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Separation Axioms on Multi Bitopological Spaces

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ABSTRACT

Background: In this paper, the concepts of lower separation axioms and regularity axioms have been introduced on multi-topological spaces and on multi bitopological spaces. The relations between these classes are also given. Some types of continuous functions between multi bitopological spaces are introduced and the inter relations between continuity and separation axioms are given.

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INTRODUCTION

The concept of a multiset is well established both in mathematics and computer science (Conder *et al*. (Conder, 2007) and Singh *et al*. (2001). The mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well- defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short). Thus, a multiset differs from a set in the sense that each element has multiplicity–a natural number not necessarily one- that indicates how many times it is a member of the multiset. In the same sense Dedekind (1963) stated that each element in the range of a function can be thought of as having a multiplicity equal to the number in the domain that are mapped to it. One of the most natural and simplest examples is the mset of prime factors of a positive integer "n", the number $4608 = 2^3\ 3^2\ 4^3$ corresponds to the mset {2, 2, 2, 3, 3, 4, 4, 4}. The notion of a bitopological space (X, τ_1, τ_2) that is of a set X equipped with two arbitrary topologies τ_1 and τ_2 was first formulated by J.C. Kelly in (1936). In 1943, Shanin (1943) introduced the separation axioms R_o and R_1 in topological spaces. Murdeshwar and Naimpally (1966) offered the notions pairwise R_o and pairwise R_1 bitopological spaces. Separation axioms and closed sets in topological spaces have been very useful in the study of certain objects in digital topology (1991). Ivan, Reilly discussed pairwise T_o, T_1, T_2, T_3, T_4 in (1992). El-Batanony *et al*. (1995)(Roy, 2013; Sreeja, 2012; Caldas, 2006; Rajesh, 2011; Al–Areefi, 2009) studied the concept of bitopological spaces, topological spaces, supra topological spaces and the continuous functions between bitopological spaces. Some generalized separation axioms on generalized bitopological spaces are introduced by Narasimhan (Narasimhan) Tantawy (2005) and others. In this paper, we introduce the concept of multi quasi–coincident and some of its properties. Moreover, we introduce some separation axioms in the multi-topological spaces using the concept. Also, we study multi supra topological spaces (msts, for short) and separation axioms of it. We extend some kinds of generalized separation axioms to multi bitopological spaces (mbts, for short), also, we study the relations between these separation axioms. The inter relations between some types of continuous functions which are introduced in (2015) and the separation axioms in this paper are studied.

2. Preliminaries and Basic Definitions:

In this section the basic definitions and notations of multisets and the different types of collections of multisets and the basic definitions of functions in multiset context are given (El–Batanony, 1995; Girish, 2012; Girish, 2012; El–Tantawy, 2005; Tantawy, 2015).

Definition 2.1: (Girish, 2012) A mset M drawn from the set X is represented by a function count M or C_M defined as $C_M : X \rightarrow N$ where N represents the set of non-negative integers.

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Here $C_M(x)$ is the number of occurrences of the element x in the mset M. Let $X = \{x_1, x_2, \ldots, x_n\}$, the mset M Drawn from the set X is represented by $M = \{m_1/x_1, m_2/x_2, \ldots\ldots m_n/x_n\}$ where m_i is the number of occurrences of the element x_i, i = 1, 2, …., n in the mset M.

Example 2.2.: Let X = {a, b, c, d, e} be any set. Then M = {2/a, 3/b, 5/d, 1/e} is an mset drawn from X. Clearly, a set is a special case of an mset.

Definition 2.3: (Girish, 2012) Let M and N be two msets drawn from a set X. then he following are defined:

1. $M = N$ if $C_M(x) = C_N(x)$, $\forall x \in X$.
2. $M \subseteq N$ if $C_M(x) \leq C_N(x)$, $\forall x \in X$.
3. $P = M \cup N$ if $C_P(x) = \text{Max}\{C_M(x), C_N(x)\}$, $\forall x \in X$.
4. $P = M \cap N$ if $C_P(x) = \text{Min}\{C_M(x), C_N(x)\}$, $\forall x \in X$.
5. $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$, $\forall x \in X$.
6. $P = M \ominus N$ if $C_P(x) = \text{Max}\{C_M(x) - C_N(x), 0\}$, $\forall x \in X$.

Where $\oplus$ and $\ominus$ represented mset addition and mset subtraction, respectively.

Definition 2.4: (Girish, 2012) Let M be any mset drawn from a set X. the support set of M denoted by M^* is a subset of X and defined as $M^* = \{x \in X : C_M(x) > 0\}$ i.e. M^* is an ordinary set. M^* is also called root set.

Definition 2.5: (Girish, 2012) An mset M is said to be an empty mset if $C_M(x) = 0$, $\forall x \in X$

Definition 2.6: (Girish, 2012) A domain X, is defined as a set of elements from which msets are constructed. The mset space $[X]^w$ is the set of all msets whose elements are in X such that no element in the mset occurs more than w times.

The set $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of the occurrences of an element in an mset.

If $X = \{x_1, x_2, \ldots\ldots, x_K\}$ then $[X]^w = \{\{m_1/x_1, m_2/x_2, \ldots., m_k/x_k\} : \text{for } i = 1, 2, \ldots., k; m_i \in \{0, 1, 2, \ldots\ldots w\}\}$.

Definition 2.7: (Girish, 2012) Let X be a support set and $[X]^w$ be the mset space defined over X. Then for any mset $M \in [X]^w$, the complement M^C of M in $[X]^w$ is an element of $[X]^w$ such that

$C_M{}^c(x) = w - C_M(x)$, $\forall x \in X$

Notation 2.8: (Girish, 2012) Let M be an mset from X with x appearing n times in M. it is denoted by $x \in^n M$.

$M = \{k_1/x_1, k_2/x_2, \ldots\ldots, k_n/x_n\}$ where M is an mset with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. $[M]_x$ denotes that the element x belongs to the mset M and $|(M)_x|$ denotes the cardinality of an element x in M.

Definition 2.9: (Girish, 2012) A submset N of M is said to be:

1. Whole submset of M if $C_N(x) = C_M(x)$, for every $x \in N$.
2. Partial whole submset of M if $C_N(x) = C_M(x)$, for some $x \in N$.
3. Full submset of M if $M^* = N^*$ and $C_N(x) \leq C_M(x)$, for every $x \in N$.

Definition 2.10: (Girish, 2012) Let $M \in [X]^w$ be an mset then:

1. The power whole mset of M denoted by PW (M) is defined as the set of all whole submsets of M.
2. The power full mset denoted by PF (M) is defined as the set of all full submsets of M.

Definition 2.11: (Girish, 2012) (Power Mset) let $M \in [X]^w$ be an mset. The power mset of M denoted by P (M) is the set of all the submsets of M. i.e. $N \in P(M)$ if and only if $N \subseteq M$. if $N = \varnothing$, then $N \in^1 P(M)$, and if $N \neq \varnothing$, then $N \in^k P(M)$ where $k = \prod_z \binom{|[M]_z|}{|[N]_z|}$, the product $\prod_z$ is taken over by distinct of z of the mset N and $|(M]_z| = m$ iff $z \in^m M$, $|[N]_z| = n$ iff $z \in^n N$, then

$$\binom{|[M]_z|}{|[N]_z|} = \binom{m}{n} = \frac{m!}{n!(m-n)!}$$

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$.

Definition 2.12: (Girish, 2012) The maximum mset is defined as Z where $C_Z(x) = \text{Max}\{C_M(x): x \in^k M; M \in [X]^w \text{ and } k \leq m\}$.

Definition 2.13: (Girish, 2012) Let $[X]^w$ be an mset space, $\{M_1, M_2, \ldots.\}$ be a collection of msets drawn from $[X]^w$, then the following operations are possible under an arbitrary collection of msets.

1. The union $\bigcup_{i\in I} M_i = \{C_{\cup M_i}(x)/x : C_{\cup M_i}(x) = \text{Max}\{C_{M_i}(x) : x \in X\}\}$.
2. The intersection $\bigcap_{i\in I} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = \text{Min}\{C_{M_i}(x) : x \in X\}\}$.
3. The mset addition $\oplus_{i\in I} M_i = \{C_{\oplus M_i}(x)/x : C_{\oplus M_i}(x) = \sum_{i\in I} C_{M_i}(x),\ x \in X\}$.
4. The mset complement $M^C = Z \ominus M = \{C_{M^c}(x)/x : C_{M^c}(x) = C_Z(x) - C_M(x),\ x \in X\}$.

Definition 2.14: (Girish, 2012) Let M_1 and M_2 be two msets drawn from a set X, then the Cartesian product of M_1 and M_2 is defined as $M_1 \times M_2 = \{(m/x, n/y)/ mn: x \in^m M_1, y \in^n M_2\}$.

Definition 2.15: (Girish, 2012) A subset R of M x M is said to be an mse relation on M if every member (m/x, n/y) of R has a count, product of C_1 (x, y) and C_2 (x, y). We denote m/x related to n/y by m/x R n/y.

The domain and range of the mset relation R on M is defined as follows:

Dom R = $\{x \in^k M: \exists\ y \in^r M$ such that k/x R r/ y$\}$ where $C_{Dom\ R}(x) = \sup\{C_1(x, y): x \in^k M\}$.

Ran R = $\{y \in^r M: \exists\ y \in^K M$ such that k/x R r/ y$\}$ where $C_{Ran\ R}(y) = \sup\{C_2(x, y): y \in^r M\}$.

Definition 2.16: (Girish, 2012) An mset relation f is called an mset function if for every element m/x in domain f, there is exactly one n/y in Ran f such that (m/x, n/y) is in f with the pair occurring as the product of C_i (x, y) and C_2 (x, y).

Definition 2.17: (Girish, 2012) Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. then τ is called a multiset topology of M if τ satisfies the following properties.

1. The mset M and the empty mset $\varnothing$ are in τ.
2. The union of elements of any sub collection of τ is in τ.
3. The intersection of elements of any finite sub collection of τ is in τ.

The pair (M, τ) is called multi topological space (for short M-topological space). The elements of τ are called open msets.

Definition 2.18: (Girish, 2012) Let (M, τ) be an M- topological space and N is a sub mset of M. the collection

$\tau_N = \{N \cap U : U \in \tau\}$ is an M- topology on N denoted by (N, τ_N) is called a subspace of (M, τ).

Definition 2.19: (Girish, 2012) A submset N of an M- topological space M in $[X]^w$ is said to be closed mset if the mset $M \ominus N$ is open.

Theorem 2.20: (Girish, 2012) Let (M, τ) be an M – topological space. Then the following conditions hold:

1. The mset M and empty mset $\varnothing$ are closed msets.
2. Arbitrary intersection of closed msets is a closed mset.
3. Finite union of closed msets is a closed mset.

Definition 2.21: (Girish, 2012) Given a submset N of M–topological space M in $[X]^w$.

