



AENSI Journals

Australian Journal of Basic and Applied Sciences

ISSN:1991-8178

Journal home page: www.ajbasweb.com



Some Stability Conditions and Synchronization of Fractional Order Pan System

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ARTICLE INFO

Article history:

Received 25 January 2014

Received in revised form

8 April 2014

Accepted 20 April 2014

Available online 10 May 2014

Keywords:

Pan system, stability, Fractional

Routh-Hurwitz Conditions,

synchronization

ABSTRACT

In this work, stability analysis of the fractional order modified autonomous Pan system are studied using the fractional Routh-Hurwitz criteria. A necessary condition for this systems to remain chaotic is obtained. Furthermore synchronization is also achieved using the unidirectional linear error feedback coupled approach.

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To Cite This Article: Ala'a Abdul-Raheem Hammodat, Sawsan Sami Ismail., Some Stability Conditions and Synchronization of Fractional Order Pan System. *Aust. J. Basic & Appl. Sci.*, 8(6): 423-428, 2014

INTRODUCTION

As a 300 year old mathematical topic, functional calculus has always attracted the interest of many famous ancient mathematicians, including Leibniz, Liouville, Riemann, Grunwald and Letnikov (Butzer, P., U. Westphal, 2000).

Although it has long history, it was not applied in our real life because it seems to be more difficult and fewer theories have been established than for classical differential equations. In recent decades, fractional order differential equations have been found to be a powerful tool in more and more fields, such as physics, chemistry, biology, economics, and other complex systems (Arena, P., R. Caponetto, 2000; Hilfer, R., others, 2000). The interest in the study of fractional order nonlinear systems lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties, which are not taken into account in the classical integer-order models.

There are several definitions for the fractional order differential operator, but the following definition is most used:

$$D^\alpha y(x) = J^{m-\alpha} y^{(m)}(x), \alpha > 0$$

Where $m = \lceil \alpha \rceil$, i.e., m is the first integer which is not less than α , $y^{(m)}$ is the general m -order derivative, J^β is the β -order Riemann - Liouville integral operator (Caputo, M., 1967), which is expressed as follows:

$$J^\beta z(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} z(t) dt, \beta > 0$$

The operator D^α is generally called " α -order Caputo differential operator". If the initial value $\alpha = 0$, D^α is denoted by D_*^α .

Chaotic attractors have been found in the following fractional order systems, Lorenz system (Grigorenko, I., E. Grigorenko, 2003), Chua (Hartly, T.T, 1995), Chen (Li, C., G. Peng, 2004) and Lu (2006). Chaos control synchronization in integer order differential systems are well understood (Lakshmanan, M., K. Murali, 1996; Matouk, A.E., H.N. Agiza, 2008), but they are still in the beginning in the case of fractional order chaotic systems. Recently, many papers about chaos control and synchronization in fractional order chaotic systems have been published by authors (Ahmed, W.M., A.M. Harb, 2003; Zhou, T., C. Li, 2005).

In this work, we study stability, chaos, control and synchronization in the pan system (Pan, L., 2010) with same fractional order. we use the Routh-Hurwitz conditions given in (Ahmed, E., 2006) to study the stability

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conditions in this system. The sufficient conditions for chaos control are derived analytically using linear feedback control technique, as well as synchronization of this system is studied, where it is shown that the global synchronization of the chaotic pan system can be achieved by unidirectional linear coupled approach.

The Fractional order Pan System:

Pan designed the Pan attractor in 2010, and the originally theoretical equations were later found to be useful in modeling equilibrium in chemical reactions. The definition equations are:

$$\left. \begin{aligned} \frac{dx}{dt} &= a(y-x) \\ \frac{dy}{dt} &= bx-xz \\ \frac{dz}{dt} &= xy-cz \end{aligned} \right\} \quad (1)$$

Where x, y, z are state variables, and a, b, c are positive integers (AL-Azzawi, S.F., 2012; Pan, L., 2010; Pan, L., 2011).

Here, we consider the fractional order system denoted as system (2) below. The standard derivative is replaced by the fractional derivative as the followings:

$$\left. \begin{aligned} \frac{d^{\alpha_1}x}{dt^{\alpha_1}} &= a(y-x) \\ \frac{d^{\alpha_2}y}{dt^{\alpha_2}} &= bx-xz \\ \frac{d^{\alpha_3}z}{dt^{\alpha_3}} &= xy-cz \end{aligned} \right\} \quad (2)$$

Where $0 < \alpha_i \leq 1$ ($i=1,2,3$) is the derivative order, and a, b, c are parameters. when $\alpha=1$ system (2) become system (1) which exhibits chaotic behaviors at the parameters value $a=10, b=16, c=8/3$ as shown in the figure (1).

