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Symbolic Simulation of the Relativistic Polytrope

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ABSTRACT

Tolman-Oppenheimer-Volkoff (TOV) equation represents the relativistic polytropic gas spheres. Up to our knowledge, no general analytical solution has been introduced to this equation. In the present article, we introduced power series solution to TOV equation. We used two different accelerating methods to improve the radius of convergence of the series, Euler-Abel transformation and Pade approximation. Applying the proposed accelerating scheme, the transformed series is found to converge to the surface of the polytrope. Comparison with the numerical solution revealed good agreement.

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INTRODUCTION

Relativistic effects play an important role in many stellar configurations such as, white dwarfs, neutron stars, black holes, supermassive stars, and star clusters, Sygne (1934), Walker (1936), Einstein (1939), Sen and Roy (1954) and Sharma (1988).

Relativistic polytropic equation of state has been studied by Tooper (1964). Tooper derived two non linear differential equations analogue to the non-relativistic Lane Emden equation (Tolman-Oppenheimer-Volkoff [TOV] equation).

TOV equation is usually solved by numerical integration techniques. But certainly, if full analytical formulae are established, they definitely become valuable for obtaining models with desired accuracy, Sharma (1981). Moreover, these analytical formulae usually offer much deeper insight into the nature of a model as compared to numerical integration. On the other hand, in the absence of a closed analytical solution of a given differential equation, the power series solution can be served as the analytical representation of its solution, Nouh et al. (2003).

Analytical solutions of TOV equation has been done by Topper (1964) for the polytropic index $n = 0$ only. Sharma (1981) presented an approximate analytical solution for the polytropic indices $n = 0, 1.0 (0.5) 3.0$ and illustrates his results for the relativistic parameters $\sigma = 0 (0.002) 0.04$ only.

Ferrari et al. (2007) and Linares et al. (2004), solved numerically the two first order differential equations obtained by Tooper and investigated the effect of increasing a specific relativistic parameter on the polytropes with indices $n=1.5$ and $n=3$ respectively.

In the present paper, we introduce an analytical solution to the relativistic gas spheres. We accelerate the power series solution of the relativistic Emden's function using a combination of two different transformations.

The structure of the paper is as follows. In section 2, TOV equation is discussed, section 3 is devoted to the power series solution of TOV equation. The results are summarized in section 4. Section 5 is devoted to the conclusion reached.

The Relativistic Polytropic Gas Sphere:

For hydrostatic equilibrium stars, Tolman-Oppenheimer-Volkoff (TOV) equation has the form

$$\frac{dP}{dr} = -\frac{G \varepsilon(r) m(r)}{c^2 r^2} \left[1 + \frac{P(r)}{\varepsilon(r)} \right] \left[1 + \frac{4 \pi^3 P(r)}{m(r) c^2} \right] \left[1 - \frac{2G m(r)}{c^2 r} \right]^{-1} \quad (1)$$

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The above equation is an extension of the Newtonian formalism with relativistic correction. The equation of state for a polytropic star is $P = k\rho^{1+\frac{1}{n}}$, where n is the polytropic index. Tooper (1964) has shown that the TOV equation together with the mass conservation equation have the form

$$\xi^2 \frac{d\theta}{d\xi} \frac{1-2\sigma(n+1)\frac{\nu}{\xi}}{1+\sigma\theta} + \nu + \sigma \xi \theta \frac{d\nu}{d\xi} = 0, \quad (2)$$

$$\text{and} \quad \frac{d\nu}{d\xi} = \xi^2 \theta^n, \quad (3)$$

With the initial conditions

$$\theta(0) = 1, \quad \nu(0) = 0, \quad (4)$$

where

$$\theta = \rho/\rho_c, \quad \xi = rA, \quad \nu = \frac{A^3 m(r)}{4\pi\rho_c}, \quad A = \left(\frac{4\pi G\rho_c}{\sigma(n+1)c^2} \right)^{1/2}, \quad \sigma = \frac{P_c}{\rho_c c^2}. \quad (5)$$

where σ is the relativistic parameter and could be related to the sound velocity in the gas. In Equations (5) θ , ξ and ν are dimensionless parameter, while A is a constant.