1. The interior of N is defined as the union of all open msets contained in N and is denoted by Int (N)

   i.e. $C_{Int(N)}(x) = Max\{C_G(x) : G \in \tau, G \subseteq N\}$.
2. The closure of N is defined as the intersection of all closed msets containing N and is denoted by Cl(N). i.e. $C_{Cl(N)}(x) = Min\{C_F(x) : F \in \tau^c, F \supseteq N\}$.

Theorem 2.22: (Girish, 2012) Let N be a subspace of an M-topological space M in $[X]^w$ and A be a submset of an mset N and Cl (A) denote the closure of an mset A in M. Then the closure of an mset A in N equals Cl (A) $\cap$ N.

Theorem 2.23: (Girish, 2012) If M_1, M_2 are submsets of the M-topological space M in $[X]^w$, then the following properties hold:

1. $C_{Int(M_1 \cap M_2)}(x) = Min\{C_{Int(M_1)}(x), C_{Int(M_2)}(x)\}$.
2. $C_{Cl(M_1 \cup M_2)}(x) = Max\{C_{Cl(M_1)}(x), C_{Cl(M_2)}(x)\}$.

Definition 2.24: (Girish, 2012) Let (M, τ) be an M-topological space, let $x \in^k M$ and $N \subseteq M$. then N is said to be a neighborhood of k/x if there is an open mset U in τ such that $x \in^k U$ and $C_U(y) \leq C_N(y)$, $\forall_y \neq x$.

Definition 2.25: (Girish, 2012) Let M and N be two M-topological spaces. The mset function f: M $\rightarrow$ N is said to be continuous if for each open submset U of N then the mset $f^{-1}(U)$ is an open submset of M, where $f^{-1}(U)$ is the mset of all points m/x in M for which f (m/x) $\in^n$ U for some n.

Theorem 2.26: (Girish, 2012) Let M and N be to M–topological space and f : M $\rightarrow$ N be an mset function then the following are equivalent:

1. The mset function f is continuous.
2. For every submset A of M, $C_{f(Cl(A))}(x) \leq C_{Cl(f(A))}(x)$.
3. For every closed mset B of N, the mset $f^{-1}(B)$ is a closed mset in M.
4. For each $x \in^m M$ and each neighborhood V of (m/x), there is a neighborhood U of m/x such that $C_{f(U)}(x) \leq C_V(x)$.

Definition 2.27:(Kelly, 1963) Let X be a non–empty set. The triple (X, τ_1, τ_2) is called a bitopological space. Where τ_1 and τ_2 are arbitrary topologies on a set x.

Definition 2.28: (Tantawy, 2015) Let (M_1, τ_1, τ_2), (M_2, τ_1, τ_2) be two mbts. A function f : (M_1, τ_1, τ_2) $\rightarrow$ (M_2, τ_1, τ_2) is said to be:

1. j–m–continuous if $\forall$ V $\in$ j–m–C (M_2), then $f^{-1}(V) \in$ j–m–C(M_1).
2. ij–m–g–continuous if $\forall$ V $\in$ j-m–C(M_2), then $f^{-1}(V) \in$ ij–m–GC(M_1).
3. ij–m–Ω-continuous if $\forall$ V $\in$ j–m–C(M_2), then $f^{-1}(V) \in$ ij–m–ΩC(M_1).
4. ij–m–αg–continuous if $\forall$ V $\in$ j–m–C(M_2), then $f^{-1}(V) \in$ ij–m–αGC(M_1).

5. ij–m–gs–continuous if $\forall$ V $\in$ j–m–C (M_2), then f^{-1} (V) $\in$ ij–m–GSC(M_1).
6. ij–m–Ω-irresolute if $\forall$ V $\in$ ij–m–ΩC (M_2), then f^{-1} (V) $\in$ ij–m–ΩC (M_1).

3. Separation Axioms on M–Topological Spaces:

In this section we study the concept of quasi–coincident in multiset, and study some separation axioms using this concept.

Definition 3.1: A multipoint m/x is a multiset on X defined by:

$$(m/x)(y)=\begin{cases} m & \text{if} \quad x=y \\ 0 & \text{otherwise} \end{cases}$$

A multipoint m/x is belongs to a submultiset A on M if $m \le C_A(x)$.

Definition 3.2: Let $[X]^w$ be the multiset space A, B $\in [X]^w$ are called multi quasi–coincident denote by AqB if there exists x $\in$ X such that:

$C_A(x) + C_B(x) > w$.

A is not quasi–coincident B (denote by $A \bar{q} B$)

$\Leftrightarrow C_A(x) + C_B(x) \le w \qquad \forall x \in X.$
$\Leftrightarrow C_A(x) \le w - C_B(x) \qquad \forall x \in X.$
$\Leftrightarrow C_A(x) \le C_B{}^c(x) \qquad \forall x \in X.$
i.e. $A \bar{q} B \Leftrightarrow A \subseteq B^C$.

Proposition 3.3: Let two multisets A, B $\in [X]^w$ and $\{A_i : i \in I\} \subseteq [X]^w$. Then

1. $m/a \bar{q} A$ iff $m/a \in A^C$.
2. $A \bar{q} A^C$.
3. $\forall$ a, b $\in$ X, a $\neq$ b $\Rightarrow$ $m/a \bar{q} n/b$, for some m, n $\in$ N (where N is the set of non negative integers. If $m/a \bar{q} n/b$, then a $\neq$ b or a = b and $m + n \le w$.
4. AqB iff $\exists$ a multipoint m/a $\in$ A and m/a qB.
5. A $\subseteq$ B iff $\forall$ multipoint m/a qA $\Rightarrow$ m/a qB.
6. m/a $q(\bigcup_{i\in I} A_i)$ iff $\exists$ i $\in$ I such that m/a qA_i.
7. m/a $q(\bigcap_{i\in I} A_i)$ iff m/a qA_i, $\forall_i \in I$.

Proof (1) Let $m/a \bar{q} A \Rightarrow m + C_A(a) \le w \Rightarrow m \le w - C_A(a) \Rightarrow m \le C_A{}^C(a) \Rightarrow m/a \in A^C$.

Conversely Let $m/a \in A^C \Rightarrow m \le C_A{}^C(a) \Rightarrow m \le w - C_A(a) \Rightarrow m + C_A(a) \le w \Rightarrow m/a \bar{q} A$.

(2) If AqA^c, then $\exists$ x $\in$ X such that $C_A(x) + C_A{}^c(x) > w \Rightarrow C_A(x) + w - C_A(x) > w$ which is contradiction. Then $A \bar{q} A^C$.

(3) Let a $\neq$ b, then:

$$C_{m/a}(x) + C_{n/b}(x) = \begin{cases} m & \text{if} \quad x=a \\ n & \text{if} \quad x=b \\ o & \text{otherwise} \end{cases}$$

Then we have $C_{m/a}(x) + C_{n/b}(x) \le w$, $\forall$x $\in$ X. Then $m/a \bar{q} n/b$.

(4) Let AqB, then $\exists$ $x_0 \in X$ such that $C_A(x_0) + C_B(x_0) > w$. Let $C_A(x_0) = m$, then $m + C_B(x_0) > w$.

Then m/x_0 qB, where $m/x_0 \in A$.

Conversely Let m/a qB, for some $m/a \in A \Rightarrow m + C_B(a) > w \Rightarrow C_A(a) + C_B(a) > w$ for some a $\in$ X $\Rightarrow$ AqB.

(5) Let $m/a \; qA \Rightarrow m + C_A(a) > w \Rightarrow m + C_B(a) > w \Rightarrow m/a \; qB$.

Conversely Suppose m/a q A $\Rightarrow$ m/ a q B. Let $m/a \in B^C \Rightarrow m \le C_B{}^c(a) = w - C_B(a) \Rightarrow m + C_B(a) \le w \Rightarrow m/a \bar{q} B \Rightarrow m/a \; \bar{q} A \Rightarrow m/ a \in A^C \Rightarrow B^C \subseteq A^C$. Consequently A $\subseteq$ B.

(6) Suppose $m/a \bar{q} A_i$, $\forall_i \in I \Rightarrow m + C_{Ai}(a) \le w$, $\forall_i \in I \Rightarrow Sup\{m + C_{Ai}(a): i \in I\} \le w \Rightarrow m + Sup\{C_{Ai}(a): i \in I\} \le w \Rightarrow m + C_{UAi}(a) \le w \Rightarrow m/a \bar{q} (\bigcup_{i\in I} A_i)$ which contradiction. Then we have the result.

Conversely Suppose $\exists$ i $\in$ I such that $m/a \; qA_i \Rightarrow m + C_{Ai}(x) > w$, for some x $\in$ X $\Rightarrow m + C_{\bigcup_{i\in I} A_i}(x) > w$, for some x $\in$ X. Then m/a $q(\bigcup_{i\in I} A_i)$.

(7) Let $m/a \; q(\bigcap_{i\in I} A_i) \Rightarrow m + C_{\bigcap_{i\in I} A_i}(x) > w$, for some x $\in$ X, and for all i $\in$ I. Then for all i $\in$ I such that $m + C_{Ai}(x) > w$, for some x $\in$ X. Then $m/a \; qA_i$, for all i $\in$ I.

Conversely Suppose $m/a \; qA_i$, for all i $\in$ I $\Rightarrow m + C_{Ai}(x) > w$ for some x $\in$ X, and for all i $\in$ I $\Rightarrow \min\{m + C_{Ai}(x) : i \in I\} > w$, for some x $\in$ X $\Rightarrow m + \min\{C_{Ai}(x) : i \in I\} > w$, for some x $\in$ X $\Rightarrow m + C_{\bigcap_{i\in I} A_i}(x) > w$. Then m/a $q(\bigcap_{i\in I} A_i)$.

Proposition 3.4: Let (M, τ) be a M-topological space, m/a is a multipoint and $B \in [X]^w$, then

1. $\forall A \in \tau$, $AqB \Leftrightarrow AqCl(B)$.
2. m/a $qCl(B) \Leftrightarrow AqB$, $\forall A \in N(m/a, \tau)$ (where $N(m/a,\tau)$ is the class of all open neighborhood of m/a).

Proof (1) Let $A \in \tau$ and let $A \overline{q} B$ then $C_A(x) \leq C_B^c(x)$, $\forall x \in X \Rightarrow C_B(x) \leq C_A^c(x)$, $\forall x \in X$, A^C is closed mset, then $C_B(x) \leq C_{Cl(B)}(x)$, $\forall x \in X \Rightarrow C_{Cl(B)}(x) \leq C_A^c(x)$, $\forall x \in X$. Then $Cl(B) \overline{q} A$.

Conversely Let $A \overline{q} Cl(B) \Rightarrow C_A(x) \leq C_{Cl(B)}^c(x)$, $\forall x \in X \Rightarrow C_{Cl(B)}(x) \leq C_A^c(x)$, $\forall x \in X \Rightarrow C_B(x) \leq C_A^c(x)$, $\forall x \in X$. Then $A \overline{q} B$.

