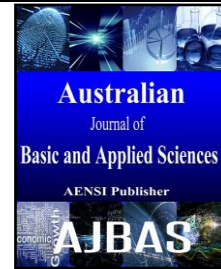




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On Certain Classes of Meromorphically P-Valent Functions With Positive Coefficients

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ABSTRACT

A purpose of this paper is to introduce the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ of meromorphically p-valent functions by introducing a new operator $\mathcal{A}_c f(z)$ associated with polylogarithm function. We study various properties such as coefficient inequality, growth and distortion theorems, closure theorems, convolution properties, radii of meromorphically p-valent starlikeness and convexity, weighted mean and arithmetic mean.

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INTRODUCTION

At last time, the classical polylogarithm function was invented in 1696, by Leibniz and Bernoulli see (Alhindi, K.R. and M. Darus, 2005), as mentioned in (EL- Ashwah, R.M., 2012). For $|z| < 1$ and c a natural number with $c \geq 2$, the polylogarithm function (which is also known as Jonquiere's function) is defined by the absolutely convergent series:

$$Li_c(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^c}$$

Later on, many mathematicians studied the polylogarithm function such as Euler, Spence, Able, Lobachevsky, Rogers, Ramanujan, and many others (Gorchakov, A.B., 1994), where they discovered many functional identities by using polylogarithm function. However, the work employing polylogarithm has been stopped many decades, later. During the past four decades, the work using polylogarithm has again been intensified vividly due to its importance in many fields of mathematics, such as complex analytic, algebra, geometry, topology and mathematical physics (quantum field theory) (Gorchakov, A.B., 1994; Oi, S., 2009; Ponnusamy, S. and S. Sabapathy, 1996) in 1996, Ponnusamy and Sabapathy discussed the geometric mapping properties of the generalized polylogarithm. Recently, Al-Shaqsi and Darus (Al-Shaqsi, K. and M. Darus, 2008) generalized Ruscheweyh and Salagean operators, using polylogarithm function on class \mathcal{D} of analytic functions in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

By making use of the generalized operator they introduced related properties.

A year later, same authors again employed the n th order Polylogarithm function to define a multiplier transformation on the class \mathcal{D} in Δ (Al-Shaqsi, K. and M. Darus, 2009).

Recall the polylogarithm function to be on meromorphic p-valent type, let Σ denote the class of normalized meromorphic p-valent functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n, \quad (1)$$

which are meromorphic and p-valent in the punctured unit disk $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

A function $f \in \Sigma$ is meromorphic starlike of order ρ , ($0 \leq \rho < 1$) iff $-\Re\left(\frac{zf'(z)}{f(z)}\right) > \rho$ ($z \in \Delta^* = \Delta \setminus \{0\}$).

The class of all such functions is denoted by $\Sigma^*(\rho)$.

A function $f \in \Sigma$ is meromorphic convex of order ρ , ($0 \leq \rho < 1$) iff $-\Re\{1 + zf''(z)f'(z)\} > \rho, z \in \Delta^* = \Delta \setminus \{0\}$. The class of all such functions is denoted by $\Sigma^k(\rho)$.

Let Σ_p be the class of functions $f \in \Sigma$ of the form $f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n, a_n \geq 0$. (2)

The subclass of Σ_p consisting of starlike functions of order ρ is denoted by $\Sigma_p^*(\rho)$, and the subclass of Σ_p consisting of convex functions of order ρ is denoted by $\Sigma_p^k(\rho)$.

For functions $f(z)$ given by (1) and $g(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} b_n z^n$ we define the Hadamard product or convolution of f and g by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n b_n z^n = (g * f)(z). \tag{3}$$

see (Raina, R.K. and H.M. Srivastava, 2006), which are analytic and univalent in Δ^* .

