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# The Kumaraswamy-Generalized Pareto Distribution

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**ABSTRACT**

The modeling and analysis of lifetimes is an important aspect of statistical work in a widevariety of scientific and technological fields. For the first time, the called Kumaraswamy generalizedPareto (Kum-GP) distribution, is introduced and studied.The new distribution can have a decreasing and upside-down bathtub failure rate function depending on the value of its parameters; it's including some special sub-model like generalized Pareto distribution and its exponentiated. Some structural properties of the proposed distribution are studied including explicit expressions for the moments and the density function of the order statistics and obtain their moments will be obtain. The method of maximum likelihood is used for estimating the model parameters and the observed information matrix is derived. The information matrix is easily numerically determined.Monte Carlo simulations and the application oftwo real data setsare performed to illustrate the potentiality of this distribution.

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## INTRODUCTION

The Pareto distribution is the most popular model for analyzing skewed data. The Pareto distribution was first proposed by (Pareto(1897) as a model for the distribution of income. It can be used to represent various other forms of distributions (other than income data) that arise in human life. Most of authorsgave an extensive historical survey of its use in the context of income distribution with many formulas. There are several forms and extensions of the Pareto distribution in the literature. (Pickands (1975) was the first to propose an extension of the Pareto distribution with the generalized Pareto (GP) distribution when analyzing the upper tail of a distribution function. The GP has been used for modeling extreme value data because of its long tail feature (see Choulakian&Stephens, (2001). Naturally, the Pareto distribution is a special case of the GP.The exponentiated Pareto (EP) distribution was introduced by (Gupta *et al*. (1998) in the same settings that the generalized exponential(GE) distribution extends the exponential distribution (see Gupta &Kundu, (1999). For more details on the GP distribution, its theory and further applications, we refer the readers to (Leadbetter *et al*. (1987), (Embrechts *et al*. (1997), (Castillo *et al*. (2005). The four parameters Pareto (generalized Pareto) distribution was introduced by (Abdul Fattah et el. The cumulative distribution function cdf of the four parameters Pareto distribution (generalized Pareto distribution) is

$$F(x;\alpha,\beta,\lambda,\delta)=1-\left(1+\left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-\alpha} \qquad x>\lambda \qquad (1)$$

A random variable $X$ is said to follow the Pareto distribution with fourparameters (generalized Pareto distribution) if the probability density function pdf of $X$ is as follows:

$$f(x;\alpha,\beta,\lambda,\delta)=\frac{\delta\alpha}{\beta}\left(\frac{x-\lambda}{\beta}\right)^{\delta-1}\left(1+\left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-(\alpha+1)} \qquad (2)$$

For $x>\lambda$ , $(\alpha,\beta,\delta)>0$ , $\lambda\geq 0$ where $\lambda$ is the location parameter, $\beta$ is scale parameter and $\alpha$ *and* $\delta$ are the shape parameter

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In this context, we propose an extension of the Generalized Pareto distribution based on the family of Kumaraswamy generalized (denoted with the prefix "Kw-G" for short) distributions introduced by Cordeiro and de Castro. Nadarajah *et al*. studied some mathematical properties of this family. The Kumaraswamy (Kw) distribution is not very common among statisticians and has been little explored in the literature. Its cdf (for 0 < x < 1) is $F(x) = 1 - (1 - x^a)^b$, where a > 0 and b > 0 are shape parameters, and the density function has a simple form$f(x) = ab\, x^{a-1}(1 - x^a)^{b-1}$, which can be unimodal, increasing, decreasing or constant, depending on the parameter values.

In this note, we combine the works of Kumaraswamy and Abdul Fattah *et al*, to derive some mathematical properties of a new model, called the Kumaraswamy Generalized Pareto (Kw-GP) distribution, which stems from the following general construction: if G denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by

$$F(x; a, b) = 1 - (1 - G(x)^a)^b \quad (3)$$

where a > 0 and b > 0 are two additional shape parameters which govern Skewness and tail weights. Because of its tractable distribution function (2), the Kw-G distribution can be used quite effectively even if the data are censored. Correspondingly, its density function is distributions has a very simple form

$$f(x; a, b) = ab\, g(x)\, G(x)^{a-1}(1 - G(x)^a)^{b-1} \quad (4)$$

The density family (3) has many of the same properties of the class of beta-G distributions (see Eugene *et al*. [4]), but has some advantages in terms of tractability, since it does not involve any special function such as the beta function.

This paper is outlined as follows. In section 2, we define the Kw-GP distribution and provide expansions for its cumulative and density functions. A range of mathematical properties of this distribution is considered in sections 3 till 7. These include quantile function, simulation, skewness and kurtosis, order statistics, L-moments and mean deviations; we also provide expansions for the moments of the order statistics. The Re´nyi entropy is calculated in section 8. Maximum likelihood estimation is performed and the observed information matrix is determined in section 9. In section 10, we provide monte Carlo simulation and application to several real data sets to illustrate the potentiality of this distribution. Finally, some conclusions are addressed in section 11.

**Table 1:** Some Special Distributions.

| Distribution | α | β | δ | λ | $\beta^\delta$ |
|---|---|---|---|---|---|
| 3-Parameters Pareto | -- | -- | 1 | -- | -- |
| 2-Parameters Burr XII | -- | 1 | -- | 0 | -- |
| 2-Parameters Lomax | -- | -- | 1 | 0 | -- |
| Beta-II | -- | 1 | 1 | 0 | -- |
| Compound Exponential-Exponential | 1 | -- | 1 | -- | $\theta$ |
| Compound Raleigh-Gamma | -- | -- | 2 | -- | $\theta$ |
| Compound Weibull-Exponential | 1 | -- | -- | -- | $\theta$ |

***2 The Kumaraswamy-Generalized Pareto Distribution:***

If $G(x; \underline{\theta})$ is the Generalized Pareto cumulative distribution with Parameter $\underline{\theta} = (\alpha, \beta, \lambda, \delta)$ then equation (2) yields the Kw-GP cumulative distribution

$$F\left(x; \underline{\xi}\right) = 1 - \left\{1 - \left[1 - \left(1 + \left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-\alpha}\right]^{a}\right\}^{b} \quad x \geq \lambda \quad (5)$$

where$\underline{\xi} = (a, b, \alpha, \beta, \lambda, \delta)$ ,$(a, b, \alpha \; and \; \delta) > 0$ are non-negative shape Parameters,$\beta > 0$ is the scale parameter is positive,and $\lambda \geq 0$ is the location parameteris real. The corresponding pdf and Hazard Rate Function are

$$f\left(x; \underline{\xi}\right) = \frac{ab\alpha\delta}{\beta}\left(\frac{x-\lambda}{\beta}\right)^{\delta-1}\left(1 + \left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-(\alpha+1)}\left[1 - \left(1 + \left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-\alpha}\right]^{a-1}$$

$$\left\{1 - \left[1 - \left(1 + \left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-\alpha}\right]^{a}\right\}^{b-1} \quad (6)$$

and

$$S\left(x; \underline{\xi}\right) = 1 - F\left(x; \underline{\xi}\right) = \left\{1 - \left[1 - \left(1 + \left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-\alpha}\right]^{a}\right\}^{b}$$

$$H\left(x; \underline{\xi}\right) = \frac{f\left(x; \underline{\xi}\right)}{S\left(x; \underline{\xi}\right)} = \frac{ab\alpha\delta\left(\frac{x-\lambda}{\beta}\right)^{\delta-1}\left(1 + \left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-(\alpha+1)}\left[1 - \left(1 + \left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-\alpha}\right]^{a-1}}{\beta\left\{1 - \left[1 - \left(1 + \left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-\alpha}\right]^{a}\right\}}$$