(2) Suppose there exists $A \in N(m/a, \tau)$ such that $A \overline{q} B \Rightarrow C_A(x) \leq C_B^c(x)$, $\forall x \in X$. Then $\exists U \in \tau$

Such that $m \leq C_U(a)$, $C_U(x) \leq C_A(x)$, $\forall x \in X$. From (1) we have $U \overline{q} B \Rightarrow U \overline{q} Cl(B) \Rightarrow m/a \overline{q} Cl(B)$.

Conversely Suppose that $\forall A \in N(m/a, \tau)$, AqB. if $m/a \overline{q} Cl(B) \Rightarrow m \leq C_{(Cl(B))^c}(a)$, $(Cl(B))^C \in \tau$, $(Cl(B))^C \in N(m/a, \tau)$, $B \overline{q} (Cl(B))^C$ from (3.3). This is contradiction. Then m/a qCl(B).

Definition 3.5: A M-topological space (M, τ) is called:

1. MT_0 iff $m/a \overline{q} n/b$ implies that $\exists A \in \tau$ such that $(m/a \in A$ and $n/b \overline{q} A)$ or $(n/b \in A$ and $m/a \overline{q} A)$.
2. MT_1 iff $m/a \overline{q} n/b$ implies that $\exists A, B \in \tau$ such that $(m/a \in A$ and $n/b \overline{q} A)$ and $(n/b \in B$ and $m/a \overline{q} B)$.
3. MT_2 (or M-Hausdorff) iff $m/a \overline{q} n/b$ implies that $\exists A, B \in \tau$ such that $m/a \in A$ and $n/b \in B$ and $A \overline{q} B$).
4. M $T_{2\frac{1}{2}}$ iff $m/a \overline{q} n/b$ implies that $\exists A, B \in \tau$ such that $m/a \in A$ and $n/b \in B$, $Cl(A) \overline{q} Cl(B)$.
5. MR_2 (or M-regular) iff $m/a \overline{q} F$, F is closed mset implies that $\exists A, B \in \tau$ such that $m/a \in A$, $F \subseteq B$, $A \overline{q} B$.
6. MR_3 (or M-normal) iff $\forall F_1, F_2$ are closed msets, with $F_1 \overline{q} F_2$, implies that $\exists A, B \in \tau$ such that $F_1 \subseteq A$ and $F_2 \subseteq B$, $A \overline{q} B$.

Proposition 3.6: Let (M, τ_1), (M, τ_2) are a M- topological space and $\tau_1 \subseteq \tau_2$. Then:

1. If (M, τ_1) is MT_i. Then (M, τ_2) is MT_i, $i = 0, 1, 2, 2\frac{1}{2}$.
2. If (M, τ_1) is MR_i. Then (M, τ_2) is MR_i, $i = 2, 3$.

Proof (1) i = 0 Let (M, τ_1) is MT_0 then $\exists A \in \tau_1$ such that $(m \leq C_A(a)$ and $n/b \overline{q} A)$ or $(n \leq C_A(b)$ and $m/a \overline{q} A)$.

Since $\tau_1 \subseteq \tau_2$, then $\exists A \in \tau_2$ such that $(m \leq C_A(a)$ and $n/b \overline{q} A)$ or $(n \leq C_A(b)$ and $m/a \overline{q} A)$.Then (M, τ_2) is MT_0. Other are similar.

(2) i = 2 Let $m/a \overline{q} F$, F is closed mset. Since (M, τ_1) is MR_2, then $\exists A, B \in \tau_1$ such that $(m \leq C_A(a)$, $C_F(x) \leq C_B(x)$, $\forall x \in X$, $A \overline{q} B$. Since $\tau_1 \subseteq \tau_2$. Then $\exists A, B \in \tau_2$ such that $m \leq C_A(a)$, $C_F(x) \leq C_B(x)$, $\forall x \in X$, $A \overline{q} B$. then (M, τ_2) is MR_2.

Other are similar.

Theorem 3.7: A M-topological space (M, τ) is MT_0 if and only if every multipoint m/a of M is closed mset.

Proof Let (M, τ) is MT_0 and let m/a be a multipoint. Let $m \leq C_{(n/b)^c}(a)$, then $n/b \overline{q} m/a$. Then there exists $A \in \tau$ such that $m \leq C_A(a)$ and $n/b \overline{q} A$. then $C_A(x) \leq C_{(n/b)^c}(x)$, $\forall x \in X$. then $(n/b)^c$ is open mset. Consequently n/b is closed mset.

Conversely Suppose any multipoint is closed mset, and let n/b, k/c with $n/b \overline{q} k/c$ implies that $k \leq C_{(n/b)^c}(c)$. Hence $(n/b)^c$ is an open mset such that $k \leq C_{(n/b)^c}(c)$ and $n/b \overline{q} (n/b)^c$. then (M, τ) is MT_0.

Theorem 3.8: A M- topological space (M, τ) is MR_3 if and only if for every closed mset F and open mset U containing F. there exists an open mset G such that $F \subseteq G \subseteq Cl(G) \subseteq U$.

Proof Let (M, τ) is MR_3 and let F is closed mset, U is open mset, $C_F(x) \leq C_U(x)$, $\forall x \in X$, Then $F \overline{q} U^C$, U^C is an closed mset. Then $\exists A, B \in \tau$ such that $C_F(x) \leq C_A(x)$, $C_U^c(x) \leq C_B(x)$, $\forall x \in X$ $A \overline{q} B$ implies that $C_A(x) \leq C_B^c(x)$, $\forall x \in X$. Since $C_U^C(x) \leq C_B(x)$, $\forall x \in X \Rightarrow C_B^C(x) \leq C_U(x)$, $\forall x \in X$. Then we have $C_F(x) \leq C_A(x) \leq C_{Cl(A)}(x) \leq C_B^C(x) \leq C_U(x)$, $\forall x \in X$.

Conversely Let F_1, F_2 are closed mset and $F_1 \overline{q} F_2$, then $C_{F_1}(x) \leq C_{F_2^C}(x)$, $\forall x \in X$, F_2^C is open mset, then $\exists$ an open mset G such that $C_{F_1}(x) \leq C_G(x) \leq C_{CL(G)}(x) \leq C_{F_2^C}(x)$, since $C_{CL(G)}(x) \leq C_{F_2^C}(x)$, $\forall x \in X$, $\Rightarrow C_{F_2}(x) \leq C_{(Cl(G))^c}(x)$, $\forall x \in X$ and $C_G(x) \leq C_{Cl(G)}(x)$, $\forall x \in X \Rightarrow G \overline{q} (Cl(G))^C$. Where $(CL(G))^C$ is an open mset. Thus $C_{F_1}(x) \leq C_G(x)$, $\forall x \in X$ and $C_{F_2}(x) \leq C_{(Cl(G))^c}(x)$, $\forall x \in X$, $G \overline{q} (Cl(G))^C$. Then (M, τ) is MR_3.

Definition 3.9: Let $M \in [X]^w$ and $\tau^* \subseteq P^*(M)$. Then τ^* is called multi supra topology on M if τ^* satisfies the

following properties:
1. $\varnothing$ and M are in τ^*.
2. $A_\alpha \in \tau^*, \forall\alpha \Rightarrow \bigcup_\alpha A_\alpha \in \tau^*$.
The pair (M, τ^*) is said to be multi supra-topological space (for short msts).
A is said to be multi supra-open iff $A \in \tau^*$. B is said to be multi supra-closed iff $B^C \in \tau^*$.

Definition 3.10: Let (M, τ^*) be msts, $A \subseteq M$. the mset
$Cl_{\tau^*}(A) = \bigcap \{ B \subseteq M : B^C \in \tau^*, B \supseteq A \}$ is called multi supra-closure submset of A.

Definition 3.11: A msts (M, τ^*) is said to be:
1. MST_0 iff m/a $\bar{q}$ n/b implies that $\exists$ A $\in \tau^*$ such that (m/a $\in$ A and n/b $\bar{q}$ A) or (n/ b $\in$ A and m/a $\bar{q}$ A).
2. MST_1 iff m/a $\bar{q}$ n/b implies that $\exists$ A, B $\in \tau^*$ such that (m/a $\in$ A and n/b $\bar{q}$ A) or (n/b $\in$ B and m/a $\bar{q}$ B).
3. MST_2 iff m/a $\bar{q}$ n/b implies that $\exists$ A $\in$ N (m/a, τ^*) and B $\in$ N (n/b, τ^*) such that A $\bar{q}$ B.
4. $MST_{2\frac{1}{2}}$ iff m/a $\bar{q}$ n/b implies that $\exists$ A $\in$ N (m/a, τ^*) and B $\in$ N (n/b, τ^*) such that (Cl(A)) $\bar{q}$ (Cl(B)).
5. MSR_0 iff m/a $\bar{q}$ (Cl (n/b)) $\Rightarrow$ n/b $\bar{q}$ (Cl (m/a)).
6. MSR_1 iff m/a $\bar{q}$ (Cl(n/b)) implies that $\exists$ A $\in$ N (m/a, τ^*) and B $\in$ N (n/b, τ^*) such that A $\bar{q}$ B.
7. MSR_2 (multi supra regular) iff m/a $\bar{q}$ F, F is multi supra closed mset, then $\exists$ A $\in$ N (m/a, τ^*) and B $\in$ N (n/b, τ^*) such that A $\bar{q}$ B.
8. MSR_3 (multi supra normal) iff $F_1 \bar{q} F_2$, F_1, F_2 are multi supra closed mset, then $\exists$ A $\in$ N (F_1, τ^*) and B $\in$ N (F_2, τ^*) such that A $\bar{q}$ B.

Theorem 3.12: Let (M, τ^*) be a multi supra topological space. The following statements are equivalent:
1. (M, τ^*) is MSR_0.
2. Cl (m/a) $\subseteq$ A, $\forall$A $\in$ N (m/a, τ^*).
3. Cl (m/a) $\subseteq$ ker (m/a) = $\bigcap \{A \in [X]^w : A \in N (m/a, \tau^*)\}$.
4. m/a $\bar{q}$ F, F is multi supra closed mset, $\exists$ B $\in$ N (F, τ^*) such that m/a $\bar{q}$ B.
5. m/a $\bar{q}$ F, F is multi supra closed mset, implies that Cl (m/a) $\bar{q}$ F.
6. m/a $\bar{q}$ (Cl (n/b)) $\Rightarrow$ (Cl(m/a)) $\bar{q}$ (Cl (n/b)).

Proof (1) $\Rightarrow$ (2) Let (M, τ^*) be MSR_0, A $\in$ N (m/a, τ^*) and let n/ b $\bar{q}$ A. From proposition (3.4) we have m/a $\bar{q}$ (Cl (n/b)). Then n/b $\bar{q}$ (Cl (m/a)). Consequently Cl (m/ a) $\subseteq$ A.
(2) $\Leftrightarrow$ (3) is obvious.
(3) $\Rightarrow$ (1) Let n/b $\bar{q}$ (Cl (m/a)) , then n $\leq C_{(Cl(m/a))^C}$ (b) . From (2) we have $C_{Cl(n/b)}(x) \leq C_{(Cl(m/a))^c}(x)$, $\forall x \in X \Rightarrow C_{Cl(n/b)}(x) \leq C_{Int(m/a)^c}(x)$, $\forall x \in X \Rightarrow C_{Cl(n/b)}(x) \leq C_{(m/a)^c}(x)$, $\forall x \in X \Rightarrow$ m/a $\bar{q}$ (Cl (n/b)). Then (M, τ^*) is MSR_0.