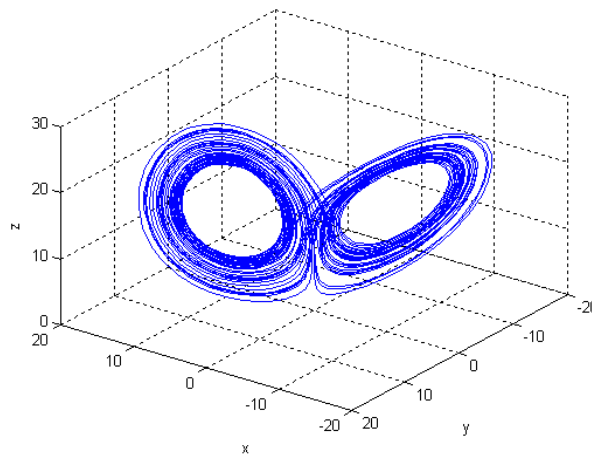


Fig. 1: $x-y-z$ Phase Plane The Pan Chaotic Attractor When $a=10, b=16, c=8/3$.

Note that The equilibrium points of system (2) are $P_0(0,0,0)$, $P_{\mp}(\mp\sqrt{bc}, \mp\sqrt{bc}, b)$.

Conditions for Stability in 3-Dimensional Fractional order Systems:

The stability conditions of fractional order systems given in (Ahmed, E., 2006) are shown in the following: Consider the 3-dimensional fractional order system

$$\frac{d^{\alpha}}{dt^{\alpha}}x(t) = f(x, y, z), \quad \frac{d^{\alpha}}{dt^{\alpha}}y(t) = g(x, y, z), \quad \frac{d^{\alpha}}{dt^{\alpha}}z(t) = h(x, y, z)$$

Where $\alpha \in (0,1]$ and (x_e, y_e, z_e) is an equilibrium solution of (3).

The eigenvalues equation of the equilibrium point (x_e, y_e, z_e) is given by the following polynomials:

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \quad (4)$$

And its discriminant $D(p)$ is given as:

$$D(p) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2 \quad (5)$$

Then (x_e, y_e, z_e) is locally asymptotically stable if all the roots of equation (4) satisfy the condition $|\arg(\lambda)| > \alpha\pi/2$. Consequently, we have the following stability conditions:

1. If $D(p) > 0$, then the necessary and sufficient condition for the equilibrium point (x_e, y_e, z_e) , to be locally asymptotically stable, is

$$a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0.$$

2. If $D(p) < 0$, $a_1 \geq 0, a_2 \geq 0, a_3 > 0$, then (x_e, y_e, z_e) is locally asymptotically stable for $\alpha < 2/3$. However if $D(p) < 0, a_1 < 0, a_2 < 0, \alpha > 2/3$, then all roots of equation (4) satisfy the condition $|\arg(\lambda)| < \alpha\pi/2$.

3. If $D(p) < 0, a_1 > 0, a_2 > 0, a_1a_2 - a_3 = 0$, then (x_e, y_e, z_e) is locally asymptotically stable for all $\alpha \in (0,1)$.

4. The necessary condition for the equilibrium point (x_e, y_e, z_e) , to be locally asymptotically stable is $a_3 > 0$.

The stability region of the fractional order system with order α is illustrated in Fig. (2) (in which σ, w refer to the real and imaginary parts of the eigenvalues, respectively, and $j = \sqrt{-1}$). From Fig. (2), it is easy to show that the stability region of the fractional order case is greater than the stability region of the integer order case.

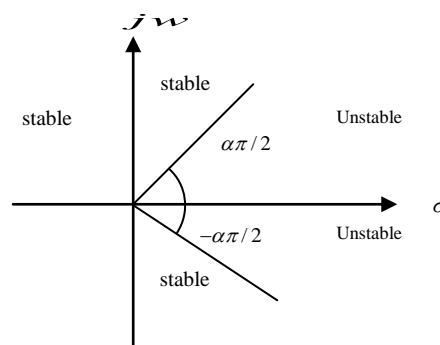


Fig. 2: Stability region of fractional order system.