Liu and Srivastava (Liu, J.L. and H.M. Srivastava, 2000) defined a function

$(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ by multiplying the well-known generalized hypergeometric function ${}_qF_s$ with z^{-p} as follows:

$$\begin{aligned} \Phi_c f(z) &= \Psi_c(z) * f(z) \\ &= \frac{1}{z^p} + \sum_{n=p}^{\infty} \frac{1}{(n-p+2)^c} a_n z^n. \end{aligned}$$

Now, we define the linear operator $\mathcal{A}_c f(z): \Sigma_p \rightarrow \Sigma_p$ was studied by K. R. Alhindi and M. Darus [1], as follows:

$$\begin{aligned} \mathcal{A}_c f(z) &= \left\{ \Phi_c f(z) - \frac{1}{2^c} a_p z^p \right\} \\ &= \frac{1}{z^p} + \sum_{n=p+1}^{\infty} \frac{1}{(n-p+2)^c} a_n z^n. \end{aligned} \tag{5}$$

Now, by making use of the operator $\mathcal{A}_c f(z)$ we define the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ of functions in Σ_p as follows.

Definition 1:

A function $f(z)$ of the form (2) is said to be in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ if it satisfies the following inequality:

where $0 \leq \alpha < p, 0 < \beta \leq 1, \frac{1}{2} \leq \gamma < 1, p \in \mathbb{N}; z \in \Delta^*$.

The following are special classes of the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$

- (1) $\mathcal{N}_{c,p}(\alpha, 1, 1) = \{f \in \Sigma_p : \Re\{-z^{p+1}(\mathcal{A}_c f(z))\} > \alpha, 0 \leq \alpha < p, p \in \mathbb{N}; z \in \Delta^*\}$, see [8].
- (2) $\mathcal{N}_{0,p}(\alpha, 1, 1) = \{f \in \Sigma_p : \Re\{-z^{p+1}f'(z)\} > \alpha, 0 \leq \alpha < p, p \in \mathbb{N}; z \in \Delta^*\}$, see [4].

Examples: Also we note some special classes of the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ as the following:

Meromorphically multivalent functions have been extensively studied (for example) by many others such as Raina and Srivastava (Raina, R.K. and H.M. Srivastava, 2006), Yang, EL- Ashwah, Saif and Kilicman, Mostafa and others. In the present paper, we obtain coefficient inequality, growth and distortion theorems, closure theorems, Hadamard product, radii of meromorphically p -valent starlikeness, convexity and weighted mean and arithmetic mean for these functions for the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

$$\begin{aligned} &h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \\ &= z^{-p} {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z), \end{aligned}$$

where $\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s$ are complex parameters and $q \leq s + 1, p \in \mathbb{N}$. Analogous to Liu and Srivastava work (Liu, J. and H.M. Srivastava, 2004) and corresponding function $\Psi_c(z)$ given by

$$\begin{aligned} \Psi_c(z) &= z^{-2} Li_c(z) = \\ &= \frac{1}{z^p} + \sum_{n=p}^{\infty} \frac{1}{(n-p+2)^c} z^n. \end{aligned} \tag{4}$$

We consider linear operator $\Phi_c f(z): \Sigma_p \rightarrow \Sigma_p$, which is defined the following Hadamard product (or convolution):

(3) $\mathcal{N}_{0,p}(\alpha, \beta,$

(1) $\mathcal{N}_{0,$

(2)

2. Coefficient Inequality:

We now give a necessary and sufficient condition for a function f to be in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Theorem 1:

Let $f \in \Sigma_p$ given by (2). Then $f \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$, if and only if

where $0 \leq \alpha < p, 0 < \beta \leq 1, \frac{1}{2} \leq \gamma < 1, p \in \mathbb{N}; z \in \Delta^*$.

Proof. Suppose that (7) is holds. Then

For $|z| = r < 1$ the left hand side of (8) is bounded above by

Thus $f \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Conversely, suppose $f \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$. Then by (6),

Since $|\Re(z)| \leq |z|$ for all z , then

Now choosing the values of z on the real axis so that the function $z^{p+1}(\mathcal{A}_c f(z))'$ is real. By clearing the denominator in (9) and letting $z \rightarrow 1^-$ through positive values, we get:

Hence the proof is complete. ■

Corollary 1:

Let the function $f(z)$ defined by (1) be in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$. Then

The result is sharp for the function:

3. Growth and Distortion Theorems:

A growth and distortion property for the function f to be in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ is given as follows:

Theorem 2:

Let the function $f(z)$ defined by (1) be in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$. Then for $0 < |z| = r < 1$, we have

$$\frac{1}{r^p} - \frac{2\beta\gamma(p-a)}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}} r^p \leq |f(z)| \leq \frac{1}{r^p} + \frac{2\beta\gamma(p-a)}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}} r^p, \quad (11)$$

and

$$\frac{p}{r^{p+1}} - \frac{2\beta\gamma(p-a)}{(1+2\beta\gamma-\beta)\frac{1}{2^c}} r^{p-1} \leq |f'(z)| \leq \frac{1}{r^{p+1}} + \frac{2\beta\gamma(p-a)}{(1+2\beta\gamma-\beta)\frac{1}{2^c}} r^{p-1}, \quad (12)$$

with equality for

$$f(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-a)}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}} z^p \quad (p \in \mathbb{N}) \quad (13)$$

Proof. By Theorem 1, we have

$$p(1+2\beta\gamma-\beta)\frac{1}{2^c} \sum_{n=p+1}^{\infty} a_n \leq \sum_{n=p+1}^{\infty} n(1+2\beta\gamma-\beta) \frac{1}{(n-p+2)^c} a_n \leq 2\beta\gamma(p-a).$$

Then

$$\sum_{n=p+1}^{\infty} a_n \leq \frac{2\beta\gamma(p-a)}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}}$$

for $0 < |z| = r < 1$,

$$\begin{aligned} |f(z)| &\leq \frac{1}{r^p} + \sum_{n=p+1}^{\infty} a_n r^n, \\ &\leq \frac{1}{r^p} + r^p \sum_{n=p+1}^{\infty} a_n \\ &\leq \frac{1}{r^p} + \frac{2\beta\gamma(p-a)}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}} r^p \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq \frac{1}{r^p} - \sum_{n=p+1}^{\infty} a_n r^n, \\ &\geq \frac{1}{r^p} - r^p \sum_{n=p+1}^{\infty} a_n \\ &\geq \frac{1}{r^p} - \frac{2\beta\gamma(p-a)}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}} r^p, \end{aligned}$$

which, together, yield (11). Furthermore, it follows from Theorem 1 that

$$\sum_{n=p+1}^{\infty} n a_n \leq \frac{2\beta\gamma(p-a)}{(1+2\beta\gamma-\beta)\frac{1}{2^c}}.$$

Hence

$$\begin{aligned} |f'(z)| &\leq \left| \frac{-p}{z^{p+1}} \right| + \sum_{n=p+1}^{\infty} n a_n z^{n-1}, \\ |f'(z)| &\leq \frac{p}{z^{p+1}} + r^{p-1} \sum_{n=p+1}^{\infty} n a_n \\ &\leq \frac{p}{z^{p+1}} + \frac{2\beta\gamma(p-a)}{(1+2\beta\gamma-\beta)\frac{1}{2^c}} r^{p-1}, \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq \left| \frac{-p}{z^{p+1}} \right| - \sum_{n=p+1}^{\infty} n a_n z^{n-1}, \\ |f'(z)| &\geq \frac{p}{z^{p+1}} - r^{p-1} \sum_{n=p+1}^{\infty} n a_n \end{aligned}$$

$$\geq \frac{p}{z^{p+1}} - \frac{2\beta\gamma(p-a)}{(1+2\beta\gamma-\beta)\frac{1}{2^c}} r^{p-1},$$

which, together, yield (12). It is clear that the function given by (13) is extremal function. Hence the proof is complete. ■

4. Closure Theorems:

We now prove the closure theorems as follows.

Theorem 3:

Let

$$f_p(z) = \frac{1}{z^p},$$

and

$$f_n = \frac{1}{z^p} + \frac{2\beta\gamma(p-a)}{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}} z^n, (n \geq p+1, p \in \mathbb{N}).$$

Then $f(z)$ is in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=p}^{\infty} \mu_n f_n(z),$$

where $\mu_n \geq 0$ and $\sum_{n=p}^{\infty} \mu_n = 1$.