(1) $\Rightarrow$ (4) Let m/a $\bar{q}$ F, F is a supra-closed mset $\Rightarrow C_{Cl(n/b)}(x) \leq C_F(x)$, $\forall x \in X \Rightarrow$ m/a $\bar{q}$ (Cl (n/b)) $\Rightarrow$ m $\leq C_{(Cl(n/b))^c}$(a), n/b $\bar{q}$ (Cl (m/a)). From proposition (3.4) $\exists$ A $\in$ N (n/b, τ^*), A $\bar{q}$ m/a.

(4) $\Rightarrow$ (5) Let m/a $\bar{q}$ F, F is a multi supra closed mset. From (4) $\exists$ A $\in$ N (F, τ^*) such that m/a $\bar{q}$ A. Then m $\leq C_{A^c}$ (a), $C_{Cl(m/a)}(x) \leq C_{A^c}(x)$, $\forall x \in X$. Therefore (Cl (m/a)) $\bar{q}$ A, and Cl (m/a) $\bar{q}$ F.
(5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (1) are obvious.

4. Separation Axioms on M-Bitopological Spaces:

In this section we introduce some kind of separation axioms in multi-bitopological spaces. Also we study separation axioms in multi supra topological space (M, τ_s) which is generated by the multi bitopological space (M, τ_1, τ_2). Where $\tau_s = \{G = G_1 \cup G_2 : G_1 \in \tau_1, G_2 \in \tau_2)$.

Definition 4.1: A mbts (M, τ_1, τ_2) is called:
1. MBT_0 iff $\forall$ m/a, n/b with m/a $\bar{q}$ n/b, $\exists$ A $\in \tau_1 \cup \tau_2$ such that (m/a $\in$ A, n/b $\bar{q}$ A) or (n/b $\in$ A, m/a $\bar{q}$ A).
2. MBT_1 iff $\forall$ m/a, n/b with m/a $\bar{q}$ n/b, $\exists$ A, B $\in \tau_1 \cup \tau_2$ such that (m/a $\in$ A, n/b $\bar{q}$ A) and (n/b $\in$ B, m/a $\bar{q}$ B).
3. MBT_2 iff $\forall$ m/a, n/b with m/a $\bar{q}$ n/b, $\exists$ A $\in \tau_i$, B $\in \tau_j$ such that (m/a $\in$ A, n/b $\in$ B, and A $\bar{q}$ B).
4. MBR_2 (or multi pairwise regular) iff $\forall$ m/a, closed mset F in τ_i (briefly i-m-closed) with m/a $\bar{q}$ F, $\exists$ A $\in \tau_i$, B $\in \tau_j$ such that (m/a $\in$ A, F $\subseteq$ B, and A $\bar{q}$ B).
5. MBR_3 (or multi pairwise normal) iff $\forall$ closed mset F_1 in τ_i, and closed F_2 in τ_j (briefly j-m-closed) with $F_1 \bar{q} F_2$, $\exists$ A $\in \tau_j$, B $\in \tau_i$ such that $F_1 \subseteq$ A and $F_2 \subseteq$ B, and A $\bar{q}$ B, i $\neq$ j.

6. MBT_3 if it is MBR_2 and MBT_1.
7. MBT_4 if it is MBR_3 and MBT_1.

Theorem 4.2: A mbts (M, τ_1, τ_2) is MBT_0 if and only if every multipoint (m/a) of M is closed mset in τ_s (briefly τ_s–m–closed).

Proof Let (M, τ_1, τ_2) is MBT_0 and let $n/b \in (m/a)^C$, then $m/a \,\bar{q}\, n/b$, then there exists $A \in \tau_1 \cup \tau_2$ such that $m \leq C_A$ (a) and $n/b \,\bar{q}\, A$. then $C_A(x) \leq C_{(n/b)^c}(x)$, $\forall x \in X$, then $(n/b)^C$ is τ_s–m–open mset. Consequenlty n/b is τ_s–m–closed mset. Or $n \leq C_A$ (b) and $m/a \,\bar{q}\, A$. then $C_A(x) \leq C_{(m/a)^c}(x)$, $\forall x \in X$. then $(m/a)^c$ is τ_s–m–open mset. Then m/a is τ_s–m–closed mset.

Conversely Let n/b, k/c with $n/b \,\bar{q}\, k/c$ implies that $k \leq C_{(n/b)^c}$ (c). Hence $(n/b)^c$ is an τ_s–m–open mset such that $k \leq C_{(n/b)^c}$ (c) and $n/b \,\bar{q}\, (n/b)^c$. Similarly $n \leq C_{(k/c)^c}$ (b). Hence $(k/c)^c$ is an τ_s–m–open mset such that $n \leq C_{(k/c)^c}$ (b) and $k/c \,\bar{q}\, (k/c)^c$. then (M, τ_1, τ_2) is MBT_0.

Theorem 4.3: Let (M, τ_1, τ_2) be mbts, if satisfies the following condition for every τ_s–m–closed mset F and τ_s–m–open mset U containing F, there exists an τ_s–m–open mset G such that $F \subseteq G \subseteq Cl(G) \subseteq U$. Then (M, τ_1, τ_2) is MBR_3.

Proof Let F_1 is τ_s-m-closed mset, F_2 is τ_s–m–closed mset and $F_1 \,\bar{q}\, F_2$, then $C_{F_1}(x) \leq C_{F_2^C}(x)$, $\forall x \in X$, F_2^C is τ_s–m–open mset, then $\exists$ an τ_s–m–open mset G such that $C_{F_1}(x) \leq C_G(x) \leq C_{Cl(G)}(x) \leq C_{F_2^C}(x)$, $\forall x \in X$. But $C_{Cl(G)}(x) \leq C_{F_2^C}(x)$, $\forall x \in X \Rightarrow C_{F_2}(x) \leq C_{Cl(G))^c}(x)$, $\forall x \in X$ and $C_G(x) \leq C_{Cl(G)}(x)$, $\forall x \in X \Rightarrow G \,\bar{q}\, (Cl(G))^C$, where $(CL(G))^C$ is an τ_s–m–open mset. Thus $C_{F_1}(x) \leq C_G(x)$, $\forall x \in X$ and $C_{F_2}(x) \leq C_{(Cl(G))^c}(x)$, $\forall x \in X$, $G \,\bar{q}\, (Cl(G))^C$. Then (M, τ_1, τ_2) is MBR_3.

Definition 4.4: A mbts (M, τ_1, τ_2) is called:

1. $MB\,T_0^*$ iff $\forall$ m/a, n/b are multi-point with $a \neq b$, $\exists A \in \tau_1 \cup \tau_2$ such that ($m/a \in A$ and $A \subseteq (n/b)^C$) or ($n/b \in A$ and $A \subseteq (m/a)^C$).
2. $MB\,T_1^*$ iff $\forall$ m/a, n/b are multi-point with $a \neq b$, $\exists A, B \in \tau_1 \cup \tau_2$ such that ($m/a \in A$, $A \subseteq (n/b)^C$) and ($n/b \in B$, $B \subseteq (m/a)^C$).
3. $MB\,T_2^*$ iff $\forall$ m/a, n/b are multi-point with $a \neq b$, $\exists A, B \in \tau_1 \cup \tau_2$ such that ($m/a \in A$ and $n/b \in B$, $A \subseteq B^C$).

Proposition 4.5: (M, τ_1, τ_2) be a mbts. Consider the following statements:

1. (M, τ_1, τ_2) is MBT_i, i = 0, 1, 2.
2. (M, τ_1, τ_2) is $MB\,T_i^*$.

Then (1) $\Rightarrow$ (2)

Proof i = 0 Let (M, τ_1, τ_2) is MBT_0 and let m/a, n/b are multi-points with $a \neq b$, then $m/a \,\bar{q}\, n/b$. Then $\exists A \in \tau_1 \cup \tau_2$ such that ($m \leq C_A$ (a), $n/b \,\bar{q}\, A$) or ($n \leq C_A$ (b), $m/a \,\bar{q}\, A$). From definition (3.2) we have that $\exists A \in \tau_1 \cup \tau_2$ such that ($m \leq C_A$ (a), $C_A(x) \leq C_{(n/b)^c}(x)$, $\forall x \in X$) or ($n \leq C_A$ (b), $C_A(x) \leq C_{(m/a)^c}(x)$, $\forall x \in X$). Then (M, τ_1, τ_2) is $MB\,T_0^*$.

i = 1 Let (M, τ_1, τ_2) is MBT_1 and let m/a, n/b are multi-points with $a \neq b$, then $m/a \,\bar{q}\, n/b$. Then $\exists A, B \in \tau_1 \cup \tau_2$ such that ($m \leq C_A$ (a), $n/b \,\bar{q}\, A$) and ($n \leq C_B$ (b), $m/a \,\bar{q}\, B$). From definition (3.2) we have that $\exists A, B \in \tau_1 \cup \tau_2$ such that ($m \leq C_A$ (a), $C_A(x) \leq C_{(n/b)^c}(x)$, $\forall x \in X$) and ($n \leq C_B$ (b) and $C_B(x) \leq C_{(m/a)^c}(x)$, $\forall x \in X$). Then (M, τ_1, τ_2) is $MB\,T_1^*$.

i= 2 Let (M, τ_1, τ_2) is MBT_2 and let m/a, n/b are multi-points with $a \neq b$, then $m/a \,\bar{q}\, n/b$. Then $\exists A \in \tau_1$, $B \in \tau_2$ such that ($m \leq C_A$ (a), $n \leq C_B$ (b), $A \,\bar{q}\, B$). From definition (3.2) we have that $\exists A \in \tau_1$, $B \in \tau_2$ such that ($m \leq C_A$ (a), $n \leq C_B$ (b) and $C_A(x) \leq C_B^c(x)$, $\forall x \in X$). Then (M, τ_1, τ_2) is $MB\,T_2^*$.

Example 4.6 Let $X = \{a, b\}$, $M = \{5/a, 3/b\}$, $\tau_1 = \{M, \varnothing, \{2/a\}\}$, $\tau_2 = \{M, \varnothing, \{2/a, 3/b\}\}$. Note that $2/a \in \{2/a\} \subseteq (\{3/b\})^C$, $\{3/b\} \in \{3/a, 3/b\} \subseteq (\{2/a\})^C$, then (M, τ_1, τ_2) is $MB\,T_1^*$ but it is not MBT_1.

Proposition 4.7 Let each (M, τ_1, τ_2), (M, τ_1^*, τ_2^*) be a mbts, and $\tau_1 \subseteq \tau_1^*$, $\tau_2 \subseteq \tau_2^*$. Then $\tau_s \subseteq \tau_s^*$.