Respectively

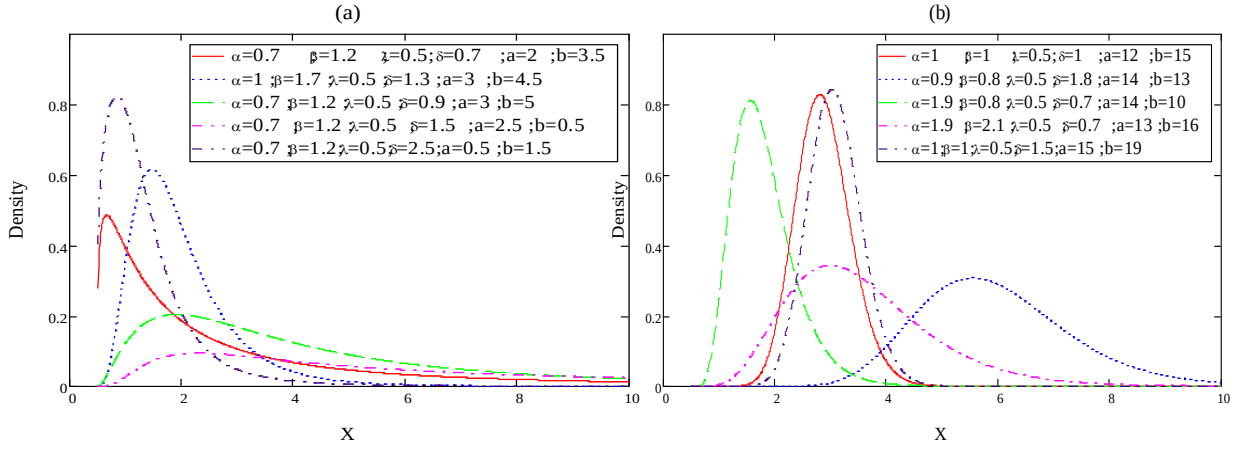


**Fig. 1:** Plots of the Kw-GP density function for some parameter values.

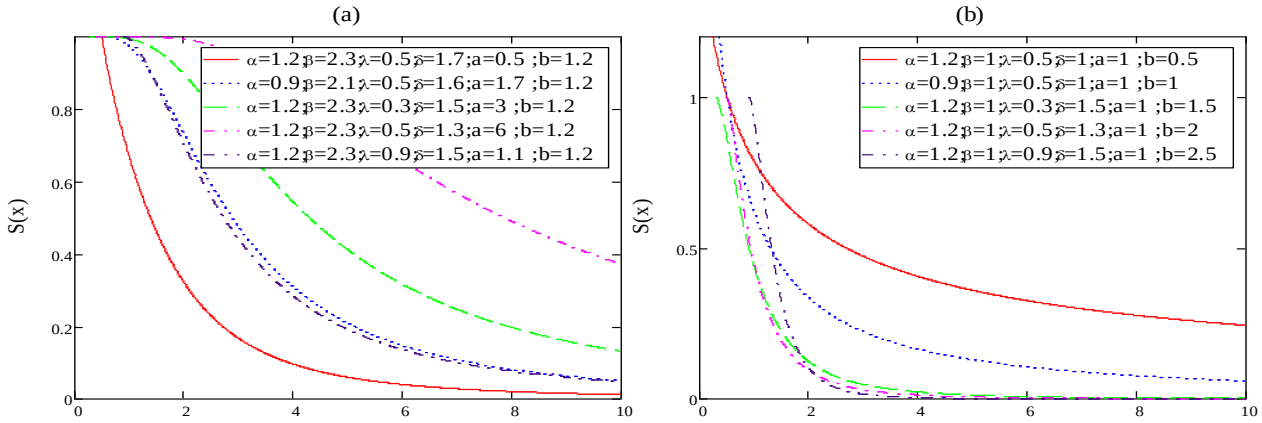


**Fig. 2:** (a) Plots of the Kum-GP survival function for some values of a. (b) Plots of the Kum-GP survival function for some values of b.

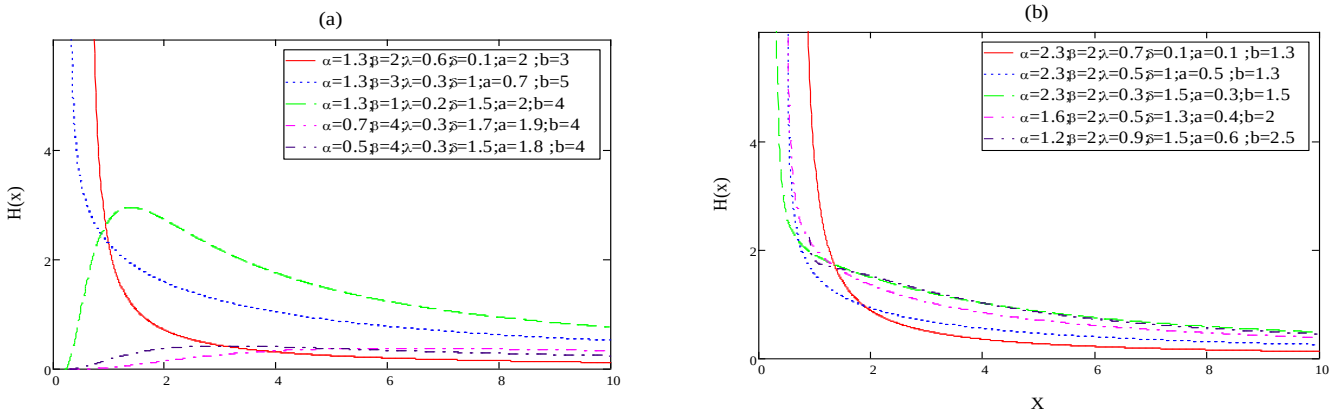


**Fig. 3:** Plots of the Kw-P hazard function for some parameter values.

### *2.1 Special Distributions:*

The following well-known and new distributions are special sub-models of the Kum-GP distribution.

- ***Exponentiated Generalized Pareto distribution:***

If $b = 1$, the Kum-GP distribution reduces to

$$f\left(x;\underline{\xi}\right) = \frac{a\alpha\delta}{\beta}\left(\frac{x-\lambda}{\beta}\right)^{\delta-1}\left(1+\left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-(\alpha+1)}\left[1-\left(1+\left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-\alpha}\right]^{a-1}$$

Which is the exponentiated generalized Pareto (EGP). For$a = b = 1$, we obtain the Generalized Pareto distribution, if $a = b = 1$ and $\beta^{\delta} = \theta$the Kum-GP distribution reduces to compound Weibull-Gamma Distribution (Abdul Fattah *et al*).