Proof. First suppose that $f(z)$ can be expressed of the form

$$\begin{aligned} f(z) &= \sum_{n=p}^{\infty} \mu_n f_n(z), \\ &= \frac{1}{z^p} + \sum_{n=p+1}^{\infty} \frac{2\beta\gamma(p-a)\mu_n}{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}} z^n. \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=p+1}^{\infty} \frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} \frac{2\beta\gamma(p-a)\mu_n}{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}} \\ \sum_{n=p+1}^{\infty} \mu_n = 1 - \mu_p \leq 1. \end{aligned}$$

which shows, that $f \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Conversely, suppose $f \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$, then

$$a_n \leq \frac{2\beta\gamma(p-a)}{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}, (n \geq p+1, p \in \mathbb{N}).$$

Setting

$$\begin{aligned} \mu_n &= \frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} a_n \\ \mu_p &= 1 - \sum_{n=p+1}^{\infty} \mu_n, \end{aligned}$$

we get

$$f(z) = \sum_{n=p}^{\infty} \mu_n f_n(z).$$

Hence the proof is complete. ■

Theorem 4:

Let the function $f_i(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_{n,i} z^n, i = 1, 2, \dots, k$ be in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Then the function

$F(z) = \sum_{i=1}^k \mu_i f_i(z)$, where $\sum_{i=1}^k \mu_i = 1$, is also in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Proof. From Theorem 1, we have

$$\sum_{n=p+1}^{\infty} n(1+2\beta\gamma-\beta) \frac{1}{(n-p+2)^c} a_n \leq 2\beta\gamma(p-a).$$

Since

$$F(z) = \frac{1}{z^p} \sum_{n=p+1}^{\infty} \left(\sum_{i=1}^k \mu_i a_{n,i} \right) z^n,$$

Then

$$\begin{aligned} \sum_{i=1}^k \mu_i \sum_{n=p+1}^{\infty} n(1+2\beta\gamma-\beta) \frac{1}{(n-p+2)^c} \left(\sum_{i=1}^k \mu_i a_{n,i} \right) \\ \leq 2\beta\gamma(p-a) \sum_{i=1}^k \mu_i \\ = 2\beta\gamma(p-a). \end{aligned}$$

This completes the proof of the theorem. ■

Theorem 5:

The class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ is convex.

Proof. In order to prove the theorem it is enough to show that the function $h(z)$ defined by

$$h(z) = \delta f(z) + (1-\delta)g(z), \quad (0 \leq \delta \leq 1)$$

is in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$,

where

$$\begin{aligned} f(z) &= \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \geq 0 \\ g(z) &= \frac{1}{z^p} + \sum_{n=p+1}^{\infty} b_n z^n, \quad b_n \geq 0, \end{aligned}$$

are in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Then

$$h(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} (\delta a_n + (1-\delta)b_n) z^n.$$

By using Theorem 1, we get

$$\begin{aligned} \sum_{n=p+1}^{\infty} n(1+2\beta\gamma-\beta) \frac{1}{(n-p+2)^c} (\delta a_n + (1-\delta)b_n) \leq \delta 2\beta\gamma(p-a) + (1-\delta)2\beta\gamma(p-a) \\ = 2\beta\gamma(p-a). \end{aligned}$$

Thus $h(z) \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Hence the proof is complete. ■

5. Convolution Properties:

For the functions

$$f_i(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_{n,i} z^n, \quad (a_{n,i} \geq 0; i = 1, 2) \quad (14)$$

, belonging to the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$, we denote by $(f_1 * f_2)(z)$ the Hadamard product (or the convolution) of the functions $f_1(z)$ and $f_2(z)$, that is

$$(f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_{n,1} a_{n,2} z^n. \quad (15)$$

Theorem 6:

Let the functions $f_i(z)$ ($i=1, 2$) defined by (14) be in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$, the

$(f_1 * f_2)(z) \in \mathcal{N}_{c,p}(\omega, \beta, \gamma)$, where

$$\omega = p - \frac{2\beta\gamma(p-a)^2}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}}$$

The result is sharp for the functions $f_i(z)$ ($i = 1, 2$) given by

$$f_i(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-a)^2}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}} z^p, \quad (i = 1, 2, p \in \mathbb{N}).$$

Proof. Since $f_i(z) \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ ($i = 1, 2$).