Proof Let $A \in \tau_s \Rightarrow A = A_1 \cup A_2$, $A_i \in \tau_I$, i = 1, 2. Since $\tau_1 \subseteq \tau_1^*$, $\tau_2 \subseteq \tau_2^* \Rightarrow A_i \in \tau_1^*$, i = 1, 2 $\Rightarrow A \in \tau_s^*$. Then $\tau_s \subseteq \tau_s^*$.

Definition 4.8: A mbts (M, τ_1, τ_2) is called:

1. $MB^*\,T_0$ iff $\forall$ m/ a, n/b with $m/a \,\bar{q}\, n/b$, $\exists A \in \tau_s$ such that ($m/a \in A$ and $n/b \,\bar{q}\, A$) or ($n/b \in A$ and $m/a \,\bar{q}\, A$).

2. $MB^*\ T_1$ iff $\forall$ m/a, n/b with m/a $\overline{q}$ n/b, $\exists$ A, B $\in \tau_s$ such that (m/a $\in$ A and n/b $\overline{q}$ A) and (n/b $\in$ B, m/a $\overline{q}$ B).
3. $MB^*\ T_2$ iff $\forall$ m/a, n/b with m/a $\overline{q}$ n/b, $\exists$ A, B $\in \tau_s$ such that (m/a $\in$ A and n/b $\in$ B, A $\overline{q}$ B).
4. M $B^*\ T_{2\frac{1}{2}}$ iff $\forall$ m/a, n/b with m/ a $\overline{q}$ n/b, $\exists$ A, B $\in \tau_s$ such that m/a $\in$ A and n/b $\in$ B and C_{12} (A) $\overline{q}$ C_{12} (B).(

Where C_{12} (A) = (Cl_1 (A)) $\cap$ (Cl_2 (A)), Cl_i(A) is the closure of A in τ_i, i=1,2.

5. $MB^*\ R_0$ iff m/a $\overline{q}$ C_{12} (n/ b) implies that n/b $\overline{q}$ C_{12} (m/ a).
6. $MB^*\ R_1$ iff m/a $\overline{q}$ C_{12} (n/ b) implies that $\exists$ A, B $\in \tau_s$ such that A $\overline{q}$ B.
7. $MB^*\ R_2$ (or MB^*-regular) iff m/a $\overline{q}$ F, F is τ_s-m–closed mset implies that $\exists$ A, B $\in \tau_s$ such that m/a $\in$ A, F $\subseteq$ B, A $\overline{q}$ B.
8. M $B^*\ R_3$ (or MB^*-normal) iff F_1 $\overline{q}$ F_2, F_1, F_2 are τ_s – m – closed msets implies that $\exists$ A, B $\in \tau_s$ such that $F_1 \subseteq$ A, $F_2 \subseteq$ B, A $\overline{q}$ B.
9. $MB^*\ T_{k+1}$ iff it is $MB^*\ T_1$ and $MB^*\ T_k$.

Proposition 4.9: Let (M, τ_1, τ_2) and (M, τ_1^*, τ_2^*) are mbts and $\tau_1 \subseteq \tau_1^*$, $\tau_2 \subseteq \tau_2^*$. Then

1. If (M, τ_1, τ_2) is M $B^*\ T_i$, then (M, τ_1^*, τ_2^*) is $MB^*\ T_i$, i = 0, 1, 2, $2_{\frac{1}{2}}$.
2. If (M, τ_1, τ_2) is M $B^*\ R_i$, then (M, τ_1^*, τ_2^*) is $MB^*\ R_i$, i = 0, 1, 2, 3.

Proof (1) i = 0 Let m/a, n/b with m/a $\overline{q}$ n/b. Since (M, τ_1, τ_2) is $MB^*\ T_0$. Then $\exists$ A $\in \tau_s$ such that (m $\leq$ C_A (a) and n/b $\overline{q}$ A) or (n $\leq$ C_A (b), m/a $\overline{q}$ A). Since $\tau_s \subseteq \tau_s^*$ Then $\exists$ A $\in \tau_s^*$ such that (m $\leq$ C_A (a) and n/b $\overline{q}$ A) or (n $\leq$ C_A (b), m/a $\overline{q}$ A). Then (M, τ_1^*, τ_2^*) is $MB^*\ T_0$.

(2) i = 0 Let m/a $\overline{q}$ Cl_{12}^* (n/b) $\Rightarrow$ m/a $\overline{q}$ C_{12}(n/b) $\Rightarrow$ n/b $\overline{q}$ C_{12}(m/ a) $\Rightarrow$ n/b $\overline{q}$ Cl_1 (m/a) or n/b $\overline{q}$ Cl_2 (m/ a) $\Rightarrow$ n/b $\overline{q}$ Cl_1^* (m/a) or n/b $\overline{q}$ Cl_2^* (m/a) $\Rightarrow$ n/b $\overline{q}$ ((Cl_1^* (m/a)) $\cap$ (Cl_2^* (m/ a))) $\Rightarrow$ n/b $\overline{q}$ Cl_{12}^* (m/a). Then (M, τ_1^*, τ_2^*) is $MB^*\ T_0$.

The other are similar.

Proposition 4.10: For a mbts (M, τ_1, τ_2) we have

1. $MBT_i \Rightarrow MB^*\ T_i$, i = 0, 1, 2.
2. $MBR_i \Rightarrow MB^*\ R_i$, i = 2, 3.

Proof (1) i = 0 Let (M, τ_1, τ_2) is MBT_0 and let m/a, n/b with m/a $\overline{q}$ n/b. Then $\exists$ A $\in \tau_1 \cup \tau_2$ such that (m $\leq$ C_A (a) and n/b $\overline{q}$ A) or (n $\leq$ C_A (b), m/a $\overline{q}$ A). Then $\exists$ A $\in \tau_s$ such that (m $\leq$ C_A (a) and n/b $\overline{q}$ A) or (n $\leq$ C_A (b), m/a $\overline{q}$ A). Then (M, τ_1, τ_2) is $MB^*\ T_0$.

Other are similar.

(2) i = 2 Let (M, τ_1, τ_2) is MBR_2 and let m/a $\overline{q}$ F, F is τ_s–m–closed $\Rightarrow$ F = $F_1 \cap F_2$, F_i is τ_i – m – closed. Then m/a $\overline{q}$ F_1 or m/a $\overline{q}$ F_2, then $\exists$ A $\in \tau_1$, B $\in \tau_2$, such that m $\leq$ C_A (a), C_{F_i} (x) $\leq$ C_B (x), $\forall$x $\in$ X, A $\overline{q}$ B or m $\leq$ C_B (a), C_{F_2} (x) $\leq$ C_A (x), $\forall$x $\in$ X, A $\overline{q}$ B implies that $\exists$ A, B $\in \tau_s$ such tha m $\leq$ C_A (a), C_F (x) $\leq$ C_B (x), $\forall$x $\in$ X, A $\overline{q}$ B. Then (M, τ_1, τ_2) is $MB^*\ R_2$.

i = 3 Let (M, τ_1, τ_2) is MBR_3 and let F_1 $\overline{q}$ F_2, F_1, F_2 are τ_s–m–closed mset implies that $\exists$ A $\in \tau_1$, B $\in \tau_2$, such that C_{F_1} (x) $\leq$ C_A (x), C_{F_2} (x) $\leq$ C_B (x), $\forall$x $\in$ X, A $\overline{q}$ B. Then $\exists$ A, B $\in \tau_s$ such that C_{F_1} (x) $\leq$ C_A (x), C_{F_2} (x) $\leq$ C_B (x), $\forall$x $\in$ X. A $\overline{q}$ B. Then (M, τ_1, τ_2) is $MB^*\ R_3$.

Lemma 4.11: Let (M, τ_1, τ_2) be a mbts. Then

1. If (M, τ_1) or (M, τ_2) is MT_i, then (M, τ_1, τ_2) is $MB^*\ T_i$, i = 0, 1, 2, $2\frac{1}{2}$.
2. If each (M, τ_1) and (M, τ_2) is MR_i, then (M, τ_1, τ_2) is $MB^*\ R_i$, i = 0, 1, 2.

Proof (1) i = 0 Let (M, τ_1) or (M, τ_2) is M T_0 and let m/a $\overline{q}$ n/b, then $\exists$ A $\in \tau_1$ or A $\in \tau_2$. Hence A $\in \tau_1 \cup \tau_2 \subseteq \tau_s$ such that (m $\leq$ C_A (a) and n/b $\overline{q}$ A) or (n $\leq$ C_A (b) and m/a $\overline{q}$ A). Then (M, τ_1, τ_2) is $MB^*\ T_0$.

i = 1 Let (M, τ_1) or (M, τ_2) is MT_1 and let m/a $\overline{q}$ n/b, then $\exists$ A, B $\in \tau_1$ or A, B $\in \tau_2$. Hence A, B $\in \tau_1 \cup \tau_2 \subseteq \tau_s$ such that (m $\leq$ C_A (a) and n/b $\overline{q}$ A) or (n $\leq$ C_B (b) and m/a $\overline{q}$ B). Then (M, τ_1, τ_2) is $MB^*\ T_1$.

i= 2 Let (M, τ_1) or (M, τ_2) is MT_2 and let m/a $\overline{q}$ n/b, then $\exists$ A, B $\in \tau_1$ or A, B $\in \tau_2$. Hence A, B $\in \tau_1 \cup \tau_2 \subseteq \tau_s$ such that (m $\leq$ C_A (a), n $\leq$ C_B (b) and A $\overline{q}$ B). Then (M, τ_1, τ_2) is $MB^*\ T_2$.

i= $2\frac{1}{2}$ Let (M, τ_1) or (M, τ_2) is M $T_{2\frac{1}{2}}$ and let m/a $\overline{q}$ n/b, then $\exists$ A, B $\in \tau_1$ or A, B $\in \tau_2$. Hence A, B $\in \tau_1 \cup \tau_2 \subseteq \tau_s$

such that ($m \leq C_A$ (a), $n \leq C_B$ (b) and Cl (A) $\overline{q}$ Cl (B). Then (M, τ_1, τ_2) is MB^* $T_{2\frac{1}{2}}$.

(2) i = 2 Let (M, τ_1) or (M, τ_2) is M R_2 and let m/a $\overline{q}$ F, F is τ_s–m–closed mset, then $C_F(x) = C_{F_1 \cap F_2}(x)$, $\forall x \in X$, F_1 is i-m–closed mset and F_2 is 2-m–closed mset. So if m/a $\overline{q}$ F_1, then $\exists$ A, B $\in \tau_1 \subseteq \tau_s$ such that $m \leq C_A$ (a) and $C_{F_1}(x) \leq C_B(x)$, $\forall$ x $\in$ X and A $\overline{q}$ B. Hence (M, τ_1, τ_2) is MB^* R_2. Or if m/a $\overline{q}$ F_2, then $\exists$ A, B $\in \tau_2 \subseteq \tau_s$ such that $m \leq C_A$ (a) and $C_{F_2}(x) \leq C_B(x)$, $\forall x \in X$ and A $\overline{q}$ B. Hence (M, τ_1, τ_2) is MB^* R_2.