If the pressure is much smaller than the energy density at the center of the star (i.e. σ tends to zero), then we turn to the well known Lane-Emden equation for Newtonian polytropic stars,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0. \quad (6)$$

Solution of TOV Equation:

Equation (2) can be rewritten in the form

$$\xi^2 \frac{d\theta}{d\xi} - 2\sigma(n+1)\xi\nu \frac{d\theta}{d\xi} + \nu + \sigma\nu\theta + \sigma\xi\theta \frac{d\nu}{d\xi} + \sigma^2\xi\theta^2 \frac{d\nu}{d\xi} = 0. \quad (7)$$

Consider a series expansion, near the origin, of the form

$$\theta(\xi) = 1 + \sum_{k=1}^{\infty} a_k \xi^{2k}, \quad (8)$$

Which satisfies the initial conditions (Equation (4)). Then

$$\xi^2 \frac{d\theta}{d\xi} = \sum_{k=1}^{\infty} 2(k+1) a_{k+1} \xi^{2k+3}. \quad (9)$$

With the help of the algebraic operations on series, we get

$$\theta^n = \sum_{k=0}^{\infty} \alpha_k \xi^{2k}, \quad (10)$$

Where

$$\alpha_0 = a_0^n,$$

$$\alpha_k = \frac{1}{ka_0} \sum_{i=1}^k (ni - k + i) a_i \alpha_{k-i}, \quad k \geq 1.$$

Equation (3) can be written in the form

$$\nu = \sum_{k=0}^{\infty} \frac{\alpha_k}{(2k+3)} \xi^{2k+3}. \quad (11)$$

Using Equations (9) and (11), we have

$$\xi \frac{d\theta}{d\xi} \nu = \sum_{k=0}^{\infty} f_k \xi^{2k+2} \sum_{k=0}^{\infty} g_k \xi^{2k+3},$$

where

$$f_k = 2(k+1) a_{k+1}, \quad g_k = \frac{\alpha_k}{(2k+3)}.$$

Using the formula of multiplication of two series, yields

$$\xi \frac{d\theta}{d\xi} = \sum_{k=0}^{\infty} \gamma_k \xi^{2k+5}, \quad (12)$$

where

$$\gamma_k = \sum_{i=0}^k f_i g_{k-i},$$

$$f_i = 2(i+1) a_{i+1}, \quad g_i = \frac{\alpha_i}{(2i+3)}.$$

From Equations (8) and (11), we obtain

$$\nu \theta = \sum_{k=0}^{\infty} \eta_k \xi^{2k+3}, \quad \eta_k = \sum_{i=0}^k a_i g_{k-i}. \quad (13)$$

The usage of Equations (3), (8) and (10) generates

$$\xi \theta \frac{d\nu}{d\xi} = \sum_{k=0}^{\infty} \beta_k \xi^{2k+3}, \quad \beta_k = \sum_{i=0}^k a_i \alpha_{k-i}. \quad (14)$$

Equation (14) gives us

$$\xi \theta^2 \frac{d\nu}{d\xi} = \sum_{k=0}^{\infty} \zeta_k \xi^{2k+3}, \quad \zeta_k = \sum_{i=0}^k a_i \beta_{k-i}. \quad (15)$$

Substitution of Equations (9) to (15) in Equation (7), evaluates

$$\begin{aligned} & \sum_{k=0}^{\infty} 2(k+1) a_{k+1} \xi^{2k+3} - 2\sigma(n+1) \gamma_k \xi^{2k+5} + \\ & + \frac{\alpha_k}{(2k+3)} \xi^{2k+3} + \sigma \eta_k \xi^{2k+3} + \sigma \beta_k \xi^{2k+3} + \\ & + \sigma^2 \zeta_k \xi^{2k+3} = 0 \end{aligned} \quad (16)$$

