,0) on n-dimensional Riemannian manifold, determined by the formula

$$K(X,Y,Z,W) = R(X,Y,Z,W) -$$

$$-\frac{1}{n-2}[g(X,W)S(Y,Z) - g(X,Z)S(Y,W) + g(Y,Z)S(X,W) - g(Y,W)S(X,Z)], \quad (1)$$

Where R – the Riemann curvature tensor, S - Ricci tensor. This tensore is invariant under conharmonictransformations, i.e. with conformal transformations of space keeping a harmony of functions.

Let's consider properties tensor conharmoniccurvature: (Shihab, 2011)

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$$1)\ K(X,Y,Z,W)=R(X,Y,Z,W)-\frac{1}{2(n-1)}\left[g(X,W)S(Y,Z)-g(X,Z)S(Y,W)\right]-$$

$$-\frac{1}{2(n-1)}\left[g(Y,Z)S(X,W)-g(Y,W)S(X,Z)\right]=-R(Y,X,Z,W)+$$

$$+\frac{1}{2(n-1)}\left[-g(X,W)S(Y,Z)+g(X,Z)S(Y,W)\right]+$$

$$+\frac{1}{2(n-1)}\left[-g(Y,Z)S(X,W)+g(Y,W)S(X,Z)\right]=-K(Y,X,Z,W).$$

Properties are similarly proved:

$$2)\ K(X,Y,Z,W)=-K(X,Y,W,Z);\quad 3)\ K(X,Y,Z,W)=K(Z,W,X,Y);$$

$$4)\ K(X,Y,Z,W)+K(Y,Z,X,W)+K(Z,X,Y,W)=0.$$

Thus, conharmoniccurvature tensor satisfies all the properties of algebraic curvature tensor:

$$1)\ K(X,Y,Z,W)=-K(Y,X,Z,W);$$

$$2)\ K(X,Y,Z,W)=-K(X,Y,W,Z);$$

$$3)\ K(X,Y,Z,W)+K(Y,Z,X,W)+K(Z,X,Y,W)=0;$$

$$4)\ K(X,Y,Z,W)=K(Z,W,X,Y),\quad X,Y,Z,W\in\mathfrak{X}(M). \tag{2}$$

tensorconharmoniccurvature looks like:

$$K(X,Y)Z=R(X,Y)Z-\frac{1}{2(n-1)}\{S(Y,Z)X-S(X,Z)Y\}-$$

$$-\frac{1}{2(n\text{-}1)}\{g(Y,Z)QX-g(X,Z)QY\}, \tag{3}$$

Where $g(QX,Y)=S(X,Y)$. By definition of a spectrum tensor,

$$K(X,Y)Z=\underset{0}{K}(X,Y)Z+\underset{1}{K}(X,Y)Z+\underset{2}{K}(X,Y)Z+\underset{3}{K}(X,Y)Z+$$

$$+\underset{4}{K}(X,Y)Z+\underset{5}{K}(X,Y)Z+\underset{6}{K}(X,Y)Z+\underset{7}{K}(X,Y)Z;\quad X,Y,Z\in\mathfrak{X}(M).$$