Then by Theorem 1 we have:

$$\sum_{n=p+1}^{\infty} \frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} a_{n,i} \leq 1 \quad (i = 1, 2).$$

Thus by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=p+1}^{\infty} \frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} \sqrt{a_{n,1}a_{n,2}} \leq 1. \quad (16)$$

To prove the theorem we need to find the largest ω such that

$$\sum_{n=p+1}^{\infty} \frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-\omega)} a_{n,1}a_{n,2} \leq 1.$$

or we must get:

$$\frac{a_{n,1}a_{n,2}}{p-\omega} \leq \frac{\sqrt{a_{n,1}a_{n,2}}}{p-a} \quad (n \geq p+1; p \in \mathbb{N}),$$

which is equivalent to

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{p-\omega}{p-a} \quad (n \geq p+1; p \in \mathbb{N}).$$

From (16), we have

$$\frac{2\beta\gamma(p-a)}{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}} \leq \frac{p-\omega}{p-a}.$$

By simplifying it, we get:

$$\omega \leq p - \frac{2\beta\gamma(p-a)^2}{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}} \quad (n \geq p+1; p \in \mathbb{N}).$$

Now, defining the function $\varphi(n)$ by

$$\varphi(n) = p - \frac{2\beta\gamma(p-a)^2}{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}} \quad (n \geq p+1).$$

This function is an increasing function of n . Thus, we have

$$\omega \leq \varphi(p) = p - \frac{2\beta\gamma(p-a)^2}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}}.$$

Hence the proof is complete. ■

Theorem 7:

Let the function $f_1(z)$ defined by (14) be in the class $\mathcal{N}_{c,p}(\alpha_1, \beta, \gamma)$ and the function $f_2(z)$ defined by

(14) be in the class $\mathcal{N}_{c,p}(\alpha_2, \beta, \gamma)$. Then $(f_1 * f_2)(z) \in \mathcal{N}_{c,p}(\psi, \beta, \gamma)$, where

$$\psi = p - \frac{2\beta\gamma(p-a_1)(p-a_2)}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}}.$$

The result is sharp for the functions $f_i(z)$ ($i = 1, 2$) given by

$$f_1(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-a_1)^2}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}} z^p \quad (p \in \mathbb{N}),$$

and

$$f_2(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-a_2)^2}{p(1+2\beta\gamma-\beta)\frac{1}{2^c}} z^p \quad (p \in \mathbb{N}).$$

Proof. By using the same technique of Theorem 6 we prove the theorem, hence it is omitted.

Theorem 8. If

$f_1(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_{n,1} z^n \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ and $f_2(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_{n,2} z^n \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ with $|a_{n,2}| \leq 1, n \geq p+1, p \in \mathbb{N}$. Then $(f_1 * f_2)(z) \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Proof. By using Theorem 1 it is enough to show that:

$$\sum_{n=p+1}^{\infty} \frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} a_{n,1} a_{n,2} \leq 1,$$

Since

$$\begin{aligned} & \sum_{n=p+1}^{\infty} \frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} |a_{n,1} a_{n,2}| \\ &= \sum_{n=p+1}^{\infty} \frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} a_{n,1} |a_{n,2}| \\ &\leq \sum_{n=p+1}^{\infty} \frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} a_{n,1} \leq 1, \end{aligned}$$

Thus $(f_1 * f_2)(z) \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$. ■

Corollary 2. If $f_1(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_{n,1} z^n \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ and $f_2(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_{n,2} z^n \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ with $h_0 \leq |a_{n,2}| \leq 1, n \geq p+1, p \in \mathbb{N}$. Then $(f_1 * f_2)(z) \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Theorem 9. Let the functions $f_i(z) (i=1, 2)$ defined by (14) be in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ and

$$p(1+2\beta\gamma-\beta)\frac{1}{2^c} - 4\beta\gamma(p-a) \geq 0,$$

then the function $h(z)$ defined by

$$h(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} (\alpha_{n,1}^2 + \alpha_{n,2}^2) z^n, \quad (17)$$

is also in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Proof. By Theorem 1 we have

$$\sum_{n=p+1}^{\infty} \frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} a_{n,1} \leq 1,$$

and

$$\sum_{n=p+1}^{\infty} \frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} a_{n,2} \leq 1.$$