Equating the like powers of ξ in both sides would determine the coefficients a_k as follows

$$a_{k+1} = \frac{\sigma}{2(k+1)} (2(n+1) \gamma_{k-1} - \eta_k - \beta_k + \sigma \zeta_k) - \frac{\alpha_k}{2(k+1)(2k+3)}, \quad k \geq 1 \quad (17)$$

where

$$\gamma_{k-1} = \sum_{i=0}^{k-1} f_i g_{k-i-1}, \quad \eta_k = \sum_{i=0}^k a_i g_{k-i},$$

$$\beta_k = \sum_{i=0}^k a_i \alpha_{k-i}, \quad \zeta_k = \sum_{i=0}^k a_i \beta_{k-i},$$

$$\alpha_k = \frac{1}{k a_0} \sum_{i=1}^k (n i - k + i) a_i \alpha_{k-i}, \quad k \geq 1$$

$$\alpha_0 = a_0^n, \quad a_0 = 1.$$

If $\sigma = 0$, we get the coefficients of Emden solution. To get a_1 , put $k = 0$ in Equation (16), then equate the coefficients of the same power in both sides yields

$$a_1 = -\frac{1}{6} - \frac{\sigma}{2} \left(\sigma + \frac{4}{3} \right).$$

When $\sigma = 0$, then $a_1 = -\frac{1}{6}$ is the first series coefficient in case of Lane-Emden solution.

Results:

A Mathematica routine is elaborated to determine the zeros of TOV equation at different polytropic index n and relativistic parameter σ (the maximum value of σ is given by $\sigma_{\max} = n/(n+1)$, where the sound velocity must be smaller than the speed of light). To obtain reliable comparison between the analytical solution and the numerical one, we integrated equation (2) and (3) using Runge-Kutta method. The integrations were started at the initial values $\xi = 0$, $\theta = 1$, and $\nu = 0$ and proceeded forward using step size $\Delta\xi$. The zero of the emden function θ (i.e. ξ_1) is determined by integrating until a negative value of θ is reached. Then, small step size $\Delta\xi$ is used to give more accurate results.

First, we run the program for the series solution only. Tables (1), (2) and (3) show the radius of convergence computed analytically ($\xi_1(A)$) for $n=0.5$, $n=1.5$ and $n=3$, compared with the numerical value ($\xi_1(N)$). In the tables, m represents the number of the terms in the series, Equation (17).

As it is clear from these tables, at $n=0.5$ the series converges for all values of σ . For $n > 0.5$ the series diverges for all values of σ even when increasing the series terms to 400 terms.

To accelerate the convergence of the series solution of Equation (8), we implemented a combination of the two different transformations schemes developed by Nouh (2004), namely Pade' approximation and Euler-Abel transformation.

First we used Pade' approximation to accelerate the series, Equation (8), and obtained the result in Tables (4) and (5) for $n=0.5$ and $n=1.5$, the results for $n=3$ are poor when compared with the numerical solution. The designations in the tables are as follows: σ is the relativistic parameter, m is the number of terms in the original series, $k \times l$ is the order of Pade' approximation, $\xi_1(A)$ is the first zero of $\theta(\xi)$ (Equation (8)) derived analytically, $\xi_1(N)$ is the first zero of $\theta(\xi)$ derived numerically and ε is the relative error such that $\varepsilon = \left| \xi_1(A) - \xi_1(N) \right| / \xi_1(N)$.

To improve the radius of convergence, we apply Pade' and Euler-Abel transformations (Nouh, 2004), the results obtained are listed in Tables (6) and (7). Here, m_1 is the number of terms in the transformed series, p is the times number of applying Euler-Abel transformation, $\xi_1(A)$ is the first zero of the Emden function ($\theta E_n(\xi)$, Equation (20) in Appendix I) derived analytically, $\xi_1(N)$ is the first zero of the Emden function derived numerically and the rest of symbols are as in Table (4).

Table 1: Radii of the convergence of $\theta(\xi)$ (Equation (8)) for $n=0.5$.

| σ | M | $\xi_1(N)$ | $\xi_1(A)$ |
|----------|-----|------------|------------|
| 0.1 | 80 | 2.2899 | 2.2893 |
| 0.2 | 80 | 2.0008 | 2.0001 |
| 0.3 | 100 | 1.8013 | 1.8010 |

Table 2: Radii of the convergence of $\theta(\xi)$ (Equation (8)) for $n=1.5$.

| σ | M | $\xi_1(N)$ | $\xi_1(A)$ |
|----------|-----|------------|------------|
| 0.1 | 100 | 3.0384 | 2.5440 |
| 0.2 | 100 | 2.6993 | 2.0824 |
| 0.3 | 200 | 2.4930 | 1.7954 |
| 0.4 | 300 | 2.3610 | 1.5976 |
| 0.5 | 300 | 2.2749 | 1.4965 |
| 0.6 | 400 | 2.2192 | 1.5934 |