- ***Kum-Compound Weibull Gamma Distribution:***

If $\beta^{\delta} = \theta$, the Kum-GP distribution reduces to

$$f\left(x;\underline{\xi}\right) = ab\alpha\delta\frac{(x-\lambda)^{\delta-1}}{\theta}\left(1+\frac{(x-\lambda)^{\delta}}{\theta}\right)^{-(\alpha+1)}\left[1-\left(1+\frac{(x-\lambda)^{\delta}}{\theta}\right)^{-\alpha}\right]\left\{1-\left[1-\left(1+\frac{(x-\lambda)^{\delta}}{\theta}\right)^{-\alpha}\right]^{a}\right\}^{b-1}a-1$$

which is the Kum-Compound Weibull Gamma Distribution, For $a = 1$ we obtain the exponentiated Weibull Gamma Distribution, if $a = b = 1$the Kum-GP distribution reduces to compound Weibull-Gamma Distribution (Abdul Fattah *et al*).

- ***Kum-Compound Rayleigh Gamma Distribution:***

If $\beta^{\delta} = \theta$, $\delta = 2$ the Kum-GP distribution reduces to

$$f\left(x;\underline{\xi}\right) = 2ab\alpha\frac{(x-\lambda)}{\theta}\left(1+\frac{(x-\lambda)^{2}}{\theta}\right)^{-(\alpha+1)}\left[1-\left(1+\frac{(x-\lambda)^{2}}{\theta}\right)^{-\alpha}\right]\left\{1-\left[1-\left(1+\frac{(x-\lambda)^{2}}{\theta}\right)^{-\alpha}\right]^{a}\right\}^{b-1}a-1$$

Which is the Kum-Compound Rayleigh Gamma Distribution, for $a = 1$ we obtain the exponentiated Rayleigh Gamma Distribution, if $a = b = 1$,the Kum-GP distribution reduces to compound Rayleigh-Gamma Distribution (Abdul Fattah *et al*)

***2.2Expansions for the cumulative and density functions:***

Here, we give simple expansions for the Kw-GP cumulative distribution. By using the generalized binomial theorem (for 0 < a < 1)

$(1+a)^{v} = \sum_{i=0}^{\infty}\binom{v}{i}a^{i}$where$\binom{v}{i} = \frac{n(n-1)\ldots(v-i+1)}{i!}$(7)

In equation (5), we can write

$$F\left(x;\underline{\xi}\right) = 1-\sum_{i=0}^{\infty}(-1)^{i}\binom{b}{i}\left[1-\left(1+\left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-\alpha}\right]^{ai} = 1-\sum_{i=0}^{\infty}\kappa_{i}\tau\left(x;\underline{\zeta}\right)$$

Where $\kappa_{i} = (-1)^{i}\binom{b}{i}$ and $\tau\left(x;\underline{\zeta}\right)$ denotes the EGP cumulative distribution with parameters $\underline{\zeta} = (\alpha,\beta,\lambda,\delta,ai)$, Now, using the power series (7) in the last term of (6), we obtain

$$f\left(x;\underline{\xi}\right) = \frac{ab\alpha(j+1)\delta}{\beta(j+1)}\left(\frac{x-\lambda}{\beta}\right)^{\delta-1}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\left(1+\left(\frac{x-\lambda}{\beta}\right)^{\delta}\right)^{-(\alpha(j+1)+1)}$$

$f\left(x;\underline{\xi}\right) = \sum_{j=0}^{\infty}w_{j}\,g(x;\underline{\vartheta})$where$w_{j} = \frac{ab}{(j+1)}\sum_{i=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}$ (8)

and$g\left(x;\underline{\vartheta}\right)$denotes the generalized Pareto distribution with parameters $\underline{\vartheta} = (\alpha(j+1),\beta,\lambda,\delta)$.

Thus, the Kw-GP density function can be expressed as an infinite linear combination of Pareto densities. Thus, some of its mathematical properties can be obtained directly from those properties of the generalized Pareto distribution. For example, the ordinary, inverse and factorial moments, moment generating function (mgf) and characteristic function of the Kw-GP distribution follow immediately from those quantities of the Pareto distribution.

***3- Moments:***

Here, and henceforth, let x be a Kw-GP random Variable following (8), the$r^{th}$ moment of $X$ can be obtained from (8) as

$$E(X^{r}) = \sum_{j=0}^{\infty}w_{j}\int_{\lambda}^{\infty}x^{r}\,g\left(x;\underline{\vartheta}\right)dx$$

$= \sum_{j=0}^{\infty} w_j\, \alpha(j+1)\varsigma_l\, \beta\left\{\left(\alpha(j+1) - \frac{r-l}{\delta}\right), \left(\frac{r-l}{\delta}+1\right)\right\}$Where$\varsigma_l = \sum_{l=0}^{\infty}\binom{r}{l}\lambda^l\beta^{r-l}$
In particular, setting $r = 1$and $a = b = 1$in (9), the mean of $X$ reduces to

$$E(X) = \mu = \beta\frac{\Gamma\left(\alpha - \frac{1}{\delta}\right)\Gamma(\frac{1}{\delta}+1)}{\Gamma(\alpha)} + \frac{\lambda}{2}$$

This is precisely the mean of the generalized Pareto distribution with Parameters

$$\underline{\vartheta} = (\alpha(j+1), \beta, \lambda, \delta).$$

***4- Quantile function and simulation:***

Here, themethod for simulating from the Kw-GP distribution (6) is presented. The quantile function corresponding to (5) is

$$Q(u) = F^{-1}(u) = \beta\left[\left\{1 - \left(1 - (1-u)^{\frac{1}{b}}\right)^{\frac{1}{a}}\right\}^{\frac{-1}{\alpha}} - 1\right]^{\frac{1}{\delta}} + \lambda$$

Simulating the Kw-GP random variable is straightforward. Let $U$ be a uniform variate on the unit interval$(0, 1)$. Thus, by means of the inverse transformation method, we consider the random variable $X$ given by

$$X = \beta\left[\left\{1 - \left(1 - (1-U)^{\frac{1}{b}}\right)^{\frac{1}{a}}\right\}^{\frac{-1}{\alpha}} - 1\right]^{\frac{1}{\delta}} + \lambda$$

which follows (6), i.e. $X \sim KW - GP\ (a, b, \alpha, \beta, \lambda, \delta)$.