Theorem 4.12: Let (M, τ_1, τ_2) be a mbts. The following statements are equivalent.

1. (M, τ_1, τ_2) is MB^* R_0.
2. C_{12} (m/a) $\subseteq$ A, $\forall$ A $\in$ N (m/a, τ_s).
3. C_{12} (m/a) $\subseteq ker_s = \bigcap \{A : A \in N(m/a, \tau_s)\}$.
4. m/a $\overline{q}$ F, F is τ_s–m–closed mset, there exists A $\in$ N (F, τ_s) such that m/a $\overline{q}$ A.
5. m/a $\overline{q}$ F, F is τ_s–m–closed mset implies that C_{12} (m/a) $\overline{q}$ F.
6. m/a $\overline{q}$ C_{12} (n/b) implies that C_{12} (m/a) $\overline{q}$ C_{12} (n/b).

Proof (1) $\Rightarrow$ (2) Let n/b $\overline{q}$ C_{12} (m/a), then from (1) m/a $\overline{q}$ C_{12} (n/b) and by proposition (3.4) we have n/b $\overline{q}$ A, $\forall$ A $\in$ N (m/a, τ_s). by proposition (3.3) we have C_{12} (m/a) $\subseteq$ A, $\forall$ A $\in$ N (m/a, τ_s).

(2) $\Rightarrow$ (3) Is obvious.

(3) $\Rightarrow$ (1) Let n/b $\overline{q}$ C_{12} (m/a) $\Rightarrow$ $n \leq C_{(C_{12}(n/b))^C}$ (b), from (2) we have $C_{C_{12}(n/b)}(x) \leq C_{(C_{12}(m/a))^C}(x) = C_{Int12((m/a)^C}(x)$, $\forall$ x $\in$ X. Then we have $C_{C_{12}(n/b)}(x) \leq C_{(m/a)^c}(x)$, $\forall$ x $\in$ X. Then m/a $\overline{q}$ C_{12} (n/ b). Therefore (M, τ_1, τ_2) is MB^* R_0.

(1) $\Rightarrow$ (4) Let m/a $\overline{q}$ F, F is τ_s–m–closed mset, then $C_{C_{12}(n/b)} \leq C_F(x)$, $\forall$ n/b $\in$ F, x $\in$ X. Then we have m/a $\overline{q}$ C_{12} (n/b), $\forall$ n/b $\in$ F. From (1) hence n/b $\overline{q}$ C_{12} (m/a). Then $\exists$ A $\in$ N (n/b, τ_s), $\forall$ n/b $\in$ F such that A $\overline{q}$ m/a.

(4) $\Rightarrow$ (5) Let m/a $\overline{q}$ F, F is τ_s–m–closed mset, From (4) $\exists$ A $\in$ N (F, τ_s) such that m/a $\overline{q}$ A $\Rightarrow$ $m \leq C_{A^c}(a) \Rightarrow C_{C_{12}(m/a)}(x) \leq C_{A^c}(x)$, $\forall$ x $\in$ X $\Rightarrow$ C_{12} (m/a) $\overline{q}$ A $\Rightarrow$ C_{12} (m/a) $\overline{q}$ F.

(5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (1) are obvious

Theorem 4.13: Let (M, τ_1, τ_2) be a mbts then

1. (M, τ_1, τ_2) is MB^* R_2 iff for all m/a is a multipoint and for all A $\in \tau_s$, m/a $\in$ A there exists B $\in \tau_s$ such that m/a $\in$ B, C_{12} (B) $\subseteq$ A.
2. (M, τ_1, τ_2) is MB^* R_3 iff for all τ_s–m–closed mset F and for all B $\in \tau_s$, F $\subseteq$ B, there exists $B_1 \in \tau_s$ such that F $\subseteq B_1 \leq C_{12}(B_1) \subseteq$ B.

Proof (1) Let (M, τ_1, τ_2) is MB^* R_2 and let A $\in \tau_s$, $m \leq C_A$ (a), m/a $\overline{q}$ $A^C \Rightarrow \exists$ G, H $\in \tau_s$ such that $m \leq C_G$ (a), $C_{A^c}(x) \leq C_H(x)$, $\forall$ x $\in$ X, G $\overline{q}$ H $\Rightarrow$ $C_G(x) \leq C_{H^c}(x) \leq C_A(x)$, $\forall$ x $\in$ X $\Rightarrow$ $C_{Cl(G)}(x) \leq C_{H^c}(x) \leq C_A(x)$, $\forall$ x $\in$ X.

Conversely Let m/a be a multipoint, F is τ_1–m–closed, m/a $\overline{q}$ F $\Rightarrow$ $m \leq C_{(F)^c}(a) \Rightarrow \exists$ B $\in \tau_s$, $m \leq C_B$ (a) such that $C_{C_{12}(B)}(x) \leq C_{(F)^c}(x)$, $\forall$ x $\in$ X. Then $C_F(x) \leq C_{(C_{12}(B))^C}(x)$, $\forall$ x $\in$ X $\Rightarrow$ B $\overline{q}$ $(C_{12}(B))^c$. Then (M, τ_1, τ_2) is MB^* R_2.

(2) Let (M, τ_1, τ_2) is MB^* R_3 and let F is τ_s–m–closed, B $\in \tau_s$ such that $C_F(x) \leq C_B(x)$, $\forall$ x $\in$ X $\Rightarrow$ F $\overline{q}$ B^C, B^C is τ_s–m–closed $\Rightarrow \exists$ G, H $\in \tau_s$ such that $C_F(x) \leq C_G(x)$, $C_B(x) \leq C_H(x)$, $\forall$ x $\in$ X, G $\overline{q}$ H $\Rightarrow$ $C_F(x) \leq C_G(x) \leq C_{H^c}(x) \leq C_B(x)$, $\forall$ x $\in$ X $\Rightarrow$ $C_F(x) \leq C_G(x) \leq C_{C_{12}(G)}(x) \leq C_{H^c}(x) \leq C_B(x)$, $\forall$ x $\in$ X.

Conversely Let F_1, F_2 are τ_s–m–closed, F_1 $\overline{q}$ $F_2 \Rightarrow$ $C_{F_1}(x) \leq C_{(F_2)^c}(x)$, $\forall$ x $\in$ X $\Rightarrow \exists$ G $\in \tau_s$ such that $C_{F_1}(x) \leq C_G(x) \leq C_{C_{12}(G)}(x) \leq C_G(x) \leq C_{(F_2)^c}(x)$, $\forall$ x $\in$ X $\Rightarrow$ $C_{F_1}(x) \leq C_G(x)$, $C_{F_2}(x) \leq C_{(C_{12}(G))^C}(x)$, $\forall$ x $\in$ X, G $\overline{q}$ $(C_{12}(G))^c$. Then (M, τ_1, τ_2) is MB^* R_3.

Proposition 4.14: (M, τ_1, τ_2) be a mbts. Consider the following statements:

1. (M, τ_1, τ_2) is MBT_2.
2. (M, τ_1, τ_2) is $MB\,T_2^*$.
3. (M, τ_1, τ_2) is MB^*T_2.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof (1) $\Rightarrow$ (2) proved in (4.5).

(2) $\Rightarrow$ (3) Let (M, τ_1, τ_2) is MB T_2^* and let m/a, n/b with m/a $\bar{q}$ n/b, then there exists A $\in \tau_1$, B $\in \tau_2$ such that (m $\leq C_A$ (a) and n $\leq C_B$ (b), C_A (x) $\leq C_{B^c}$ (x), $\forall$ x $\in$ X). Hence A, B $\in \tau_s$ such that (m $\leq C_A$ (a) and n $\leq C_B$ (b), A $\bar{q}$ B). Then (M, τ_1, τ_2) is MB* T_2.

5. Generalized Separation Axioms on M-Bitopological Spaces:

In this section we extend the separation axioms , $T_{1/2}$ T_b,, -αT_b, T_c, αT_c, …..) to the cases of the mbts. Also we study the relations between them.

Definition 5.1: A mbts (M, τ_1, τ_2) is said to be.

1. ij-m-$T_{1/2}^{\Omega}$ if ij–m-ΩC(M)=j-m-C(M).
2. ij–m-$^{\Omega}T_{1/2}$ if ij-m-GC(M)=ij-m-ΩC(M).
3. ij–m-$T_{1/2}$ if ij-m-GC(M)=j-m-C(M).
4. ij–m−T_c if ij-m-GSC(M)=ij-m-ΩC(M).
5. ij–m−αT_c ij-m-αGC(M)=ij-m-ΩC(M).
6. ij–m−T_b if ij-m-GSC(M)=j-m-C(M).
7. ij–m−αT_b if ij-m-αGC(M)=j-m-C(M).
8. ij–m−T_d if ij-m-GSC(M)=ij-m-GC(M).
9. ij–m−αT_d if ij-m-αGC(M)=ij-m-GC(M).

Theorem 5.2: The relation between above mentioned separation axioms are stated in the diagram (5.2).

Proof (1) ij–m-$T_{1/2}$ $\rightarrow$ ij-m-$T_{1/2}^{\Omega}$.

Let N is ij–m-ΩC(M), then N is ij–m–GC(M)([17], theorem (3.5)). Since (M, τ_1, τ_2) is ij–m-$T_{1/2}$ then N is j–m–C(M). Therefore ij–m-$T_{1/2}$ is ij–m-$T_{1/2}^{\Omega}$.

(2) ij–m-$T_{1/2}$ $\rightarrow$ ij–m-$^{\Omega}T_{1/2}$.

Let N is ij–m–GC(M). Since (M, τ_1, τ_2) is ij–m-$T_{1/2}$ then N is j–m–C(M). from ([17], theorem (3.5)) we have N is ij–m-Ω-closed. Therefore ij–m-$T_{1/2}$ is ij–m$^{\Omega}T_{1/2}$- .

(3) ij–m-αT_b $\rightarrow$ ij–m-$T_{1/2}$.

Let N is ij–m–GC(M). From ([17], theorem (3.5)) we have N is ij–m-αGC(M). Since (M, τ_1, τ_2) is ij–m-αT_b, then N is j–m–closed. Therefore ij–m-αT_b, is ij–m-$T_{1/2}$.

(4) ij–m-αT_c, $\rightarrow$ ij–m-$^{\Omega}T_{1/2}$.

Let N is ij–m–GC(M). From ([17], theorem (3.5)) we have N is ij–m-αGC(M). Since (M, τ_1, τ_2) is ij–m−αT_c then N is ij–m-ΩC(M). Therefore ij–m-αT_c, is ij–m-$^{\Omega}T_{1/2}$.

The other are similar.

In the diagram (5.2) the arrows ar not reversible as one may notice in the following some examples.