Table 3: Radii of the convergence of $\theta(\xi)$ (Equation (8)) for $n=3$.

| σ | M | $\xi_1(N)$ | $\xi_1(A)$ |
|----------|-----|------------|------------|
| 0.1 | 100 | 6.8258 | 1.90 |
| 0.2 | 100 | 7.9508 | 1.67 |
| 0.3 | 100 | 10.8337 | 1.49 |
| 0.4 | 200 | 17.8197 | 1.31 |
| 0.5 | 200 | 37.2058 | 1.36 |
| 0.6 | 300 | 91.0723 | 1.17 |
| 0.7 | 400 | 162.5832 | 1.80 |
| 0.75 | 400 | 180.4300 | 1.0 |

In deriving the parameters ($m, m_1, p, k \times l$), we applied a trial and error procedure controlled by the best relative error. We illustrate the results for $n=1.5$ and $n=3$. For $n=1.5$, $p=3$ for all values of σ except at $\sigma=0.4$,

where $p=5$. The number of series terms (m and m_1) required to reach the surface of the polytrope is almost increasing as σ increases. The maximum number of series terms is 40 with maximum relative error 0.0077.

For $n=3$, the effect of increasing σ on m and m_1 is clear, as σ goes high m and m_1 also go high. The maximum number of series terms is 100 with maximum relative error 0.004.

Table 4: Radii of the convergence of $\theta(\xi)$ (using Pade' approximation) for $n=0.5$.

| σ | m | $k \times l$ | $\xi_1(N)$ | $\xi_1(A)$ | ε |
|----------|-----|--------------|------------|------------|---------------|
| 0.1 | 100 | 6,6 | 2.2899 | 2.2875 | 0.0024 |
| 0.2 | 100 | 6,6 | 2.0008 | 2.0002 | 0.0006 |
| 0.3 | 200 | 6,6 | 1.8013 | 1.8099 | 0.0086 |

Table 5: Radii of the convergence of $\theta(\xi)$ (using Pade' approximation) for $n=1.5$.

| σ | m | $k \times l$ | $\xi_1(N)$ | $\xi_1(A)$ | ε |
|----------|-----|--------------|------------|------------|---------------|
| 0.1 | 250 | 10,10 | 3.0384 | 3.0282 | 0.0033 |
| 0.2 | 300 | 10,10 | 2.6993 | 2.6397 | 0.0220 |
| 0.3 | 300 | 10,10 | 2.4930 | 2.4966 | 0.0014 |
| 0.4 | 300 | 16,16 | 2.3610 | 2.3400 | 0.0088 |
| 0.5 | 300 | 6,6 | 2.2749 | 2.1800 | 0.0417 |
| 0.6 | 400 | 13,13 | 2.2192 | 2.1561 | 0.0284 |

Table 6: Radii of the convergence of $\theta E_n(\xi)$ (using using Pade' and Euler- Abel transformation) for $n=1.5$.

| σ | m | m_1 | P | $k \times l$ | $\square \square \square \square$ | $\square \square \square \square$ | \square |
|----------|-----|-------|---|--------------|-----------------------------------|-----------------------------------|-----------|
| 0.1 | 20 | 20 | 3 | 6,6 | 3.0384 | 3.0429 | 0.0014 |
| 0.2 | 20 | 20 | 3 | 8,8 | 2.6993 | 2.6966 | 0.0010 |
| 0.3 | 12 | 8 | 3 | 6,6 | 2.4930 | 2.4964 | 0.0013 |
| 0.4 | 32 | 26 | 5 | 20,20 | 2.3610 | 2.3664 | 0.0023 |
| 0.5 | 30 | 30 | 3 | 12,12 | 2.2749 | 2.2806 | 0.0025 |
| 0.6 | 40 | 40 | 3 | 12,12 | 2.2192 | 2.2020 | 0.0077 |

Table 7: Radii of the convergence of $\theta E_n(\xi)$ (using using Pade' and Euler- Abel transformation) for $n=3$.