A. Stability Conditions of The Equilibrium Point P_0 :

The eigenvalues equation of the equilibrium point P_0 is given by:

$$f(\lambda) = \lambda^3 + (a+c)\lambda^2 + a(c-b)\lambda - abc = 0 \quad (6)$$

When $b < 0$, $a, c > 0$ and $D(p) > 0$, then using the stability condition (1), the equilibrium point P_0 is locally asymptotically stable for all $\alpha \in (0,1]$. However if $D(p) < 0$ then P_0 is locally asymptotically stable for $\alpha < 2/3$ or if $D(p) < 0$ and $b = \frac{c(a+c)}{a}$, then P_0 is locally asymptotically stable for all $\alpha \in (0,1)$ [using the stability conditions (2) and (3) respectively].

When $b > 0$, then $a_3 < 0$ and by using the condition (4), we can easily see that P_0 is unstable.

B. Stability Conditions of The Equilibrium Point P_{\pm} :

When $a, b, c > 0$ two other equilibrium points P_+ and P_- appear and they have the following eigenvalues equations:

$$f(\lambda) = \lambda^3 + (a+c)\lambda^2 + c(a+b)\lambda + 2abc = 0 \quad (7)$$

If $D(p) > 0$, then the necessary and sufficient condition for the equilibrium points P_+ and P_- to be locally asymptotically stable for all $\alpha \in (0,1]$ is $b < \frac{a(a+c)}{a-c}$ and $a > c$ [from stability condition (1)]. But when

$D(p) < 0$, then P_+ and P_- are locally asymptotically stable for $\alpha < 2/3$ or if $D(p) < 0$ and $b = \frac{a(a+c)}{a-c}$,

$a > c$, then P_+ and P_- are locally asymptotically stable for all $\alpha \in (0,1)$ [using stability condition (2) and (3) respectively]. While if $b > \frac{a(a+c)}{a-c}$ or $c > a$ then P_+ and P_- is unstable.

Synchronization of Fractional order Pan System:

In the following, the unidirectional linear error feedback coupled scheme is used to achieve chaos synchronization between two identical order pan systems. The derive (master) system is described by:

$$\left. \begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= a(y_m - x_m) \\ \frac{d^\alpha y}{dt^\alpha} &= bx_m - x_m z_m \\ \frac{d^\alpha z}{dt^\alpha} &= x_m y_m - cz_m \end{aligned} \right\} \quad (8)$$

And the response (slave) system is given by:

$$\left. \begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= a(y_s - x_s) + u_1 \\ \frac{d^\alpha y}{dt^\alpha} &= bx_s - x_s z_s + u_2 \\ \frac{d^\alpha z}{dt^\alpha} &= x_s y_s - cz_s + u_3 \end{aligned} \right\} \quad (9)$$

Where the controller $u = [u_1 \ u_2 \ u_3]^T$ is define as follows:

$$u_1 = k_1(x_m - x_s) \quad , \quad u_2 = k_2(y_m - y_s) \quad , \quad u_3 = k_3(z_m - z_s) \quad (10)$$

Consider the case of the integer – order systems (when $\alpha = 1$) and then by subtracting (9) from (8), we obtain:

$$\left. \begin{aligned} \frac{de_1}{dt} &= a(e_2 - e_1) - k_1 e_1 \\ \frac{de_2}{dt} &= be_1 - x_m z_m + x_s z_s - k_2 e_2 \\ \frac{de_3}{dt} &= -(c + k_3)e_3 + x_m y_m - x_s y_s \end{aligned} \right\} \quad (11)$$

And we can rewrite (11) as follows:

$$\frac{de}{dt} = (A - K - M_{x_m, x_s})e \quad (12)$$

Where

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} x_m - x_s \\ y_m - y_s \\ z_m - z_s \end{bmatrix}, \quad A = \begin{bmatrix} -a & a & 0 \\ b & 0 & 0 \\ 0 & 0 & -c \end{bmatrix}, \quad K = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}$$

$$M_{x_m, x_s} = \begin{bmatrix} 0 & 0 & 0 \\ -z_m & 0 & -x_s \\ y_m & x_s & 0 \end{bmatrix}, \quad P = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}$$

By selecting the feedback control gains k_1, k_2 and k_3 that must satisfy the condition (10) given in [Jian G.P., Tang K.S., Chen G., 2003], the two coupled integer-order pan systems are asymptotically synchronized, i.e. synchronization is achieved if the feedback control gains satisfy the following inequalities:

$$\left. \begin{aligned} k_1 &\geq \frac{1}{2}(-2a + |a + b - z_m| + |y_m| - \mu) \\ k_2 &\geq \frac{1}{2}(|a + b - z_m| - \mu) \\ k_3 &\geq \frac{1}{2}(-2c + |y_m| - \mu) \end{aligned} \right\} \quad (13)$$

Theorem 1:

For the two coupled chaotic pan systems (8) and (9) if k_1, k_2 and k_3 are selected such that the inequality (13) holds and $\alpha = 1$, then these two coupled chaotic pan systems (8) and (9) are globally asymptotically synchronized.