***5- Skewness and Kurtosis:***

The shortcomings of the classical kurtosis measure are well-known. There are many heavy tailed distributions for which this measure is infinite. So, it becomes uninformative precisely when it needs to be. Indeed, our motivation to use quantile-based measures stemmed from the non-existance of classical kurtosis for many of the Kw distributions

The Bowley'sskewness (see Kenney and Keeping [15]) is based on quartiles:

$$\mathrm{S_k} = \frac{\mathrm{Q}(3/4) - 2\mathrm{Q}(1/2) + \mathrm{Q}(1/4)}{\mathrm{Q}(3/4) - \mathrm{Q}(1/4)}$$

And the Moors' kurtosis (see Moors (28)) is based on octiles:

$$\mathrm{K_u} = \frac{\mathrm{Q}(7/8) - \mathrm{Q}(5/8) - \mathrm{Q}(3/8) + \mathrm{Q}(1/8)}{\mathrm{Q}(6/8) - \mathrm{Q}(2/8)}$$

Where $Q(\cdot)$ represents the quantile function

***6- Order statistics:***

Moments of order statistics play an important role in quality control testing and reliability, where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictors are often based on moments of order statistics. We now derive an explicit expression for the density function of the ith order statistic$\mathrm{X_{i:n}}$, say $\mathrm{f_{i:n}(x)}$, in a random sample of size n from the Kw-GP distribution,written as

$$f_{\mathrm{i:n}}(\mathrm{x}) = \frac{n!}{(i-1)!\,(n-i)!} f(x) F^{i-1}(x)[1-F(x)]^{n-i}$$

where $(.)\ and\ F(.)$, are the pdf and cdf of the Kw-GP distribution, respectively. From the above equation and using the series representation (7) repeatedly, we obtain a useful expansion for $\mathrm{f_{i:n}(x)}$, given by

$$f_{\mathrm{i:n}}(\mathrm{x}) = \sum_{r=0}^{\infty} V_{i:n}^{(r)}\, g\left(x; \underline{\vartheta}\right) \qquad (14)$$

where

$$V_{i:n}^{(r)} = \frac{n!\,ab}{(i-1)!\,(n-i)!\,(r+1)}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{l+m+r}\binom{i-1}{l}\binom{b(l+n-i+1)-1}{m}\binom{a(m+1)-1}{m}$$

and$g\left(x; \underline{\vartheta}\right)$ denotes the generalized Pareto distribution with parameters $\underline{\vartheta} = (\alpha(j+1), \beta, \lambda, \delta)$. So, the density function of the order statistics is simply an infinite linear combination of generalized Pareto densities. The pdf of the $i^{th}$ order statistic from a random sample of the generalized Pareto distribution comes by setting $a = b = 1$ in (14). Evidently, equation (14) plays an important role in the derivation of the main properties of the Kw-GP order statistics. For example, the $\mathrm{S^{th}}$raw moment of $\mathrm{X_{i:n}}$ can be expressed as

$$E(X_{i:n}^{s}) = \sum_{r=0}^{\infty} V_{i:n}^{(r)} \int_{\lambda}^{\infty} X_{i:n}^{s}\, g(x;\underline{\vartheta})dx$$

$= \sum_{r=0}^{\infty} V_{i:n}^{(r)}\, \alpha(j+1) v_l\, \beta \left\{\left(\alpha(j+1) - \frac{s-l}{\delta}\right), \left(\frac{s-l}{\delta}+1\right)\right\}$ where $v_l = \sum_{l=0}^{\infty}\binom{s}{l}\lambda^{l}\beta^{s-l}$ (15)

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They are linear functions of expected order statistics defined by

$$\lambda_{m+1} = \frac{1}{m+1}\sum_{k=0}^{m}(-1)^{k}\binom{m}{k} E(X_{m+1-k:m+1}), \qquad m = 0,1,\ldots$$

The first four L-moments are:

$$\lambda_1 = E(X_{1:1}), \lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2})$$

$$\lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3}), \qquad \text{and} \quad \lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$$

The L-moments have the advantage that they exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. From equation (15) with s = 1, we can easily obtain explicit expressions for the L-moments of $X$.

***7- Mean Deviations:***

The mean deviations about the mean can be used as measure of spread in a population. Let $\mu = E(X)$ is the mean of the Kw-GP distribution. The mean deviations about the mean can be calculated as

$$E(X-\mu)^{r} = \sum_{j=0}^{\infty} w_j \int_{\lambda}^{\infty} (X-\mu)^{r}\, g(x;\underline{\vartheta})dx$$

$$= \sum_{i=0}^{\infty}\sum_{m=0}^{\infty} w_j \binom{r}{m}\beta^{r-m}\left(\frac{\lambda}{2} - \beta\,\frac{\Gamma\left(\alpha-\frac{1}{\delta}\right)\Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma(\alpha)}\right)^{m}\left(\frac{\Gamma\left(\alpha(j+1)-\frac{r-m}{\delta}\right)\Gamma\left(\frac{r-m}{\delta}+1\right)}{\Gamma(\alpha(j+1))}\right)$$

where

$$w_j = \frac{ab}{(j+1)}\sum_{i=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}$$

***8- Re´nyi entropy:***

The entropy of a random variable X is a measure of uncertainty variation. The Re´nyi entropy is defined as $I_R(C) = \frac{1}{1-C}\log[I(C)]$ Where $I(C) = \int_{\mathbb{R}} f^{C}(x)\,dx \;;\; C > 0, C \neq 1,$

we have

$$I(C) = \frac{a^{C}b^{C}\alpha^{C}\delta^{C}}{\beta^{C}}\int_{\lambda}^{\infty}(Z_i-1)^{C\left(1-\frac{1}{\delta}\right)}(Z_i)^{-C(\alpha+1)}[1-Z_i{}^{-\alpha}]^{C(a-1)}\{1-[1-Z_i{}^{-\alpha}]^{a}\}^{C(b-1)}dx$$

By using expanding theorem and Transforming Variables we obtain

$I(C) = (ab)^{C}\alpha^{C}\beta^{1-C}\delta^{-(1-C)}\eta_j$ where $Z_i = 1 + \left(\frac{x-\lambda}{\beta}\right)^{\delta}$ and

$\eta_j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{C(b-1)}{i}\binom{a(C+i)-C}{j}(-1)^{i+j}\,\beta\left(j - C(\alpha+1)+1\right), \left(\frac{C(\delta-1)-\delta+1}{\delta}\right)$ Then

$$I_R(C) = \frac{C}{1-C}\log(ab) + \frac{C}{1-C}\log(\alpha) + log(\beta) - \log(\delta) + \frac{1}{1-C}\log(\eta_j)$$

***9 Estimation and information matrix:***

In this section, we discuss maximum likelihood estimation and inference for the Kw-GP distribution. Let $x_1, x_2, \ldots, x_n$ be a random sample from $X \sim KW - GP\left(\underline{\xi}\right)$ where $\underline{\xi} = (a, b, \alpha, \beta, \lambda, \delta)$ be the vector of the model Parameters, the log-likelihood function for $\underline{\xi}$ reduces to

$$\ell\left(\underline{\xi}\right) = n\log a + n\log b + n\log\alpha + n\log\delta - n\log\beta$$

$$+(\delta-1)\sum_{i=1}^{\infty}\log\left(\frac{x_i-\lambda}{\beta}\right) - (\alpha+1)\sum_{i=1}^{\infty}\log\left(1+\left(\frac{x_i-\lambda}{\beta}\right)^{\delta}\right) + (a-1)\sum_{i=1}^{\infty}\log\left[1-\left(1+\left(\frac{x_i-\lambda}{\beta}\right)^{\delta}\right)^{-\alpha}\right]$$

$$+ (b-1)\sum_{i=1}^{\infty}\log\left\{1-\left[1-\left(1+\left(\frac{x_i-\lambda}{\beta}\right)^{\delta}\right)^{-\alpha}\right]^{a}\right\} \qquad (16)$$