It agrees proposition 1: tensor $\underset{0}{K}(X,Y)Z$ as nonzero the component can have only components of the form $\left\{\underset{0}{K}{}^{a}{}_{bcd},\underset{0}{K}{}^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\right\}=\left\{K^{a}{}_{bcd},K^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\right\}$; tensor $\underset{1}{K}(X,Y)Z$ - components of the form $\left\{\underset{1}{K}{}^{a}{}_{bc\hat{d}},\underset{1}{K}{}^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\right\}=\left\{K^{a}{}_{bc\hat{d}},K^{\hat{a}}{}_{\hat{b}\hat{c}d}\right\}$; tensor $\underset{2}{K}(X,Y)Z$ - components of the form $\left\{\underset{2}{K}{}^{a}{}_{b\hat{c}d},\underset{2}{K}{}^{\hat{a}}{}_{\hat{b}c\hat{d}}\right\}=\left\{K^{a}{}_{b\hat{c}d},K^{\hat{a}}{}_{\hat{b}c\hat{d}}\right\}$; tensor $\underset{3}{K}(X,Y)Z$ - components of the form $\left\{\underset{3}{K}{}^{a}{}_{b\hat{c}\hat{d}},\underset{3}{K}{}^{\hat{a}}{}_{\hat{b}cd}\right\}=\left\{K^{a}{}_{b\hat{c}\hat{d}},K^{\hat{a}}{}_{\hat{b}cd}\right\}$; tensor $\underset{4}{K}(X,Y)Z$ - components of the form $\left\{\underset{4}{K}{}^{a}{}_{\hat{b}cd},\underset{4}{K}{}^{\hat{a}}{}_{b\hat{c}\hat{d}}\right\}=\left\{K^{a}{}_{\hat{b}cd},K^{\hat{a}}{}_{b\hat{c}d}\right\}$; tensor $\underset{5}{K}(X,Y)Z$ - components of the form $\left\{\underset{5}{K}{}^{a}{}_{\hat{b}c\hat{d}},\underset{5}{K}{}^{\hat{a}}{}_{b\hat{c}d}\right\}=\left\{K^{a}{}_{\hat{b}c\hat{d}},K^{\hat{a}}{}_{b\hat{c}d}\right\}$; tensor $\underset{6}{K}(X,Y)Z$ - components of the form

$\left\{\underset{6}{K}^{a}{}_{\hat{b}\hat{c}d}, \underset{6}{K}^{\hat{a}}{}_{bc\hat{d}}\right\}=\left\{K^{a}{}_{\hat{b}\hat{c}d}, K^{\hat{a}}{}_{bc\hat{d}}\right\}$; tensor $\underset{7}{K}(X,Y)Z$ - components of the form $\left\{\underset{7}{K}^{a}{}_{\hat{b}\hat{c}\hat{d}}, \underset{7}{K}^{\hat{a}}{}_{bcd}\right\}=\left\{K^{a}{}_{\hat{b}\hat{c}\hat{d}}, K^{\hat{a}}{}_{bcd}\right\}$.

Tensors $K_0=\underset{0}{K}(X,Y)Z$ $K_1=\underset{1}{K}(X,Y)Z$, …, $K_7=\underset{7}{K}(X,Y)Z$ we shall namethe basic invariants conharmonic *NK*-manifold.

Definition 2:

NK - manifold for which*Ki=0*, we shall name*NK - manifold* of class*Ki*,*i*= 0, 1, …, 7.

The manifold of class*K0* is characterized by a condition $\underset{0}{K}^{a}{}_{bcd}=0$, or $K^{a}{}_{bcd}=0$, i.e $\left[K(\varepsilon_c,\varepsilon_d)\varepsilon_b\right]^{a}\varepsilon_a=0$. As σ- a projector on $D_J^{\sqrt{-1}}$, that $\sigma\circ\{K(\sigma X,\sigma Y)\sigma Z\}=0$, i.e $\left(id-\sqrt{-1}J\right)\left\{K\left(X-\sqrt{-1}JX,Y-\sqrt{-1}JY\right)\left(Z-\sqrt{-1}JZ\right)\right\}=0$. Removing the brackets we shall receive:

$$K(X,Y)Z-K(X,JY)JZ-K(JX,Y)JZ-K(JX,JY)Z-JK(X,Y)JZ-JK(X,JY)Z-$$
$$-JK(JX,Y)Z+JK(JX,JY)JZ-\sqrt{-1}\{K(X,Y)JZ+K(X,JY)Z+K(JX,Y)Z-K(JX,JY)JZ\}-$$
$$-\{JK(X,Y)Z-JK(X,JY)JZ-JK(JX,Y)JZ-JK(JX,JY)Z\}=0,$$

i.e.

$$1)\ K(X,Y)Z-K(X,JY)JZ-K(JX,Y)JZ-K(JX,JY)Z-JK(X,Y)JZ-JK(X,JY)Z-$$
$$-JK(JX,Y)Z+JK(JX,JY)JZ=0;$$

Thus, *NK* - manifold of class*K0* are characterized by identity

$$2)\ K(X,Y)JZ+K(X,JY)Z+K(JX,Y)Z-K(JX,JY)JZ+JK(X,Y)Z-$$
$$-JK(X,JY)JZ-JK(JX,Y)JZ-JK(JX,JY)Z=0,\ \ X,Y,Z\in\mathfrak{X}(M).$$