Example 5.3: Let X = {a, b, c}, M = {2/a, 3/b, 4/c}, τ_1 = {M, ∅, {2/a}, {3/b}, {2/a, 3/b}, {2/a, 2/c}}, τ_2 = {M, ∅, {4/c}, {3/b, 4/c}}. Since 12–m-ΩC(M) = {M, ∅, {2/a}, {2/a, 3/b}} = 2–m–C(M). Then (M, τ_1, τ_2) is 12–m-$T_{1/2}^{\Omega}$. But it is not 12–m-$T_{1/2}$. Since {3/b, 4/c} ∈ 12–m–GC(M), but {3/b, 4/c} ∉ 2–m–C(M).

Example 5.4: Let X = {a, b, c}, M = {2/a, 3/b, 2/c}, τ_1 = {M, ∅, {2/a}, {3/b}, {2/a, 3/b}, {3/b, 2/c}}, τ_2 = {M, ∅, {3/b}, {2/c}, {2/a, 2/c}, {3/b, 2/c}}. Since 12–m-αGC(M) = 12–m–GC(M), then (M, τ_1, τ_2) is 12–m-αT_d. But it is not 12–m-αT_c, because {2/a, 3/b, 1/c} ∈ 12–m-αGC(M), but {2/a, 3/b, 1/c} ∉ 12–m–ΩC(M).

Also 12–m–GSC(M) = 12–m–GC(M), then (M, τ_1, τ_2) is 12–m–T_d. But it is not 12–m–T_c because {1/a, 3/b} ∈ 12–m–GSC(M), but {1/a, 3/b} ∉ 12–m-ΩC(M).

Example 5.5: Let X = {a, b, c}, M = {2/a, 3/b, 1/c}, τ_1 = {M, ∅, {2/b}, {2/a, 1/c}, {2/b, 2/a, 1/c}}, τ_2 = {M, ∅, {2/a}, (1/c}, {2/a, 1/c}}. Since 12–m-GC(M) = {{3/b}, {2/a, 3/b}, {1/a, 3/b}, {3/b, 1/c}, ∅, M} = 12–m–ΩC(M). Then (M, τ_1, τ_2) is 12–m-$^{\Omega}T_{1/2}$But it is not 12–m-$T_{1/2}$. Because {1/a, 3/b} ∈ 12–m–GC(M), but {1/a, 3/b} ∉ 2–m–C(M).

Example 5.6: Let X = {a, b, c}, M = {2/a, 3/b, 1/c}, τ_1 = {M, ∅, {2/b}, {2/a, 1/c}, {2/b, 2/a, 1/c}}, τ_2 = {M, ∅, {2/b}, {1/c}, {2/a, 1/c}}. Since 12–m-GC(M) = {{3/b}, {2/a, 3/b}, {1/a, 3/b}, {3/b, 1/c}, ∅, M} = 12–m–ΩC(M), then (M, τ_1, τ_2) is 12–m-$^{\Omega}T_{1/2}$. But it is not 12–m-αT_c because {1/a, 3/b, 1/c} ∈ 12–m– αGC(M), but {1/a, 3/b, 1/c} ∉ 12–m–ΩC(M).

Example 5.7: Let X = {a, b, c}, M = {2/a, 3/b, 2/c}, τ_1 = {M, ∅, {2/a}, {3/b}, {2/a, 3/b}, {2/a, 2/c}}, τ_2 = {M, ∅, {2/a}, {2/a, 3/b}}. Since 12–m-αGC(M) = 12–m–GC(M). Then (M, τ_1, τ_2) is 12–m-αT_d. But it is not 12–m-T_d. Because {3/b} ∈ 12–m–GSC(M), but {3/b} ∉ 12–m–GC(M).

Example 5.8: Let X = {a, b, c}, M = {1/a, 3/b, 1/c}, τ_1 = {M, ∅, {2/b}, {1/a, 1/c}, {1/a, 2/b, 1/c}}, τ_2 = {M,

∅, {1/a}, {1/c}, {1/a, 1/c}}. Since 12–m-αGC(M) = {{3/b}, {1/a, 3/b}, {3/b, 1/c}, ∅, M} = 2–m–C(M), then (M, τ_1, τ_2) is 12–m-αT_b. But it is not 12–m-T_b, because {1/a, 2/b} ∈ 12–m-GSC(M), but {1/a, 2/b} ∉ 2–m–C(M).

Also since ij–m-αGC(M) = 12–m–ΩC(M), then (M, τ_1, τ_2) is 12–m–αT_c. But it is not 12–m–T_c because {2/b, 1/c} ∈ 12–m–GSC(M), but {2/b, 1/c} ∉ 12–m-ΩC(M).

Theorem 5.9: A mbts (M, τ_1, τ_2) is:

1. ij–m-$T_{1/2}$ iff it is ij–m- ${}^{\Omega}T_{1/2}$ and ij–m-$T_{1/2}^{\Omega}$
2. ij–m-αT_b iff it is ij–m-αT_c and ij–m-$T_{1/2}^{\Omega}$.
3. ij–m–T_b iff it is ij–m-T_c and ij–m-$T_{1/2}^{\Omega}$.
4. ij–m-αT_c iff it is ij–m- ${}^{\Omega}T_{1/2}$ and ij–m-αT_d.
5. ij–m–T_c iff it is ij–m-${}^{\Omega}T_{1/2}$ and ij–m-T_d.
6. ij–m–T_c iff it is ij–m-T_d and ij–m-αT_c.

Proof Theorem (5.2) implies the necessity of the previous equivalents. So we prove the sufficiency.

Sufficiency (1) Suppose that (M, τ_1, τ_2) is both ij–m- ${}^{\Omega}T_{1/2}$ and ij–m-$T_{1/2}^{\Omega}$. Let N ∈ ij–m–GC(M). Since (M, τ_1, τ_2) is ij–m- ${}^{\Omega}T_{1/2}$, then N ∈ ij–m-ΩC(M). Since (M, τ_1, τ_2) is ij–m-$T_{1/2}^{\Omega}$, then N ∈ j–m–C(M). Therefore (M, τ_1, τ_2) is ij–m-$T_{1/2}$.

(2) Suppose that (M, τ_1, τ_2) is both ij–m-αT_c and ij–m-$T_{1/2}^{\Omega}$. Let N ∈ ij–m–αGC(M). Since (M, τ_1, τ_2) is ij– m-αT_c, then N ∈ ij–m-ΩC(M). Since (M, τ_1, τ_2) is ij–m-$T_{1/2}^{\Omega}$, then N ∈ j–m–C(M). Therefore (M, τ_1, τ_2) is ij–m-αT_b.

(3) Suppose that (M, τ_1, τ_2) is both ij–m-T_c and ij–m-$T_{1/2}^{\Omega}$. Let N ∈ ij–m–GSC(M). Since (M, τ_1, τ_2) is ij–m-T_c then N ∈ ij–m-ΩC(M). Since (M, τ_1, τ_2) is ij–m-$T_{1/2}^{\Omega}$, then N ∈ j–m–C(M). Therefore (M, τ_1, τ_2) is ij–m-T_b.

(4) Suppose that (M, τ_1, τ_2) is both ij–m- ${}^{\Omega}T_{1/2}$ and ij–m-αT_d. Let N ∈ ij–m–αGC(M). Since (M, τ_1, τ_2) is ij–m-αT_d, then N ∈ ij–m-GC(M). Since (M, τ_1, τ_2) is ij–m-${}^{\Omega}T_{1/2}$, then N ∈ ij–m–ΩC(M). Therefore (M, τ_1, τ_2) is ij–m-αT_c.

(5) Suppose that (M, τ_1, τ_2) is both ij–m- ${}^{\Omega}T_{1/2}$ and ij–m-T_d. Let N ∈ ij–m–GSC(M). Since (M, τ_1, τ_2) is ij–m-T_d, then N ∈ ij–m-GC(M). Since (M, τ_1, τ_2) is ij–m-${}^{\Omega}T_{1/2}$, then N ∈ ij–m–ΩC(M). Therefore (M, τ_1, τ_2) is ij–m-T_c.

(6) Suppose that (M, τ_1, τ_2) is both ij–m-T_d and ij–m-αT_c. Let N ∈ ij–m–GSC(M). Since (M, τ_1, τ_2) is ij–m-T_d, then N ∈ i j – m - GC(M). By ([17], theorem (3.5)) we have N∈ i j – m - αGC(M). Since (M, τ_1, τ_2) is ij–m-αT_c, then N ∈ ij–m–ΩC(M). Therefore (M, τ_1, τ_2) is ij–m-T_c.

Remark 5.10: For any mbts (M, τ_1, τ_2) we note that:

1. ij–m- ${}^{\Omega}T_{1/2}$ and ij – m -$T_{1/2}^{\Omega}$ are independent concepts.
2. ij–m-αT_c and ij–m-$T_{1/2}^{\Omega}$ are independent concepts.
3. ij–m-T_c and ij–m-$T_{1/2}^{\Omega}$ are independent concepts.
4. ij–m–T_d and ij–m-αT_c are independent concepts.
5. ij–m- ${}^{\Omega}T_{1/2}$ and ij–m-αT_d are independent concepts.
6. ij–m- ${}^{\Omega}T_{1/2}$ and ij-m-T_d are independent concepts.

In the following we give some examples to show the previous remark.

Example 5.11: (1) Let X = {a, b, c}, M = {2/a, 3/b, 4/c}, τ_1 = {M, ∅, {2/a}, {3/b}, {2/a, 3/b}, {2/a, 4/c}}, τ_2 = {M, ∅, {4/c}, {3/b, 4/c}}. Since 12–m-ΩC(M) = {M, ∅, {2/a}, {2/a, 3/b}} = 2–m–C(M). Then (M, τ_1, τ_2) is 12–m-$T_{1/2}^{\Omega}$. But (M, τ_1, τ_2) is not 12–m-${}^{\Omega}T_{1/2}$ because {3/b, 4/c} ∈ 12– m–GC(M), but {3/b, 4/c} ∉ 12–m–ΩC (M).

(2) Let X = {a, b, c}, M = {2/a, 3/b, 1/c}, τ_1 = {M, ∅, {2/b}, {2/a, 1/c}, {2/a, 2/b, 1/c}}, τ_2 = {M, ∅, {2/a}, {1/c}, {2/a, 1/c}}. Since 12– m - GC(M) = {{3/b}, {2/a, 3/b}, {1/a, 3/b}, {3/b, 1/c}, ∅, M} = 12–m–ΩC(M), then (M, τ_1, τ_2) is 12–m- ${}^{\Omega}T_{1/2}$. But (M, τ_1, τ_2) is not 12–m-$T_{1/2}^{\Omega}$ because {1/a, 3/b} ∈ 12–m–ΩC(M), but {1/a, 3/b} ∉ 2 – m –closed.

Example 5.12: (1) Let X={a, b, c}, M ={2/a, 3/b, 2/c}, τ_1 ={M, ∅, {2/a}, {3/b}, {2/a, 3/b}, {3/b, 2/c}}, τ_2={M, ∅, {3/b}, {2/c}, {2/a, 2/c}, {3/b, 2/c}}. Since 12–m-GC(M) =12–m–αGC(M), then (M, τ_1, τ_2) is 12 – m - αT_d. But (M, τ_1, τ_2) is not 12–m- ${}^{\Omega}T_{1/2}$, because {2/a, 2/b} ∈ 12–m-GC(M), but {2/a, 2/b} ∉ 12–m–ΩC (M).