Since $x > \lambda$ then the estimate value for $\lambda$ equal the first order statistic as $\lambda = x_{(1)}$.The score vector $U\left(\underline{\xi}\right) = (\partial\ell/\partial a\,, \partial\ell/\partial b\,, \partial\ell/\partial\alpha\,, \partial\ell/\partial\beta\,, \partial\ell/\partial\delta)^T$, where the components corresponding to the parameters in $\underline{\xi}$ are given by differentiating (16). By setting $z_i = 1 + \left(\frac{x_i-\lambda}{\beta}\right)^{\delta}$

$$\frac{\partial\ell}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\log(1 - {z_i}^{-\alpha}) - (b-1)\sum_{i=1}^{n}\frac{(1-{z_i}^{-\alpha})^a\log(1-{z_i}^{-\alpha})}{1-(1-{z_i}^{-\alpha})^a}$$

$$\frac{\partial\ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log(1-(1-{z_i}^{-\alpha})^a)$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log z_i + (a-1)\sum_{i=1}^{n}\frac{\log z_i}{({z_i}^{\alpha}-1)} - a(b-1)\sum_{i=1}^{n}\frac{{z_i}^{-\alpha}\log z_i}{(1-{z_i}^{-\alpha})^{-(a-1)} - (1-{z_i}^{-\alpha})}$$

$$\frac{\partial\ell}{\partial\beta} = -\frac{n\delta}{\beta} + \frac{\delta(\alpha+1)}{\beta}\sum_{i=1}^{n}(1-{z_i}^{-1}) - \frac{\delta\alpha(a-1)}{\beta}\sum_{i=1}^{n}\frac{{z_i}^{-\alpha}-{z_i}^{-(\alpha+1)}}{1-{z_i}^{-\alpha}} + \frac{\alpha\delta a(b-1)}{\beta}\sum_{i=1}^{n}\frac{{z_i}^{-\alpha}-{z_i}^{-(\alpha+1)}}{(1-{z_i}^{-\alpha})^{-(a-1)}-(1-{z_i}^{-\alpha})}$$

and

$$\frac{\partial\ell}{\partial\delta} = \frac{n}{\delta} + \sum_{i=1}^{n}\log(z_i-1) - \frac{(\alpha+1)}{\delta}\sum_{i=1}^{n}(1-{z_i}^{-1})\log(z_i-1) + \frac{\alpha(a-1)}{\delta}\sum_{i=1}^{n}\frac{(z_i-1)\log(z_i-1)}{{z_i}^{\alpha+1}-z_i} - \frac{a\alpha(b-1)}{\delta}\sum_{i=1}^{n}\frac{\left({z_i}^{-\alpha}-{z_i}^{-(\alpha+1)}\right)\log(z_i-1)}{(1-{z_i}^{-\alpha})^{-a+1}-(1-{z_i}^{-\alpha})}$$

The maximum likelihood estimates (MLEs) of the parameters are the solutions of the nonlinear equations$\nabla\ell = 0$, which are solved iteratively. The observed information matrix given

$$J_n\left(\underline{\xi}\right) = n\begin{pmatrix} J_{aa} & J_{ab} & J_{a\alpha} & J_{a\beta} & J_{a\delta} \\ J_{ba} & J_{bb} & J_{b\alpha} & J_{b\beta} & J_{b\delta} \\ J_{\alpha a} & J_{\alpha b} & J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\delta} \\ J_{\beta a} & J_{\beta b} & J_{\beta\alpha} & J_{\beta\beta} & J_{\beta\delta} \\ J_{\delta a} & J_{\delta b} & J_{\delta\alpha} & J_{\delta\beta} & J_{\delta\delta} \end{pmatrix}$$

whose elements are,

$$J_{aa} = -\frac{n}{a^2} - (b-1)\sum_{i=1}^{n}\frac{(1-{z_i}^{-\alpha})^a(\log(1-{z_i}^{-\alpha}))^2}{(1-(1-{z_i}^{-\alpha})^a)^2}$$

$$J_{ab} = -\sum_{i=1}^{n}\frac{(1-{z_i}^{-\alpha})^a\log(1-{z_i}^{-\alpha})}{1-(1-{z_i}^{-\alpha})^a}$$

$$J_{a\alpha} = \sum_{i=1}^{n}({z_i}^{\alpha}-1)\log z_i - (b-1)\sum_{i=1}^{n}\frac{((1-{z_i}^{-\alpha})^{-a}-1)\frac{\log z_i}{({z_i}^{\alpha}-1)} + a{z_i}^{-\alpha}(1-{z_i}^{-\alpha})^{-(a+1)}\log(1-{z_i}^{-\alpha})\log z_i}{((1-{z_i}^{-\alpha})^{-a}-1)^2}$$

$$J_{a\beta} = -\frac{\delta\alpha}{\beta}\sum_{i=1}^{n}\frac{(z_i-1)}{({z_i}^{(a+1)}-z_i)} - \frac{\delta\alpha(b-1)}{\beta}\sum_{i=1}^{n}\frac{\left({z_i}^{-\alpha}-{z_i}^{-(\alpha+1)}\right)[(1-{z_i}^{-\alpha})^{-a}\{a\log(1-{z_i}^{-\alpha})+1\}-1]}{(1-{z_i}^{-\alpha})[(1-{z_i}^{-\alpha})^{-a}-1]^2}$$

$$J_{a\delta} = \frac{\alpha}{\delta}\sum_{i=1}^{n}\frac{\left({z_i}^{-\alpha}-{z_i}^{-(\alpha+1)}\right)}{(1-{z_i}^{-\alpha})} - \frac{\alpha(b-1)}{\delta}\sum_{i=1}^{n}\frac{\left({z_i}^{-\alpha}-{z_i}^{-(\alpha+1)}\right)\log(z_i-1)\,[(1-{z_i}^{-\alpha})^{-a}(a\log(1-{z_i}^{-\alpha})+1)-1]}{(1-{z_i}^{-\alpha})[(1-{z_i}^{-\alpha})^{-a}-1]^2}$$

$$J_{bb} = -\frac{n}{b^2}J_{b\alpha} = -a\sum_{i=1}^{n}\frac{{z_i}^{-\alpha}\log z_i}{(1-{z_i}^{-\alpha})^{-a+1}-(1-{z_i}^{-\alpha})}$$

$$J_{b\beta} = \frac{a\alpha\delta}{\beta}\sum_{i=1}^{n}\frac{z_i^{-\alpha} - z_i^{-(\alpha+1)}}{(1-z_i^{-\alpha})^{-(a-1)} - (1-z_i^{-\alpha})} \qquad J_{b\delta} = -\frac{a\alpha}{\beta}\sum_{i=1}^{n}\frac{\left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)\log(z_i-1)}{(1-z_i^{-\alpha})^{-(a-1)} - (1-z_i^{-\alpha})}$$

$$J_{\alpha\alpha} = -\frac{n}{\alpha^2} - (a-1)\sum_{i=1}^{n}\frac{z_i^{\alpha}(\log z_i)^2}{(z_i^{\alpha}-1)^2}$$
$$+a(b-1)\sum_{i=1}^{n}\frac{z_i^{-\alpha}(\log z_i)^2\{(1-z_i^{-\alpha})^{-a}[1-(a-1)z_i^{-\alpha}] + z_i^{-2\alpha} - 1\}}{[(1-z_i^{-\alpha})^{-(a-1)} - (1-z_i^{-\alpha})]^2}$$