These equality are equivalent, the second equality turns out from the first replacement *Z* on *JZ*.

$$K(X,Y)Z-K(X,JY)JZ-K(JX,Y)JZ-K(JX,JY)Z-JK(X,Y)JZ-$$
$$-JK(X,JY)Z-JK(JX,Y)Z+JK(JX,JY)JZ=0;\ \ X,Y,Z\in\mathfrak{X}(M).$$

Similarly, considering*NK* - manifold of classes*K1* -*K7*, we shall receive the following theorem.

Theorem3:

NK - manifold of class*K0*are characterized by identity

$$K(X,Y)Z-K(X,JY)JZ-K(JX,Y)JZ-K(JX,JY)Z-JK(X,Y)JZ-$$
$$-JK(X,JY)Z-JK(JX,Y)Z+JK(JX,JY)JZ=0;\ \ X,Y,Z\in\mathfrak{X}(M).\qquad(4)$$

2)*NK* - manifold of class*K1* are characterized by identity

$$K(X,Y)Z+K(X,JY)JZ-K(JX,Y)JZ+K(JX,JY)Z+JK(X,Y)JZ-$$
$$-JK(X,JY)Z-JK(JX,Y)Z-JK(JX,JY)JZ=0;\ \ X,Y,Z\in\mathfrak{X}(M).\qquad(5)$$

3)*NK* - manifold of class*K2* are characterized by identity

$$K(X,Y)Z-K(X,JY)JZ+K(JX,Y)JZ+K(JX,JY)Z-JK(X,Y)JZ-$$
$$-JK(X,JY)Z+JK(JX,Y)Z-JK(JX,JY)JZ=0;\ \ X,Y,Z\in\mathfrak{X}(M).\qquad(6)$$

4)*NK* - manifold of class*K3* are characterized by identity

$$K(X,Y)Z+K(X,JY)JZ+K(JX,Y)JZ-K(JX,JY)Z-JK(X,Y)JZ+$$
$$+JK(X,JY)Z+JK(JX,Y)Z+JK(JX,JY)JZ=0;\ \ X,Y,Z\in\mathfrak{X}(M). \tag{7}$$

5)*NK* - manifold of class*K4* are characterized by identity

$$K(X,Y)Z+K(X,JY)JZ+K(JX,Y)JZ-K(JX,JY)Z+JK(X,Y)JZ-$$
$$-JK(X,JY)Z-JK(JX,Y)Z-JK(JX,JY)JZ=0;\ \ X,Y,Z\in X(M). \tag{8}$$

6)*NK* - manifold of class*K5* are characterized by identity

$$K(X,Y)Z-K(X,JY)JZ+K(JX,Y)JZ+K(JX,JY)Z+JK(X,Y)JZ+$$
$$+JK(X,JY)Z-JK(JX,Y)Z+JK(JX,JY)JZ=0;\ \ X,Y,Z\in\mathfrak{X}(M). \tag{9}$$

7)*NK* - manifold of class*K6* are characterized by identity

$$K(X,Y)Z+K(X,JY)JZ-K(JX,Y)JZ+K(JX,JY)Z+JK(X,Y)JZ-$$
$$-JK(X,JY)Z+JK(JX,Y)Z+JK(JX,JY)JZ=0;\ \ X,Y,Z\in\mathfrak{X}(M). \tag{10}$$

8)*NK* - manifold of class*K7* are characterized by identity

$$K(X,Y)Z-K(X,JY)JZ-K(JX,Y)JZ-K(JX,JY)Z+JK(X,Y)JZ+$$
$$+JK(X,JY)Z+JK(JX,Y)Z-JK(JX,JY)JZ=0;\ \ X,Y,Z\in\mathfrak{X}(M). \tag{11}$$

Definition4:

Let's calculate components of the components tensor curvature on space of the adjoint G-structure for NK-manifolds. In terms of the covariant component of the formula (3) [article 1]we shall write down as

$$K_{ijkl}=R_{ijkl}-\frac{1}{2n-2}\left(g_{ik}S_{jl}+g_{jl}S_{ik}-g_{il}S_{jk}-g_{jk}S_{il}\right). \tag{12}$$

With the account (1), (2), (3) Shihab A.A. (2011) we find, that on space of the adjoint G-structure relationships (12) take the form:

$$1)\ K_{\hat{a}bc\hat{d}}=A^{ad}_{bc}-B^{adh}B_{hbc}-\frac{1}{2(n-1)}\left(\delta^a_c S^d_b+\delta^d_b S^a_c\right),$$

$$2)\ K_{ab\hat{c}\hat{d}}=2B^{cdh}B_{abh}-\frac{1}{2(n-1)}\left(\delta^c_a S^d_b+\delta^d_b S^c_a-\delta^d_a S^c_b-\delta^c_b S^d_a\right), \tag{13}$$

Plus the relation received in view of materiality and properties of symmetry of it тensoras algebraic tensor curvature. Other components tensor conharmoniccurvature are equal to zero.

Thus, from told follows, that for *NK*- manifold is no more than two of the basic conharmonicinvariants can be distinct from zero, namely, conharmonicinvariants $K_1=\underset{(1)}{K}=-\underset{(2)}{K}$ with components $\left\{K^a_{bc\hat{d}},K^{\hat{a}}_{\hat{b}\hat{c}d}\right\}$ and $K_2=\underset{(4)}{K}$ with components $\left\{K^a_{\hat{b}cd},K^{\hat{a}}_{b\hat{c}\hat{d}}\right\}$.

Theorem 5:

NK- manifold belongs to classes $\overline{K}_2$ and $\overline{K}_3$, and it belongs to a class $\sim$ if and only if, when invariant $K_2=\underset{(4)}{K}$ it is equal to zero. ,

It is natural to name *NK* -structure, at which $K_2=0$, conharmonic -parakähleriane. According to (5) and (1), Shihab A.A. [16] the condition $K_2=0$ is equivalent to

$$B^{cdh}B_{abh}=\frac{1}{4(n-1)}\left(\delta^c_a S^d_b+\delta^d_b S^c_a-\delta^d_a S^c_b-\delta^c_b S^d_a\right). \tag{14}$$

Let's turn this relation on indexes *b* and *d*:

$$B^{cdh}B_{adh}=\frac{1}{4(n-1)}\left\{(n-2)S_a^c+\frac{1}{2}\chi\delta_a^c\right\}. \tag{15}$$

But, according to (1) Shihab A.A. [16] $S_{\hat{a}b}=S_{b\hat{a}}=S_b^a=A_{bh}^{ah}+3B^{adh}B_{bdh}$. Therefore, curtailing (15), we shall receive, that

$$B=\frac{n-2}{8(n-1)}\chi+\frac{n}{8(n-1)}\chi=\frac{1}{4}\chi.$$

As $\chi=6B+2A,$ we have from here:

$$1)\ A=-B,\quad 2)\ \chi=4B. \tag{16}$$

Here $A_a^b=A_{ac}^{bc},\ A=A_{ab}^{ab},\ B_a^b=B^{bch}B_{ach},\ B=B^{abc}B_{abc}=\sum_{a,b,c=1}^{n}\left|B_{abc}\right|^2\geq 0$. We shall notice, that, with the account (32),

$$S_a^c=\frac{4(n-1)}{n-2}B_a^c-\frac{2B}{n-2}\delta_a^c. \tag{17}$$

Note 1 It is known ([9, page 398], see also [8]), that B = const in view of what it is fair

Theorem 6:

Conharmonic - parakähleriane the NK-manifold is manifold of constant non-negative scalar curvature. Thus it is manifold of zero scalar curvature if and only if, when it Kahler. ,

Theorem 7:

We have the following inclusion relations:
1)K4⊃K1=K2; 2)*K0=K3=K5=K6*.