| σ | M | m_1 | P | $k \times l$ | $\xi_1(N)$ | $\xi_1(A)$ | ε |
|----------|-----|-------|---|--------------|------------|------------|---------------|
| 0.1 | 20 | 14 | 5 | 9,9 | 6.8258 | 6.8420 | 0.0024 |
| 0.2 | 72 | 70 | 0 | 22,22 | 7.9508 | 7.9831 | 0.0040 |
| 0.3 | 72 | 70 | 2 | 22,22 | 10.8337 | 10.8414 | 0.0008 |
| 0.4 | 70 | 70 | 1 | 22,22 | 17.8197 | 17.800 | 0.0011 |
| 0.5 | 100 | 100 | 5 | 22,22 | 37.2058 | 37.200 | 0.00015 |
| 0.6 | 100 | 100 | 5 | 24,24 | 91.0723 | 91.080 | 0.000084 |
| 0.7 | 100 | 100 | 0 | 6,6 | 162.5832 | 162.200 | 0.0023 |
| 0.75 | 100 | 100 | 1 | 6,6 | 180.4300 | 180.5000 | 0.00038 |

Conclusion:

In the present paper, we presented an accelerated power series solution to the relativistic polytropic gas sphere. We used two different accelerating methods to improve the radius of convergence of the series, Euler-Abel transformation and Pade' approximation. Applying the proposed accelerating scheme, the transformed series is found to converge to the surface of the polytrope. Comparison with the numerical solution revealed good agreement i.e. the maximum relative error is 0.0077.

Some physical parameters could be established using the series solution of $\theta E_n(\xi)$. In Figures (1) and (2), we plot the density profiles of the stellar matter for $n=1.5$ and $n=3$ and for different values of σ as a function of the radius R/R_\odot . The lower panel of each figure illustrates the variation of the relative error with R/R_\odot .

For $n=1.5$ we see that, when σ increases, the stellar matter density is more concentrated in the center of the star. For $n=3$ (the ultra-relativistic case) the effect is much stronger. The result reflect good agreement between the analytical solution presented here and the numerical one.

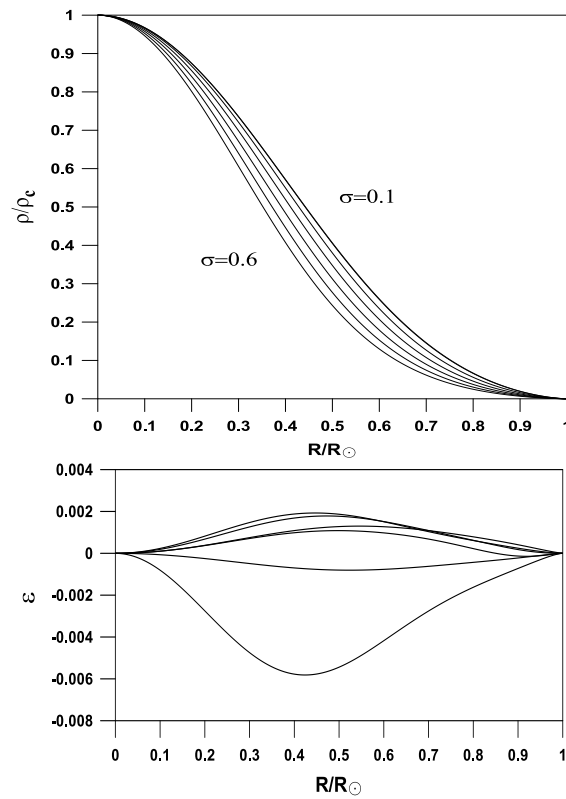


Fig. 1: Upper panel: the density profile for $n=1.5$. Lower panel: the variation of the relative error with radius.