$$J_{\alpha\beta} = \frac{\delta}{\beta}\sum_{i=1}^{n}(1-z_i^{-1}) + \frac{\delta(a-1)}{\beta}\sum_{i=1}^{n}\frac{\left(z_i^{\alpha} - z_i^{(\alpha-1)}\right)(\alpha\log z_i - 1) + (1-z_i^{-1})}{(z_i^{\alpha}-1)^2}$$
$$-\frac{a\delta(b-1)}{\beta}\sum_{i=1}^{n}\frac{z_i^{\alpha}(M_i - L_i)}{[(1-z_i^{-\alpha})^{-(a-1)} - (1-z_i^{-\alpha})]^2}$$

where

$$M_i = \left((1-z_i^{-\alpha})^{-(a-1)} - (1-z_i^{-\alpha})\right)(\alpha\log z_i - 1)(1-z_i^{-1})$$

$$L_i = \left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)\log z_i\,[(a-1)(1-z_i^{-\alpha})^{-a} + 1]$$

$$J_{\alpha\delta} = \frac{1}{\beta}\sum_{i=1}^{n}(1-z_i^{-1})\log(z_i-1) + \frac{(a-1)}{\beta}\sum_{i=1}^{n}\frac{(z_i^{\alpha}-1)(1-z_i^{-1})\log(z_i-1) - \alpha\left(z_i^{\alpha} - z_i^{(\alpha-1)}\right)(\log z_i)^2}{(z_i^{\alpha}-1)^2}$$
$$-\frac{a(b-1)}{\beta}\sum_{i=1}^{n}\frac{A_i + D_i}{[(1-z_i^{-\alpha})^{-(a-1)} - (1-z_i^{-\alpha})]^2}$$

where

$$A_i = \left[(1-z_i^{-\alpha})^{-(a-1)} - (1-z_i^{-\alpha})\right]\left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)\log(z_i-1)\,(1-\alpha\log z_i)$$

$$D_i = \alpha z_i^{-\alpha}\log z_i\left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)\log(z_i-1)\left((a-1)(1-z_i^{-\alpha})^{-a} + 1\right)$$

$$J_{\beta\beta} = \frac{n\delta}{\beta^2} + \frac{\delta(\alpha+1)}{\beta^2}\left[\delta\sum_{i=1}^{n}z_i^{-2}(z_i-1) + \sum_{i=1}^{n}(1-z_i^{-1})\right]$$
$$-\frac{\alpha\delta(a-1)}{\beta^2}\sum_{i=1}^{n}\frac{\left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)\left\{(1-z_i^{-\alpha})[\alpha\delta - \delta(\alpha+1)z_i^{-1} + 1] + \alpha\delta\left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)\right\}}{(1-z_i^{-\alpha})^2}$$
$$+\frac{a\alpha\delta(b-1)}{\beta^2}\sum_{i=1}^{n}\frac{\left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)\{T_i - R_i\}}{[(1-z_i^{-\alpha})^{-(a-1)} - (1-z_i^{-\alpha})]^2}$$

Where

$$T_i = \left[(1-z_i^{-\alpha})^{-(a-1)} - (1-z_i^{-\alpha})\right](\alpha\delta - \delta(\alpha+1)z_i^{-1} + 1)$$

$$R_i = \left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)[\alpha\delta(a-1)(1-z_i^{-\alpha})^{-a} + \alpha\delta]$$

$$J_{\beta\delta}$$
$$= -\frac{n}{\beta} + \frac{\alpha+1}{\beta}\sum_{i=1}^{n}(1 + (z_i^{-1} - z_i^{-2})(\log(z_i-1)) - z_i^{-1})$$
$$-\frac{\alpha(a-1)}{\beta}\sum_{i=1}^{n}\frac{\left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)\left\{(1-z_i^{-\alpha})\left[1 + (\log(z_i-1))\left((\alpha+1)z_i^{-1} - \alpha\right)\right] - \alpha\left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)(\log(z_i-1))\right\}}{(1-z_i^{-\alpha})^2}$$
$$+\frac{a\alpha(b-1)}{\beta}\sum_{i=1}^{n}\frac{\left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)\{G_i + P_i\}}{[(1-z_i^{-\alpha})^{-(a-1)} - (1-z_i^{-\alpha})]^2}$$

where

$$G_i = \left[(1-z_i^{-\alpha})^{-(a-1)} - (1-z_i^{-\alpha})\right]\left\{1 - (\log(z_i-1))\left((\alpha+1)z_i^{-1} + \alpha\right)\right\}$$

$$P_i = \alpha\left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)(\log(z_i-1))[1 + (a-1)(1-z_i^{-\alpha})^{-a}]$$

$$J_{\delta\delta}$$
$$= -\frac{n}{\delta^2} + \frac{1}{\delta}\sum_{i=1}^{n}\log(z_i - 1) - \frac{(\alpha+1)}{\delta^2}\sum_{i=1}^{n}(z_i^{-1} - z_i^{-2})\log(z_i - 1)$$
$$+ \frac{\alpha(a-1)}{\delta^2}\sum_{i=1}^{n}\frac{(z_i-1)\log(z_i-1)\{({z_i}^{\alpha+1} - z_i)(1+\log(z_i-1)) - (z_i-1)\log(z_i-1)\left((\alpha+1)z_i^{\alpha} - 1\right) - ({z_i}^{\alpha+1} - z_i)\}}{({z_i}^{\alpha+1} - z_i)^2}$$
$$- \frac{a(b-1)\alpha}{\delta^2}\sum_{i=1}^{n}\frac{\left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)(\log(z_i-1))\left\{Y_i + O_i - \left((1-{z_i}^{-\alpha})^{-(a-1)} - (1-{z_i}^{-\alpha})\right)\right\}}{\left((1-{z_i}^{-\alpha})^{-(a-1)} - (1-{z_i}^{-\alpha})\right)^2}$$

Where$Y_i = \left((1-{z_i}^{-\alpha})^{-(a-1)} - (1-{z_i}^{-\alpha})\right)\left\{1 + \left((\alpha+1)z_i^{-1} - \alpha\right)\log(z_i-1)\right\}$

$$O_i = \left(z_i^{-\alpha} - z_i^{-(\alpha+1)}\right)(\log(z_i-1))[\alpha + \alpha(a-1)(1-z_i^{-\alpha})^{-a}]$$

Explicit expressions for the remaining elements of J follow by symmetry.

### *10 - Empirical Applications:*

In this section, we illustrate the usefulness of the *KumGP* distribution.