Theproof:

For an example we shall prove equality*K5=K6*. Let (*M,J,g*) -AH- manifold of class*K5*, i.e $K_{\hat{b}cd\hat{}}^a=0$. Then according to (2) we have $K_{\hat{b}\hat{c}d}^a=0$, i.e. The*NK* - manifold is manifold of class*K6*. Back, let*M*-NK - manifold of class*K6*, then $K_{\hat{b}\hat{c}d}^a=0$, so, according to (2) and $K_{\hat{b}cd\hat{}}^a=0$. Thus, classes*K5*and*K6NK* - manifold coincide.

Other equality are similarly proved. We shall prove inclusion*K4⊃K1 =K2*.

Let (*the M, J, g*) -*NK* - manifold of class*K2*, i.e. take place equality $K_{bcd\hat{}}^a=K_{bd\hat{}c}^a=0$. According to property (2) we have:

$K_{\hat{b}cd}^a+K_{cd\hat{b}}^a+K_{d\hat{b}c}^a=0$, i.e $K_{\hat{b}cd}^a=0$. Thus, the*NK* - manifold of a class *K* 1 =*K2* is*NK* - manifold of class*K4*, i.e.*K1 =K2⊂K4*.

Putting(Folding) equality (5) and(6) we shall receive identity describing*NK* - *manifold* of class*K1 =K2*:

$$K(X,Y)Z+K(JX,JY)Z+JK(X,Y)JZ-JK(JX,JY)JZ=0;\ \ X,Y,Z\in\mathfrak{X}(M). \tag{18}$$

From equality (1), (4), (5), (6) we shall receive the identity describing*NK* - *manifold* of classes*K0 =K3 =K5 =K6*:

$$K(X,Y)Z+JK(JX,JY)JZ=0;\ \ X,Y,Z\in\mathfrak{X}(M). \tag{19}$$

In the theory *NK* structures there is a principle of classification of such structures on differential-geometric invariants of the second order (symmetry properties of Riemann-Christoffel tensor). The principle put forward by A. Gray and generated in a number of his works is put in his basis (Gray, 1976; Gray, 1970; Gray, 1976), according to which key to understanding of differential -geometrical properties Kahlermanifolds identities with which satisfies them Riemann curvature tensor are:

$$R_1:\langle R(X,Y)Z,W\rangle=\langle R(X,Y)JZ,JW\rangle;$$
$$R_2:\langle R(X,Y)Z,W\rangle=\langle R(JX,JY)Z,W\rangle+\langle R(JX,Y)JZ,W\rangle+\langle R(JX,Y)Z,JW\rangle;$$
$$R_3:\langle R(X,Y)Z,W\rangle=\langle R(JX,JY)JZ,JW\rangle.$$

Definition 8:

NK -structures, tensor *R*which satisfies to identity*Ri*, called the *structures of class Ri*. If $\mathfrak{S}\subset AH$- any subclass*of AH-structures*designation $\mathfrak{S}\cap Ri=\mathfrak{S}i$ is accepted;*i*= 1, 2, 3 (Kirichenko, 2003).

Well-known, that *K*⊂*R1*⊂*R2*⊂*R3* [3]. So it is natural to expect, that amongAH-manifolds for differential - geometrical and topological properties closest to the kahler manifold classmanifold class *R1* manifold class *R2* and, at last, manifold of class*R3* .

The manifold of class *R2* while having no special items, were introdused into consideration by A.Gray's in connection with studying *nearly*Kahlermanifold whereas *NK*⊂ *R2*, and were considered we Gray and Vanhecke[5], Watson and Vanhecke[18], and other authors.

Let (*M, J, g*) - *nearly*Kahler manifold of dimension *2n*, *K* - conharmoniccurvature tensor.