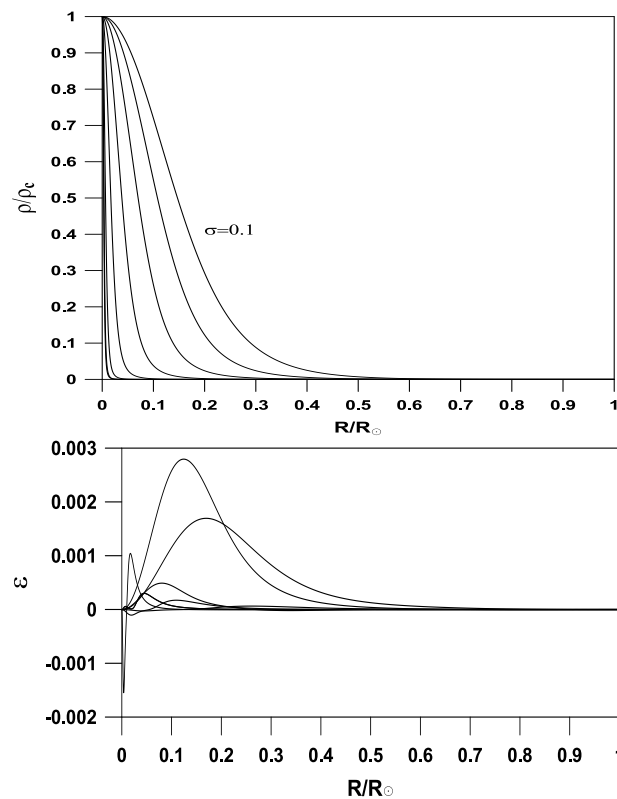


Fig. 2: Upper panel: the density profile for $n=3$. Lower panel: the variation of the relative error with radius.

Appendix I: Euler-Abel Transformation:

To accelerate the convergence of the series solution of Equation (8), we followed the scheme developed by Nough (2004). As a first step of this scheme, the alternating series is accelerated by Euler-Abel transformation (Demodovich and Maron, 1973).

Let us write

$$\theta(\xi) = a_0 + \xi \phi(\xi),$$

Where

$$\phi(\xi) = \sum_{k=0}^{\infty} a_k \xi^{k-1} = \sum_{k=1}^{\infty} a_{k+1} \xi^k,$$

then

$$\begin{aligned} (1-\xi) \phi(\xi) &= \sum_{k=0}^{\infty} a_{k+1} \xi^k - \sum_{k=1}^{\infty} a_k \xi^k, \\ &= a_0 + \sum_{k=0}^{\infty} \Delta a_k \xi^k, \end{aligned}$$

where

$$\Delta a_k = a_{k+1} - a_k, \quad k = 0, 1, 2, \dots$$

are the finite differences of the first order of the coefficients a_k . Applying Euler-Abel transformation to the

power series $\sum_{k=0}^{\infty} \Delta a_k \xi^k$, p times, and after some manipulations we obtain

$$\sum_{k=0}^{\infty} a_k \xi^k = \sum_{i=0}^{\infty} \Delta^i a_0 \frac{\xi^i}{(1-\xi)^{i+1}} + \left(\frac{\xi}{1-\xi} \right)^p \sum_{k=0}^{\infty} \Delta^p a_k \xi^k, \quad (18)$$

where $\Delta^0 a_0 = a_0$. Equation (18) becomes meaningless when $\xi = 1$, so, by setting $\xi = -t$, we obtain the Euler-Abel transformed series as

$$\theta E_n(t) = \sum_{k=0}^{\infty} \Delta^i a_0 \frac{t^i}{(1-t)^{i+1}} + \left(\frac{t}{1-t} \right)^p \sum_{k=0}^{\infty} \Delta^p [(-1)^k a_k] t^k. \quad (19)$$

Returning to the earlier variable, ξ , we obtain

$$\theta E_n(\xi) = \sum_{i=0}^{p-1} (-1)^i \Delta^i a_0 \frac{\xi^i}{(1+\xi)^{i+1}} + \left(\frac{\xi}{1+\xi} \right)^p \sum_{k=0}^{\infty} (-1)^{k+p} [\Delta^p a_k] \xi^k, \quad (20)$$

where

$$\Delta^p a_k = \Delta^{p-1} a_{k+1} - \Delta^{p-1} a_k$$

Any order difference $\Delta^p a_k$ can be written as linear combination as

$$\Delta^p a_k = \sum_{i=0}^p (-1)^{p-i} \binom{p}{i} a_{k+1},$$

where

$$\binom{p}{i} = \frac{p!}{i!(p-i)!}.$$

The second step is to apply Pade' approximation to the Euler-Abel transformed series, Equation (20). Pade' approximation is implemented by replacing Equation (20) truncated at some degree $M + N$ by a rational function $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials of degree M and N .

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