### *10.1 Simulation study:*

We conducted Monte Carlo simulation studies to assess on the finite sample behavior of the maximum likelihood estimators of$a\,, b, \alpha,\ \beta$and $\delta$ All results were obtained from 1000 Monte Carlo replications and the simulations were carried out using the statistical software Mathcad , The true parameter values used in the data generating processes are$a = 3.1,\ b = 1.2,\ \alpha = 1.7,\ \beta = 4$and $\delta = 3.2$ , Table 1 presents the mean maximum likelihood estimates of the parameters that index the *KumGP*distribution along with the respective root mean squared errors (RMSE) and bias for sample sizes$n = 30\,, 50, 80, 100$

**Table 1:** Mean estimates, root mean squared errorsandbiasof$a\,, b, \alpha,\ \beta$and $\delta$ the maximum likelihood Estimators of the KW-GP parameters.

| $n$ | Parameter | Mean | RMSE | bias |
|---|---|---|---|---|
| 30 | $a$ | 2.632 | 0.201 | 0.632 |
| | $b$ | 0.793 | 1.624 | -2.207 |
| | $\alpha$ | 1.186 | 1.097 | -1.814 |
| | $\beta$ | 3.611 | 1.314 | 0.806 |
| | $\delta$ | 3.669 | 1.403 | 1.669 |
| 50 | $a$ | 2.63 | 0.199 | 0.63 |
| | $b$ | 0.791 | 1.621 | -2.205 |
| | $\alpha$ | 1.184 | 1.095 | -1.813 |
| | $\beta$ | 3.418 | 1.015 | 0.709 |
| | $\delta$ | 3.667 | 1.402 | 1.667 |
| 80 | $a$ | 2.632 | 0.21 | 0.632 |
| | $b$ | 0.785 | 1.611 | -2.123 |
| | $\alpha$ | 1.183 | 1.091 | -1.811 |
| | $\beta$ | 3.411 | 1.00 | 0.707 |
| | $\delta$ | 3.527 | 2.613 | 1.527 |
| 100 | $a$ | 2.63 | 0.198 | 0.63 |
| | $b$ | 0.677 | 1.799 | -2.323 |
| | $\alpha$ | 1.187 | 1.095 | -1.808 |
| | $\beta$ | 3.27 | 0.81 | 0.706 |
| | $\delta$ | 3.612 | 2.692 | 1.612 |

From the results in Table 1, we notice that the biases and root mean squared errors of the maximum likelihood estimators of $a\,, b, \alpha,\ \beta$and $\delta$decay toward zero as the sample size increases, as expected. We also note that there is small sample bias in the estimation of the parameters that index the *KWGP* distribution. Future research should obtain bias corrections for these estimators.

### *10.2Real Data Applications:*

In this section we use several real data sets to compare the fits of a Kw-GP distribution with those of other sub-models, i.e., the Exponentiated Generalized Pareto (EGP), GP and Pareto distributions. In each case parameters are estimated via the MLE method described in Section 9 using the MATHCAD software. First we describe the data sets. Then we report the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion) and BIC (Bayesian Information Criterion) statistics

.

$$AIC = -2\ell(\hat{\theta}) + 2q, \qquad BIC = -2\ell(\hat{\theta}) + qlog(n) CAIC = -2\ell(\hat{\theta}) + \frac{2qn}{n-q-1}$$

Where $\ell(\hat{\theta})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, q is the number of parameters, and n is the sample size.Next, we shall compare the proposed KwGP distribution (and their sub-models) with several other lifetime distributions data set,KumaraswamyFréchet distributionKwF (Mead, *et al*. (2014)),the beta Fréchet(BF) (Nadarajah and Gupta, (2004) and Souza *et al*., (2011)).Finally, we perform the Kolmogorov-Smirnov (K-S) statistic and $-2\ell(\hat{\theta})$ tests andplot histograms of each data set to provide a visual comparison of the fitted density functions.

### *10-1The Strengths of 1.5 Cm Glass Fibres:*

Here,the data set is obtained from (Fatone*t al*.(2013)).The data are consisting of 63 of the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. The data are listed in the next table

**Table 2:** The Strengths of 1.5 cm Glass Fibres Data Set.

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.55 | 0.93 | 1.25 | 1.36 | 1.49 | 1.52 | 1.58 |
| 1.61 | 1.64 | 1.68 | 1.73 | 1.81 | 2.00 | 0.74 |
| 1.04 | 1.27 | 1.39 | 1.49 | 1.53 | 1.59 | 1.61 |
| 1.66 | 1.68 | 1.76 | 1.82 | 2.01 | 0.77 | 1.11 |
| 1.28 | 1.42 | 1.5 | 1.54 | 1.6 | 1.62 | 1.66 |
| 1.69 | 1.76 | 1.84 | 2.24 | 0.81 | 1.13 | 1.29 |
| 1.48 | 1.5 | 1.55 | 1.61 | 1.62 | 1.66 | 1.7 |
| 1.77 | 1.84 | 0.84 | 1.24 | 1.3 | 1.48 | 1.51 |
| 1.55 | 1.61 | 1.63 | 1.67 | 1.7 | 1.78 | 1.89 |

### *10-2 Uncensored Data "Carbon Fibers":*

Here, the real data set will use here to compare the fits of the Kum-GP distribution and those of other sub-models, i.e., the Exponentiated Generalized Pareto (EGP), GP and Pareto distributions. Considering an uncensored data set correspondingan uncensored data set from Nichols and Padgett (2006) consisting of 100 observations on breaking stress of carbon fibers (in Gba):

**Table 3:** On breaking stress of carbon fibers set.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1.41 | 0.39 | 2.97 | 1.36 | 0.98 | 2.76 | 4.91 | 3.68 | 1.84 | 1.59 |
| 1.57 | 1.08 | 2.03 | 1.61 | 2.12 | 1.89 | 2.88 | 2.82 | 2.05 | 3.65 |
| 1.84 | 1.17 | 3.68 | 2.48 | 0.85 | 1.61 | 2.79 | 4.7 | 2.03 | 1.8 |
| 2.17 | 1.57 | 5.08 | 2.48 | 1.18 | 3.51 | 2.17 | 1.69 | 1.25 | 4.38 |
| 3.15 | 2.35 | 2.55 | 2.59 | 2.38 | 2.81 | 2.77 | 2.17 | 2.83 | 1.92 |
| 3.19 | 2.41 | 0.81 | 5.56 | 1.73 | 1.59 | 2 | 1.22 | 1.12 | 1.71 |
| 3.39 | 2.43 | 4.2 | 3.33 | 2.55 | 3.31 | 3.31 | 2.85 | 2.56 | 3.56 |
| 3.7 | 2.74 | 2.73 | 2.5 | 3.6 | 3.11 | 3.27 | 2.87 | 1.47 | 3.11 |
| 3.75 | 2.81 | 2.95 | 2.97 | 3.39 | 2.96 | 2.53 | 2.67 | 2.93 | 3.22 |
| 4.42 | 3.68 | 3.19 | 3.22 | 1.69 | 3.28 | 3.09 | 1.87 | 3.15 | 4.9 |

**Table 4:** MLEs of the model parameters, the corresponding SEs (given in parentheses) and the statistics AIC, BIC and CAIC.