Definition 9:

The manifold (*M,J,g*) refers to as manifold of a class:

$$1.\ \overline{K}_1 \text{ if } \langle K(X,Y)Z,W\rangle=\langle K(X,Y)JZ,JW\rangle;$$
$$2.\ \overline{K}_1 \text{ if } \langle K(X,Y)Z,W\rangle=\langle K(JX,JY)Z,W\rangle+\langle K(JX,Y)JZ,W\rangle+\langle K(JX,Y)Z,JW\rangle;$$
$$3.\ \overline{K}_1 \text{ if } \langle K(X,Y)Z,W\rangle=\langle K(JX,JY)JZ,JW\rangle.$$

***Note 2*:**

From (19) follows, that*NK* - manifold of class*K0=K3=K5=K6*are also manifold of a class $\overline{K}_3$.
Sense of the specified identities of curvature it is most transparent it is shown in terms of a spectrum conharmoniccurvature tensor.

Theorem 10:

Let $\mathfrak{S}$ = (*J*,*g*= <×, ×>) - *nearly*Kahlerstructure. Then the following statements are equivalent:

(1) $\mathfrak{S}$ - Structure of a class $\overline{K}_3$;

(2) $\underset{(0)}{K}=0$;

(3) On space of the adjoint G-structure identities $K^a_{bcd}=0$ are fair.

The proof**.**

Let $\mathfrak{S}$ - structure of a class $\overline{K}_3$. Obviously, it is equivalent to identity $K(X,Y)Z+JK(JX,JY)JZ=0;\ X,Y,Z\in\mathfrak{X}(M)$. By definition of a spectrum tensor

$$K(X,Y)Z=\underset{(0)}{K}(X,Y)Z+\underset{(1)}{K}(X,Y)Z+\underset{(2)}{K}(X,Y)Z+\underset{(3)}{K}(X,Y)Z+$$
$$+\underset{(4)}{K}(X,Y)Z+\underset{(5)}{K}(X,Y)Z+\underset{(6)}{K}(X,Y)Z+\underset{(7)}{K}(X,Y)Z,\ \ X,Y,Z\in\mathfrak{X}(M),$$
$$J\circ K(JX,JY)JZ=J\circ\underset{(0)}{K}(JX,JY)JZ+J\circ\underset{(1)}{K}(JX,JY)JZ+J\circ\underset{(2)}{K}(JX,JY)JZ+$$
$$+J\circ\underset{(3)}{K}(JX,JY)JZ+J\circ\underset{(4)}{K}(JX,JY)JZ+J\circ\underset{(5)}{K}(JX,JY)JZ+J\circ\underset{(6)}{K}(JX,JY)JZ+$$
$$+J\circ\underset{(7)}{K}(JX,JY)JZ=\underset{(0)}{K}(X,Y)Z-\underset{(1)}{K}(X,Y)Z-\underset{(2)}{K}(X,Y)Z+\underset{(3)}{K}(X,Y)Z-\underset{(4)}{K}(X,Y)Z+$$
$$+\underset{(5)}{K}(X,Y)Z+\underset{(6)}{K}(X,Y)Z-\underset{(7)}{K}(X,Y)Z,\ \ X,Y,Z\in\mathfrak{X}(M),$$

Putting term by term these identities, we shall receive:

$$K(X,Y)Z + JK(JX,JY)JZ = 2\left\{\underset{(0)}{K}(X,Y)Z + \underset{(3)}{K}(X,Y)Z + \underset{(5)}{K}(X,Y)Z + \underset{(6)}{K}(X,Y)Z\right\}$$

Means, the identity $K(X,Y)Z + JK(JX,JY)JZ = 0$ is equivalent to that

$$\underset{(0)}{K}(X,Y)Z + \underset{(3)}{K}(X,Y)Z + \underset{(5)}{K}(X,Y)Z + \underset{(6)}{K}(X,Y)Z = 0$$

$$\underset{(0)}{K} = \underset{(3)}{K} = \underset{(5)}{K} = \underset{(6)}{K} = 0$$

And this identity is equivalent to identities . According to proposition 1, the received identities on space of the adjointG-structure are equivalent to relations $K^a_{bcd} = K^a_{b\hat{c}\hat{d}} = K^a_{\hat{b}c\hat{d}} = K^a_{\hat{b}\hat{c}d} = 0$. By virtue of materiality tensor *K* and his properties (2) received relations are equivalent to a relations $K^a_{bcd} = 0$, i.e. identity $\underset{(0)}{K}(X,Y)Z = 0$.