Then

$$\sum_{n=p+1}^{\infty} \left[\frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} \right]^2 \alpha_{n,1}^2 \leq 1,$$

and

$$\sum_{n=p+1}^{\infty} \left[\frac{n(1+2\beta\gamma-\beta)\frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} \right]^2 \alpha_{n,2}^2 \leq 1.$$

Hence

$$\sum_{n=p+1}^{\infty} \frac{1}{2} \left[\frac{n(1+2\beta\gamma - \beta) \frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} \right]^2 (\alpha_{n,1}^2 + \alpha_{n,2}^2) \leq 1.$$

To proof the theorem it is sufficient to show that

$$\sum_{n=p+1}^{\infty} \left[\frac{n(1+2\beta\gamma - \beta) \frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} \right] (\alpha_{n,1}^2 + \alpha_{n,2}^2) \leq 1. \quad (18)$$

Thus the inequality (18) will be satisfied if, for $n \geq p$

$$\frac{n(1+2\beta\gamma - \beta) \frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} \leq \frac{1}{2} \left[\frac{n(1+2\beta\gamma - \beta) \frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} \right]^2,$$

or if

$$n(1+2\beta\gamma - \beta) \frac{1}{(n-p+2)^c} - 4\beta\gamma(p-a) \geq 0, n = p, (p+1), (p+2), \dots$$

The left hand side of the above inequality is an increasing function of n , so itsatisfied for all n if

$$p(1+2\beta\gamma - \beta) \frac{1}{2^c} - 4\beta\gamma(p-a) \geq 0,$$

which is given by our hypothesis.

This completes the proof of the theorem. ■

Theorem 10:

Let the functions $f_i(z)$ ($i = 1, 2$) defined by (14) be in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$. Then the function $h(z)$ defined by (17) belongs to the class $\mathcal{N}_{c,p}(\Omega, \beta, \gamma)$, where

$$\Omega = p - \frac{4\beta\gamma(p-a)^2}{p(1+2\beta\gamma - \beta) \frac{1}{2^c}}.$$

The result is sharp for the functions $f_i(z)$ ($i = 1, 2$) given by

$$f_i(z) = \frac{1}{z^p} + \frac{2\beta\gamma(p-a)^2}{p(1+2\beta\gamma - \beta) \frac{1}{2^c}} z^p, (i = 1, 2, p \in \mathbb{N}).$$

Proof. From Theorem 1, we have

$$\sum_{n=p+1}^{\infty} \frac{n(1+2\beta\gamma - \beta) \frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} a_{n,i} \leq 1 (i = 1, 2).$$

Now

$$\sum_{n=p+1}^{\infty} \left[\frac{n(1+2\beta\gamma - \beta) \frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} \right]^2 \alpha_{n,i}^2 \leq \sum_{n=p+1}^{\infty} \left[\frac{n(1+2\beta\gamma - \beta) \frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} a_{n,i} \right]^2 \leq 1 (i = 1, 2),$$

for $f_i(z) \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$ ($i = 1, 2$), we have

$$\sum_{n=p+1}^{\infty} \frac{1}{2} \left[\frac{n(1+2\beta\gamma - \beta) \frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)} \right]^2 (\alpha_{n,1}^2 + \alpha_{n,2}^2) \leq 1. \quad (19)$$

Hence, we have to find the largest Ω such that

$$\frac{1}{p-\Omega} \leq \frac{n(1+2\beta\gamma - \beta) \frac{1}{(n-p+2)^c}}{4\beta\gamma(p-a)^2} (n \geq p+1),$$

or

$$\Omega = p - \frac{4\beta\gamma(p-a)^2}{n(1+2\beta\gamma - \beta) \frac{1}{(n-p+2)^c}} (n \geq p+1).$$