(2) Let X = {a, b, c}, M = {2/a, 3/b, 1/c}, τ_1 = {M, ∅, {2/b}, {2/a, 1/c}, {2/a, 2/b, 1/c}}, τ_2 = {M, ∅, {2/a}, (1/c}, {2/a, 1/c}}. Since 12 – m - GC (M) = {{3/b}, {2/a, 3/b}, {1/a, 3/b}, {3/b, 1/c}, ∅, M} = 12–m–ΩC (M). Then (M, τ_1, τ_2) is 12–m- ${}^{\Omega}T_{1/2}$. But (M, τ_1, τ_2) is not 12–m-αT_d, because {2/a, 1/c} ∈ 12–m–αGC (M), but {2/a, 1/c} ∉ 12–m–GC (M).

Theorem 5.13: For a mbts (M, τ_1, τ_2)

1. If (M, τ_1, τ_2) is an ij–m-αT_c we have for all m/a ∈ M, either {m/ a} ∈ ji–m-αGC(M) or {m/ a} ∈ ij–m-ΩO(M).
2. If (M, τ_1, τ_2) is an ij–m-$^{\Omega}T_{1/2}$ we have for all m/a∈ M, either {m/a} ∈ i–m-C(M) or {m/a} ∈ ij–m-ΩO(M).

Proof (1) Let m a ∈ M. Suppose that m/a ∉ ji–m-αGC(M). Then m/a ∉ i–m–C(M). So $(m/a)^c$ is not i–m–O(M). The only i–m–O(M) containing $(m/a)^c$ is M, implies that $(m/a)^c$ ∈ ij–m-αGC(M). Since (M, τ_1, τ_2) is ij–m-αT_c, then $(m/ a)^c$ ∈ ij–m–ΩC(M). Consequently m/a ∈ ij–m-ΩO(M).
(2) Let m/a ∈ M. Suppose that m/a ∉ i–m-C(M). So $(m/a)^c$ is not i–m–O(M). The only i–m–O(M) containing $(m/a)^c$ is M, implies that $(m/a)^c$ ∈ ij–m-GC(M). Since (M, τ_1, τ_2) is ij–m- $^{\Omega}T_{1/2}$, then $(m/ a)^c$ ∈ ij–m–ΩC(M). Consequently m/a ∈ ij–m-ΩO(M).

The converse of the above theorem is not true.

Examples 5.14 Let (M, τ_1, τ_2) be as in example (5.4). We have {3/b} ∈ 12–m-ΩO(M) and {3, b} ∈ 12–m-αGO(M). But (M, τ_1, τ_2) is not ij–m-αT_c.

Definition 5.15: A function f : (M_1, τ_1, τ_2) → (M_2, σ_1, σ_2) is said to be:
1. ij–m–gc–irresolute if ∀ V ∈ ij–m–GC(M_2), then f^{-1} (V) ∈ ij–m–GC(M_1).
2. ij–m–gs-irresolute if ∀ V ∈ ij–m–GSC(M_2), then f^{-1} (V) ∈ ij–m–GSC(M_1).
3. ij–m-αg–irresolute if ∀ V ∈ ij–m-αGC(M_2), then f^{-1} (V) ∈ ij-m-αGC(M_1).

Definition 5.16: A function f: (M_1, τ_1, τ_2) → (M_2, σ_1, σ_2). Is called ij–m–pre-Ω-closed if U ∈ ij–m-ΩC(M_1) implies that f (U) ∈ ij–m-ΩC(M_2).

Theorem 5.17: Let f be a function from (M_1, τ_1, τ_2) to (M_2, σ_1, σ_2).
1. If f is ij–m-Ω-continuous and (M_1, τ_1, τ_2) is ij–m -$T^{\Omega}_{1/2}$ then f is j–m–continuous.
2. If f is ij–m-αg-continuous and (M_1, τ_1, τ_2) is ij–m-αT_c then f is ij–m–Ω-continuous.
3. If f is ij–m-g-continuous and (M_1, τ_1, τ_2) is ij–m- $^{\Omega}T_{1/2}$ then f is ij–m–Ω-continuous.
4. If f is ij–m-gs-continuous and (M_1, τ_1, τ_2) is ij-m- T_c then f is ij–m–Ω-continuous.

Proof (1) Let V ∈ j–m–C(M_2). Since f is ij–m-Ω-continuous, then f^{-1} (V) ∈ ij–m-ΩC (M_1). Since (M_1, τ_1, τ_2) is ij–m-$T^{\Omega}_{1/2}$ then f^{-1} (V) ∈ j–m–C(M_1). Consequently f is j–m–continuous.
(2) Let V ∈ j–m–C(M_2). Since f is ij–m-αg-continuous, then f^{-1} (V) ∈ ij–m-αGC(M_1). Since (M_1, τ_1, τ_2) is ij –m-αT_c then f^{-1} (V) ∈ ij–m–ΩC (M_1). Consequently f is ij–m–Ω-continuous.
(3) Let V ∈ j–m–C(M_2). Since f is ij-m-g-continuous, then f^{-1} (V) ∈ ij–m-GC(M_1). Since (M_1, τ_1, τ_2) is ij–m-$^{\Omega}T_{1/2}$ then f^{-1} (V) ∈ ij–m–ΩC (M_1). Consequently f is ij–m–Ω-continuous.
(4) Let V ∈ j–m–C(M_2). Since f is ij–m-gs-continuous, then f^{-1} (V) ∈ ij–m-GSC (M_1). Since (M_1, τ_1, τ_2) is ij–m-T_c then f^{-1} (V) ∈ ij–m–ΩC(M_1). Consequently f is ij–m–Ω-continuous.

Theorem 5.18: Let f : (M_1, τ_1, τ_2) → (M_2, σ_1, σ_2) surjection.
1. Let f is ij–m-Ω- irresolute and j–m– closed. If (M_1, τ_1, τ_2) is ij–m-$T^{\Omega}_{1/2}$, then (M_2, σ_1, σ_2) is also ij–m-$T^{\Omega}_{1/2}$.
2. Let f is ij–m-gc-irresolute and ij–m–pre-Ω- closed. If (M_1, τ_1, τ_2) is ij–m- $^{\Omega}\mathcal{T}_{1/2}$, then (M_2, σ_1, σ_2) is also ij–m- $^{\Omega}\mathcal{T}_{1/2}$.
3. Let f is ij–m-αg-irresolute and ij–m–pre-Ω-closed. If (M_1, τ_1, τ_2) is ij–m-αT_C, then (M_2, σ_1, σ_2) is also ij–m-$\alpha\mathcal{T}_c$.
4. Let f is ij–m-gs-irresolute and ij–m–pre-Ω- closed. If (M_1, τ_1, τ_2) is ij–m–$\mathcal{T}_c$, then (M_2, σ_1, σ_2) is also ij–m-$\mathcal{T}_c$.

Proof (1) Let V ∈ ij–m–ΩC(M_2). Since f is ij–m-Ω-irresolute then f^{-1} (V) ∈ ij–m-ΩC(M_1). Since (M_1, τ_1, τ_2) is ij– m-$\mathcal{T}^{\Omega}_{1/2}$-, then f^{-1} (V) ∈ j–m–C(M_1). Since f is j-m-closed and onto, then V ∈ j–m-C(M_2). Therefore (M_2, σ_1, σ_2) is also ij–m-$\mathcal{T}^{\Omega}_{1/2}$.
(2) Let V ∈ ij–m–GC(M_2). Since f is ij–m-gc-irresolute then f^{-1} (V) ∈ ij–m-GC(M_1). Since (M_1, τ_1, τ_2) is ij–m-$^{\Omega}\mathcal{T}_{1/2}$, then f^{-1} (V) ∈ ij–m–ΩC(M_1). Since f is ij-m-pre-Ω-closed and onto, then V = f (f^{-1} (V)) ∈ ij–m -ΩC(M_2). Therefore (M_2, σ_1, σ_2) is also ij–m–$^{\Omega}\mathcal{T}_{1/2}$.
(3) Let V ∈ ij–m–αGC(M_2). Since f is ij–m-αg-irresolute then f^{-1} (V) ∈ ij–m-αGC(M_1). Since (M_1, τ_1, τ_2) is ij–m-$\alpha\mathcal{T}_c$, then f^{-1}(V) ∈ ij–m–ΩC (M_1). Since f is ij-m-pre-Ω-closed and onto, then V ∈ ij–m–ΩC(M_2). Therefore (M_2, σ_1, σ_2) is also ij–m–$\alpha\mathcal{T}_c$.
(4) Let V ∈ ij–m–GSC(M_2). Since f is ij–m-gs-irresolute then f^{-1} (V) ∈ ij–m-GSC(M_1). Since (M_1, τ_1, τ_2) is ij–m-$\mathcal{T}_c$, then f^{-1} (V) ∈ ij–m–ΩC(M_1). Since f is j-m-pre-Ω-closed and onto, then V ∈ ij–m–ΩC (M_2). Therefore (M_2, σ_1, σ_2) is also ij–m– $\mathcal{T}_c$.

Conclusion:

This paper begins with a brief survey of the notion of msets, collection of different types of separation axioms and operations under M-

topological space and M- bitopological space introduced. It also gives the definition of the notion of some kinds of separation axioms on M – bitopological space. We have found that there is a relationships between these classes. In future work we will study other M-bitopological properties, for example the compactness and connectedness in M–bitopological spaces and their properties.

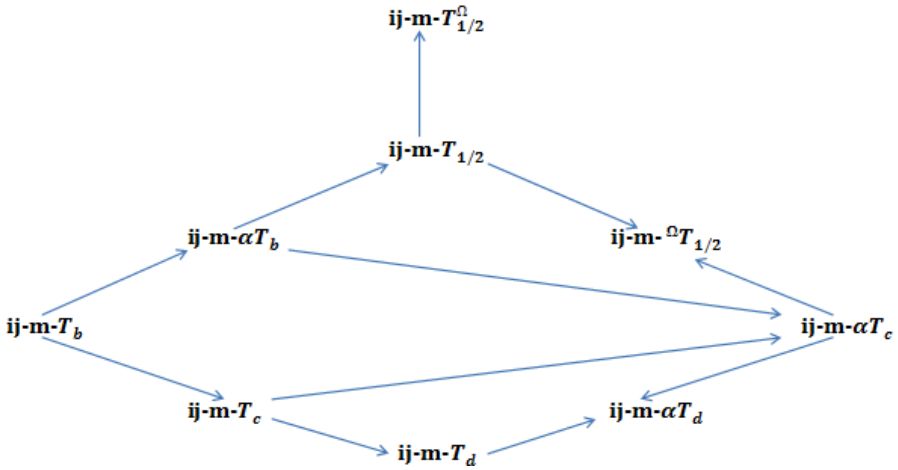

Diagram (5.2)

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