The opposite, according to (19), obviously.

Theorem 11:

Let $\mathfrak{S}$ = (*J*,*g*= <·,·>) - *nearly*Kahlerstructure. Then the following statements are equivalent:

(1) $\mathfrak{S}$- Structure of a class $\overline{K}_2$;

(2) $\underset{(0)}{K} = \underset{(7)}{K} = 0$;

(3) On space of the attached *G-structure* identities $K^a_{bcd} = K^a_{\hat{b}\hat{c}\hat{d}} = 0$ are fair.

The proof:

Let $\mathfrak{S}$ - structure of a class $\overline{K}_2$. We shall copy identity $\overline{K}_2$ in the following form. With everyone composed this identity we shall paint according to definition of a spectrum тensor:

$$1)\ K(X,Y)Z = \underset{(0)}{K}(X,Y)Z + \underset{(1)}{K}(X,Y)Z + \underset{(2)}{K}(X,Y)Z + \underset{(3)}{K}(X,Y)Z +$$
$$+ \underset{(4)}{K}(X,Y)Z + \underset{(5)}{K}(X,Y)Z + \underset{(6)}{K}(X,Y)Z + \underset{(7)}{K}(X,Y)Z;$$

$$2)\ K(JX,JY)Z = \underset{(0)}{K}(JX,JY)Z + \underset{(1)}{K}(JX,JY)Z + \underset{(2)}{K}(JX,JY)Z + \underset{(3)}{K}(JX,JY)Z +$$
$$+ \underset{(4)}{K}(JX,JY)Z + \underset{(5)}{K}(JX,JY)Z + \underset{(6)}{K}(JX,JY)Z + \underset{(7)}{K}(JX,JY)Z = -\underset{(0)}{K}(X,Y)Z + \underset{(1)}{K}(X,Y)Z +$$
$$+ \underset{(2)}{K}(X,Y)Z - \underset{(3)}{K}(X,Y)Z - \underset{(4)}{K}(X,Y)Z + \underset{(5)}{K}(X,Y)Z + \underset{(6)}{K}(X,Y)Z - \underset{(7)}{K}(X,Y)Z;$$

$$3)\ K(JX,Y)JZ = \underset{(0)}{K}(JX,Y)JZ + \underset{(1)}{K}(JX,Y)JZ + \underset{(2)}{K}(JX,Y)JZ + \underset{(3)}{K}(JX,Y)JZ +$$
$$+ \underset{(4)}{K}(JX,Y)JZ + \underset{(5)}{K}(JX,Y)JZ + \underset{(6)}{K}(JX,Y)JZ + \underset{(7)}{K}(JX,Y)JZ = -\underset{(0)}{K}(X,Y)Z - \underset{(1)}{K}(X,Y)Z +$$
$$+ \underset{(2)}{K}(X,Y)Z + \underset{(3)}{K}(X,Y)Z + \underset{(4)}{K}(X,Y)Z + \underset{(5)}{K}(X,Y)Z - \underset{(6)}{K}(X,Y)Z - \underset{(7)}{K}(X,Y)Z;$$

$$4)\ JK(JX,Y)Z = J\circ\underset{(0)}{K}(JX,Y)Z + J\circ\underset{(1)}{K}(JX,Y)Z + J\circ\underset{(2)}{K}(JX,Y)Z + J\circ\underset{(3)}{K}(JX,Y)Z +$$

$$+ J\circ\underset{(4)}{K}(JX,Y)Z + J\circ\underset{(5)}{K}(JX,Y)Z + J\circ\underset{(6)}{K}(JX,Y)Z + J\circ\underset{(7)}{K}(JX,Y)Z = -\underset{(0)}{K}(X,Y)Z - \underset{(1)}{K}(X,Y)Z +$$

$$+\underset{(2)}{K}(X,Y)Z + \underset{(3)}{K}(X,Y)Z - \underset{(4)}{K}(X,Y)Z - \underset{(5)}{K}(X,Y)Z + \underset{(6)}{K}(X,Y)Z + \underset{(7)}{K}(X,Y)Z.$$