We observe that the right hand side of the above inequality is an increasing function of n , we get

$$\Omega = p - \frac{4\beta\gamma (p - a)^2}{p(1 + 2\beta\gamma - \beta)\frac{1}{2^c}}$$

Which completes the proof of Theorem 10. ■

6. Radii of Meromorphically p -Valent Starlikeness and Convexity:

Theorem 11:

Let the function $f(z)$ defined by (2) be in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$. Then $f(z)$ is meromorphically p -valent starlike of order δ ($0 \leq \delta < p$) in the disk $|z| < r_1$, where

$$r_1 = \inf_{n \geq p+1} \left[\frac{n(1 + 2\beta\gamma - \beta) \frac{1}{(n-p+2)^c} (p - \delta)}{2\beta\gamma (p - a)(n - \delta)} \right]^{\frac{1}{p+n}}$$

The result is sharp.

Proof. From Theorem 1, we have:

$$\sum_{n=p+1}^{\infty} n(1 + 2\beta\gamma - \beta) \frac{1}{(n-p+2)^c} a_n \leq 2\beta\gamma (p - a),$$

and $f(z)$ is said to be meromorphically p -valent starlike of order δ ($0 \leq \delta < p$), if

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta,$$

or

$$\left| \frac{zf'(z) + pf(z)}{f(z)} \right| \leq p - \delta \quad (0 \leq \delta < p).$$

Now

$$\begin{aligned} \left| \frac{zf'(z) + pf(z)}{f(z)} \right| &= \left| \frac{\sum_{n=p+1}^{\infty} (p+n)a_n z^n}{z^{-p} + \sum_{n=p+1}^{\infty} a_n z^n} \right| \\ &\leq \frac{\sum_{n=p+1}^{\infty} (p+n)a_n |z|^{p+n}}{1 + \sum_{n=p+1}^{\infty} a_n |z|^{p+n}}. \end{aligned}$$

To prove the theorem the above inequality must be less than or equal to $p - \delta$, so

$$\sum_{n=p+1}^{\infty} \frac{(n - \delta)}{(p - \delta)} a_n |z|^{p+n} \leq (p - \delta). \quad (20)$$

Then by Corollary 1 the inequality (20) will be true if

$$|z|^{p+n} \leq \frac{n(1 + 2\beta\gamma - \beta) \frac{1}{(n-p+2)^c} (p - \delta)}{2\beta\gamma (p - a)(n - \delta)},$$

that is,

$$|z| \leq \left[\frac{n(1 + 2\beta\gamma - \beta) \frac{1}{(n-p+2)^c} (p - \delta)}{2\beta\gamma (p - a)(n - \delta)} \right]^{\frac{1}{p+n}},$$

The infimum of the above quantity is the radii of starlikeness of the function $f(z)$ in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$. The sharpness follows by choosing the same extremal function (10).

Which completes the proof of theorem. ■

Theorem 12:

Let the function $f(z)$ defined by (2) be in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$. Then $f(z)$ is meromorphically p -valent convex of order ζ ($0 \leq \zeta < p$) in the disk $|z| < r_2$, where

$$r_2 = \inf_{n \geq p+1} \left[\frac{p(p - \zeta)(1 + 2\beta\gamma - \beta) \frac{1}{(n-p+2)^c}}{2\beta\gamma (p - a)(2p + n - \zeta)} \right]^{\frac{1}{p+n}}$$

The result is sharp.

Proof. It is enough to show that

$$-\Re\left(1 + \frac{zf'(z)}{f'(z)}\right) > \zeta \quad (0 \leq \zeta < p, |z| < r_2, p \in \mathbb{N}),$$

or

$$\left| \frac{(zf'(z))' + pf'(z)}{f'(z)} \right| = \left| \frac{\sum_{n=p+1}^{\infty} n(p+n)a_n z^{n-1}}{-p + z^{-p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1}} \right|$$

$$\leq \frac{\sum_{n=p+1}^{\infty} n(p+n)a_n |z|^{p+n}}{p - \sum_{n=p+1}^{\infty} na_n |z|^{p+n}}.$$