| | ***Model*** | ***Estimates*** | | | | | ***Statistic*** | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{a}$ | $\hat{b}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\delta}$ | *AIC* | *BIC* | *CAIC* |
| Glass Fibres | $Kum-GP$ | 0.9686 | 54.555 | 49.658 | 20.9979 | 2.6843 | 32.296 | 43.012 | 33.349 |
| | | (0.141) | (60.634) | (4.563) | (11.666) | (0.031) | | | |
| | $EGP$ | 1.0388 | --- | 205.9286 | 8.09 | 2.6449 | 32.496 | 43.211 | 33.548 |
| | | (0.185) | | (18.245) | (1.441) | (0.045) | | | |
| | $GP$ | | | 227.0884 | 7.6791 | 2.7201 | 62.218 | 75.076 | 63.718 |
| | | --- | --- | (20.244) | (3.6) | (0.046) | | | |
| | KwF | 5.50397 | 857.34273 | | 2.11623 | 0.74044 | 47.621 | 56.193 | 45.306 |
| | | (7.982) | (153.948) | ---- | (4.555) | (0.071) | | | |
| | BF | 19.59068 | 30.41091 | | 1.33081 | 0.6849 | 69.735 | 78.307 | 67.421 |
| | | (18.115) | (18.238) | | (1.085) | (0.181) | | | |
| Carbon Fibers | $Kum-GP$ | 1.523 | 2.858 | 0.408 | 1.837 | 6.836 | 172.265 | 187.896 | 173.168 |
| | | (0.00371) | (0.1111) | (0.00074) | (0.00001) | (0.000002) | | | |
| | $EGP$ | 1.518 | --- | 0.393 | 1.838 | 6.835 | 201.747 | 217.378 | 202.65 |
| | | (0.00609) | | (0.00113) | (0.00015) | (0.00017) | | | |
| | $GP$ | --- | --- | 0.291 | 1.847 | 6.829 | 196.432 | 212.063 | 197.335 |
| | | | | (0.01829) | (0.00015) | (0.01579) | | | |
| | KwF | 6.76357 | 904.34345 | --- | 2.90998 | **0.38843** | 292.926 | 303.347 | 291.035 |
| | | (2.393) | (61.863) | | (2.259) | **(0.028)** | | | |
| | BF | 0.42934 | 138.06644 | --- | 34.38484 | 0.72474 | 293.733 | 304.154 | 291.842 |
| | | (0.236) | (113.552) | | (21.52) | (0.19) | | | |

Since the values of the AIC, BIC and CAIC are smaller for the Kum-GP distribution compared with those values of the other models, the Kum-GP distribution seems to be a very competitive model to

these data. In summary, the proposed KumGP distribution produces better fits to the data than its sub-models.

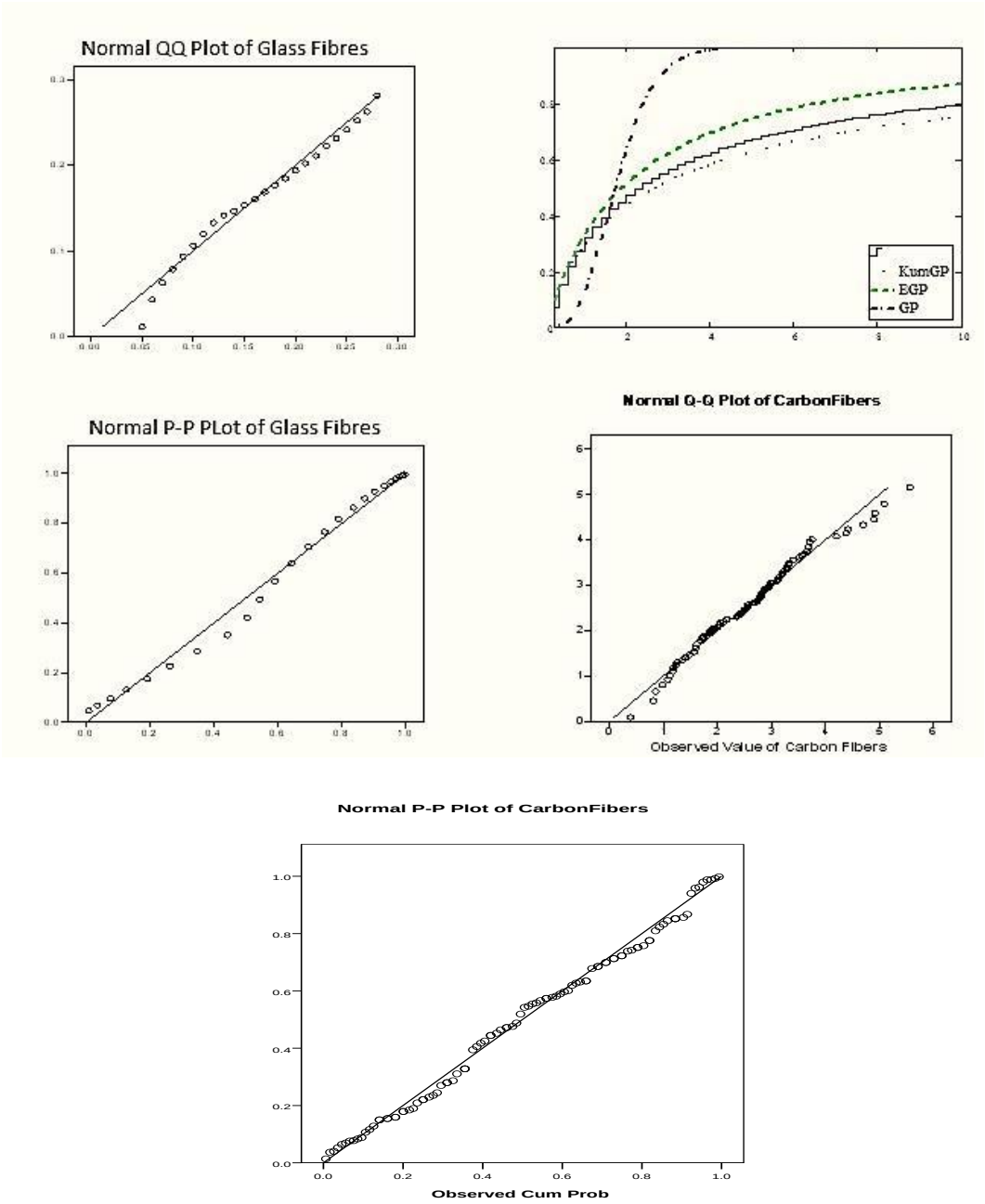


**Fig. 4:** The FittedQ-Q Plots and P-P Plots for the 63 ofthe strengths of 1.5 cm glass fibres data set& 100 observations on breaking stress of carbon fibers and Empirical CDF.

**Table 5:** K-S and $-2\ell(\hat{\theta})$statistics for the chosen Real data.

| Data | Model | *KumGP* | *EGP* | *GP* |
|---|---|---|---|---|
| *Glass Fibres* | $K-S$ | 0.224 | 0.185 | 0.225 |
| | $-2\ell(\hat{\theta})$ | 22.296 | 22.496 | 21.809 |
| *Carbon Fibers* | $K-S$ | 0.217 | 0.298 | 0.231 |
| | $-2\ell(\hat{\theta})$ | 188.337 | 238.592 | 172.745 |

***Concluding Remarks:***

The well-known generalized Pareto distribution is extended by introducing two extra shape parameters, thus defining the Kumaraswamy generalized Pareto (Kum-GP) distribution having a broader class of hazard rate and density functions. This is achieved by taking (5) as the baseline cumulative distribution of the generalized class of

beta distributions. A detailed study on the mathematical properties of the new distribution is presented. The new model includes as special sub-models as exponentiated generalized Pareto (EGP) and itssub models. We obtain the ordinary moments, and Rènyi entropy. The estimation of the model parameters is approached by maximum likelihood and the observed information matrix is obtained. An application to a real data set indicates that the fit of the new model is superior to the fits of its principal sub-models. We hope that the proposed model may be interesting for a wider range of statistical research.

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