Substituted these decomposition in the previous equality, we shall receive:

$$K(X,Y)Z - K(JX,JY)Z - K(JX,Y)JZ + JK(JX,Y)Z =$$

$$= 2\left\{\underset{(0)}{K}(X,Y)Z + \underset{(3)}{K}(X,Y)Z - \underset{(5)}{K}(X,Y)Z + \underset{(6)}{K}(X,Y)Z + \underset{(7)}{K}(X,Y)Z\right\} = 0.$$

This identity is equivalent to that

$$\underset{(0)}{K}(X,Y)Z = \underset{(3)}{K}(X,Y)Z = \underset{(5)}{K}(X,Y)Z = \underset{(6)}{K}(X,Y)Z = \underset{(7)}{K}(X,Y)Z = 0.$$

And these identities on space of the adjoint *G-structure* are equivalent to identities $K^a_{bcd} = K^a_{b\hat{c}\hat{d}} = K^a_{\hat{b}c\hat{d}} = K^a_{\hat{b}\hat{c}d} = K^a_{\hat{b}\hat{c}\hat{d}} = 0$. By virtue of materiality tensor *K* and his properties (2), the received relations are equivalent to relations: $K^a_{bcd} = K^a_{\hat{b}\hat{c}\hat{d}} = 0$, i.e. to identities $\underset{(0)}{K}(X,Y)Z = \underset{(7)}{K}(X,Y)Z = 0$.

Back, let forAH-manifold identities $\underset{(0)}{K}(X,Y)Z = 0,\ \underset{(7)}{K}(X,Y)Z = 0$ are executed. Then from (3) and (11) we have:

$$K(X,Y)Z - K(X,JY)JZ - K(JX,Y)JZ - K(JX,JY)Z = 0,$$

I.e.

$$K(X,Y)Z = K(JX,Y)JZ + K(JX,JY)Z + K(X,JY)JZ.$$

In the received identity instead of $K(X,JY)JZ$ we shall put the value received from (11) replacement $Y\to JY$ and $Z\to JZ$, i.e $K(X,JY)JZ = -JK(JX,Y)Z$. Then

$$K(X,Y)Z = K(JX,JY)Z + K(JX,Y)JZ - JK(JX,JY)Z,$$

I.e.

$$<K(X,Y)Z,W> = <K(JX,JY)Z,W> + <K(JX,Y)JZ,W> +$$

$$+ <K(JX,JY)Z,JW>,\ X,Y,Z,W\in\mathfrak{X}(M).$$

Thus, the manifold satisfies to identity $\overline{K}_2$. □

The following theorem is similarly proved.

Theorem 12:

Let $\mathfrak{S}$ = (*J*,*g*= <×, ×>) - *nearly*Kahlerstructure. Then the following statements are equivalent:

(1) $\mathfrak{S}$ - structure of a class $\overline{K}_1$;

(2) $\underset{(0)}{K} = \underset{(4)}{K} = \underset{(7)}{K} = 0$;

(3) On space of the attached *G-structure* identities $K^a_{bcd} = K^a_{\hat{b}cd} = K^a_{\hat{b}\hat{c}\hat{d}} = 0$ are fair.

Theorem 13:

Let $\mathfrak{S}$ = (*J*,*g*= <×, ×>) - *nearly*Kahlerstructure. Then the following inclusions of classes $\overline{K}_1 \subset \overline{K}_2 \subset \overline{K}_3$ are fair.

The proof:

Let $\mathfrak{S}$ - structure of a class $\bar{K}_1$. Obviously, it is equivalent to$K_{(0)} = K_{(4)} = K_{(7)} = 0$(By theorem 12) .

So (By theorem 11) $K_{(0)} = K_{(7)} = 0$ is equivalent to class $\bar{K}_2$.

Then $\bar{K}_1 \subset \bar{K}_2$.

Also the class $\bar{K}_3$ is equivalent to $K_{(0)} = 0$that is clear from(theorem 10).

Thus $\bar{K}_1 \subset \bar{K}_2 \subset \bar{K}_3$. □

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