To prove the theorem the above inequality must be less than or equal to $p - \zeta$, or

$$\sum_{n=p+1}^{\infty} \frac{n(2p+n-\zeta)}{p(p-\zeta)} a_n |z|^{p+n} \leq 1.$$

From Theorem 1, we get

$$|z|^{p+n} \leq \frac{p(p-\zeta)(1+2\beta\gamma-\beta) \frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)(2p+n-\zeta)}.$$

Thus

$$|z| \leq \left[\frac{p(p-\zeta)(1+2\beta\gamma-\beta) \frac{1}{(n-p+2)^c}}{2\beta\gamma(p-a)(2p+n-\zeta)} \right]^{\frac{1}{p+n}} \quad (n \geq p+1, p \in \mathbb{N}).$$

By choosing r_2 to be the infimum of the above quantity we get the result. The sharpness follows by choosing the same extremal function (10). This completes the proof of the theorem. ■

7. Weighted Mean and Arithmetic Mean:

Definition 2:

If the functions $f(z)$ and $g(z)$ defined by (2) are in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$, then the weighted mean $h_i(z)$ of the two functions is defined as follows

$$h_i(z) = \frac{1}{2}[(1-i)f(z) + (1+i)g(z)].$$

Theorem 13:

Let the functions $f(z)$ and $g(z)$ defined by (2) are in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$. Then their weighted mean is also in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Proof. the weighted mean of $f(z)$ and $g(z)$ is:

$$h_i(z) = \frac{1}{2}[(1-i)f(z) + (1+i)g(z)],$$

$$= \frac{1}{2} \left[(1-i) \left(f(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_n z^n \right) + (1+i) \left(g(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} b_n z^n \right) \right]$$

$$= \frac{1}{z^p} + \sum_{n=p+1}^{\infty} \frac{1}{2} ((1-i)a_n + (1+i)b_n) z^n.$$

By using Theorem 1, it is sufficient to show that

$$\sum_{n=p+1}^{\infty} n(1+2\beta\gamma-\beta) \frac{1}{(n-p+2)^c} \left[\frac{1}{2} ((1-i)a_n + (1+i)b_n) z^n \right]$$

$$= \frac{1}{2} (1-i) \sum_{n=p+1}^{\infty} n(1+2\beta\gamma-\beta) \frac{1}{(n-p+2)^c} a_n$$

$$+ \frac{1}{2} (1+i) \sum_{n=p+1}^{\infty} n(1+2\beta\gamma-\beta) \frac{1}{(n-p+2)^c} b_n$$

$$\begin{aligned} &\leq \frac{1}{2}(1-i)(2\beta\gamma(p-a)) + \frac{1}{2}(1+i)(2\beta\gamma(p-a)) \\ &= 2\beta\gamma(p-a). \end{aligned}$$

Hence $h_i(z) \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$

Which completes the proof of theorem. ■

Theorem 14:

If the functions $f_i(z)$ ($i = 1, \dots, d$) defined by

$$f_i(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_{n,i} z^n, (a_{n,i} \geq 0, n \geq p+1, i = 1, 2, \dots, d),$$

belongs to the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$, then their arithmetic mean defined by

$$h(z) = \frac{1}{d} \sum_{i=1}^d f_i(z)$$

is also in the class $\mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

Proof. Since

$$h(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} \left(\frac{1}{d} \sum_{i=1}^d a_{n,i} \right) z^n.$$

Then by using Theorem 1, we must show that

$$\begin{aligned} \sum_{n=p+1}^{\infty} n(1+2\beta\gamma-\beta) \frac{1}{(n-p+2)^c} \left(\frac{1}{d} \sum_{i=1}^d a_{n,i} \right) &= \frac{1}{d} \sum_{i=1}^d \sum_{n=p+1}^{\infty} n(1+2\beta\gamma-\beta) \frac{1}{(n-p+2)^c} a_{n,i} \\ &\leq \frac{1}{d} \sum_{i=1}^d 2\beta\gamma(p-a) = 2\beta\gamma(p-a). \end{aligned}$$

Hence $h(z) \in \mathcal{N}_{c,p}(\alpha, \beta, \gamma)$.

This completes the proof of the theorem. ■

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