Numerical simulations:

Fourth-order Runge-Kutta method is used to solve the systems of differential equations (8) and (9). In addition a time step size 0.001 is employed. We selected the parameters of chaotic pan system as $a=10, b=16$ and $c=8/3$, the initial states of the master systems are $x_{m0}=1, y_{m0}=2$ and $z_{m0}=5$ and the initial states of the slave systems are $x_{s0}=5, y_{s0}=-3$ and $z_{s0}=8$, hence the error systems has the initial values $e_{10}=-4, e_{20}=5$ and $e_{30}=-3$, selecting $\mu=-0.5, p_1=p_2=p_3=1$

And the coupled coefficients as $k_1=10, k_2=40$ and $k_3=8$ which satisfies the inequality (13). According to theorem 1, we know that the two coupled chaotic pan systems (8) and (9) are globally asymptotically synchronization. The simulation results are shown in fig (3).

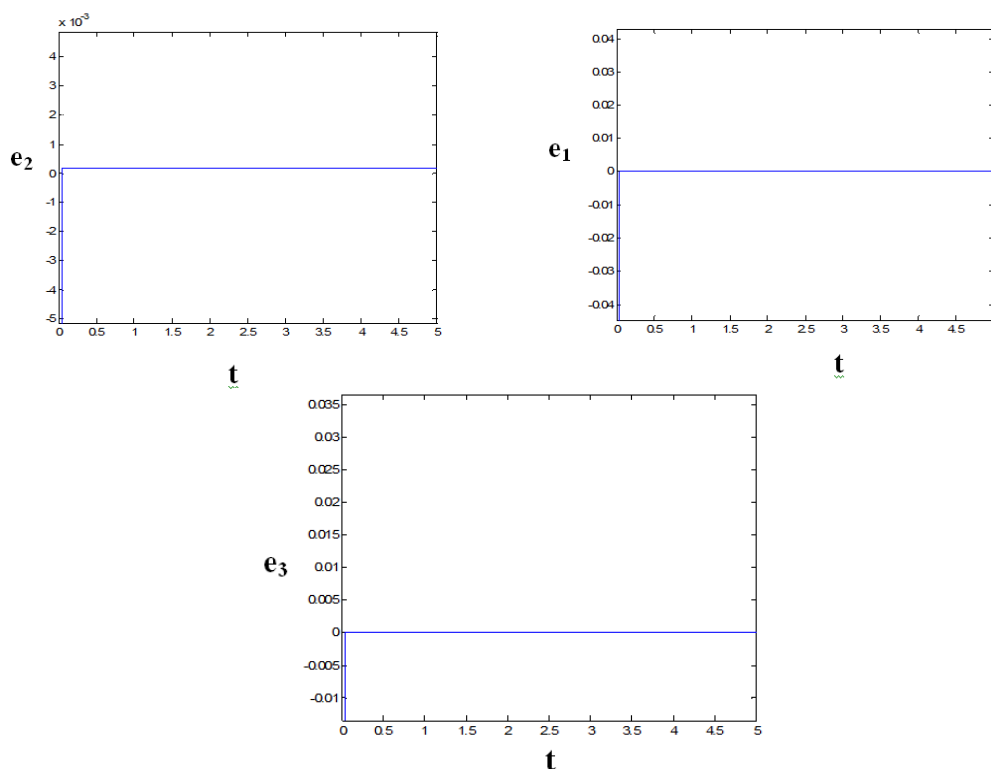


Fig. 3: The difference signal $e_1 = x_m - x_s, e_2 = y_m - y_s, e_3 = z_m - z_s$ in two coupling Pan systems parameters $k_1=10, k_2=40$ and $k_3=8$.

Conclusion:

Some dynamical behaviors of the fractional order pan system have been analyzed. The local stability of the equilibria using the fractional Routh-Hurwitz conditions. Analytically conditions for linear feedback control have been achieved. Chaos synchronization has also been demonstrated numerically by using the unidirectional error feedback coupled scheme.

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