Australian Journal of Basic and Applied Sciences, 6(12): 61-68, 2012
ISSN 1991-8178

A survey on discrete inner products of Chebyshev polynomials of the first kind

A. Arzhang, H. Derili

Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.

Abstract: In this paper, we present explicit formulas for discrete inner products of Chebyshev polynomials of four kinds in the set of the zeros of the first kind. We establish some relations step by step that are used for proving the next relations. Finally, we give an application of the obtained formulas.

Key words: Chebyshev polynomials; Discrete orthogonality; Fredholm integral equations.

INTRODUCTION

Chebyshev polynomials have a great role in approximation theory. These polynomials are widely used in many areas of numerical analysis such as, uniform approximation, numerical integration, least-squares approximation, numerical solution of ordinary and partial differential equations, numerical solution of integral equations, and so on (Delves, L.M., J.L. Mohammed, 1985; Gradshteyn, I.S., I.M. Ryzhik, 2007; Jolley, L.B.W., et al., 1961; Kythe, P.K., P. Puri, 2002; Mason, J.C., D.C. Handscomb, 2003; Rivlin, T.J., 1990; Rivlin, T.J., 1974). Many properties and applications of These polynomials are known (Arfken, G.B., H.J. Weber, 2001; Brychkov, Y.A., 2008; Delves, L.M., J.L. Mohammed, 1985; Gradshteyn, I.S., I.M. Ryzhik, 2007; Mason, J.C., D.C. Handscomb, 2003; Rivlin, T.J., 1990; Rivlin, T.J., 1974; Snyder, M.A., 1966). Chebyshev polynomials of the first kind are also known as the optimal approximating polynomials on the interval (Abramowitz, M., I.A. Stegun, 1973), because the zeros of these polynomials are used as nodes in polynomial interpolation. The resulting interpolation polynomial is close to a continuous function under the maximum norm. (Mason, J.C., D.C. Handscomb, 2003; Rivlin, T.J., 1990; Rivlin, T.J., 1974).

Chebyshev polynomials of the first, second, third and fourth kinds of degree n, are defined by

$$T_n(x) = \cos n\theta,$$

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},$$

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}},$$

$$W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}, \tag{1}$$

respectively, where $x = \cos\theta$ and $0 \le \theta \le \pi$. There are many alternative ways to define These polynomials which lead to the same polynomials.

The four kinds of Chebyshev polynomials T_n, U_n, V_n, W_n are orthogonal with respect to the following weight functions

$$\frac{1}{\sqrt{1-x^2}}, \quad \sqrt{1-x^2}, \quad ,\sqrt{\frac{1+x}{1-x}}, \quad \sqrt{\frac{1-x}{1+x}},$$

on the interval $[-1,1]$ respectively, i.e.

Corresponding Author: A. Arzhang, Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
E-mail Address: arzhang@kiau.ac.ir

$$\int_{-1}^{1}\frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx=\begin{cases}0 & m\neq n,\\ \pi & m=n=0,\\ \frac{\pi}{2} & m=n\neq 0,\end{cases} \tag{2}$$

$$\int_{-1}^{1}\sqrt{1-x^2}U_m(x)U_n(x)dx=\frac{\pi}{2}\delta_{mn},\quad m,n=0,1,\cdots, \tag{3}$$

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}V_m(x)V_n(x)dx=\pi\delta_{mn},\quad m,n=0,1,\cdots, \tag{4}$$

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}W_m(x)W_n(x)dx=\pi\delta_{mn},\quad m,n=0,1,\cdots, \tag{5}$$

where δ_{mn} is Kronecker delta (Mason, J.C., D.C. Handscomb, 2003). In the following section we give explicit formulas for discrete inner products

$$< f,g >=\sum_{k=1}^{n+1}f(x_k)g(x_k), \tag{6}$$

where x_k is the kth zero of $T_{n+1}(x)$ and
$f,g\in\{T,U,V,W\}$.

First, we introduce the following fundamental theorems as essential tools for the next section.

Theorem 1.1. *For every real* θ *and natural* n *the following Trigonometrical summations hold.*

$$(i)\quad \cos\theta+\cos 2\theta+\cdots+\cos n\theta=\cos\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}\csc\frac{\theta}{2}=$$

$$\sin\frac{(n+1)\theta}{2}\cos\frac{n\theta}{2}\csc\frac{\theta}{2}-1 \tag{7}$$

$$(ii)\quad \sin\theta+\sin 2\theta+\cdots+\sin n\theta=\sin\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}\csc\frac{\theta}{2}=$$

$$\cot\frac{\theta}{2}-\cos\frac{(n+1)\theta}{2}\cos\frac{n\theta}{2}\csc\frac{\theta}{2}, \tag{8}$$

$$(iii)\quad \cos\frac{\theta}{2}+\cos\frac{3\theta}{2}+\cdots+\cos(n+\frac{1}{2})\theta=\frac{1}{2}\sin(n+1)\theta\csc\frac{\theta}{2}. \tag{9}$$

Proof. (i):

$$2(\cos\theta+\cos 2\theta+\cdots+\cos n\theta)=\sum_{k=0}^{n}e^{ik\theta}+\sum_{k=0}^{n}e^{-ik\theta}-2=\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}+$$

$$\frac{1-e^{-i(n+1)\theta}}{1-e^{-i\theta}}-2=\frac{2-e^{i(n+1)\theta}-e^{-i(n+1)\theta}+e^{in\theta}+e^{-in\theta}-e^{i\theta}-e^{-i\theta}}{2-e^{i\theta}-e^{-i\theta}}-2=$$

$$\frac{2(1-cos\theta)+2(\cos n\theta-\cos(n+1)\theta)-8\sin^2\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}}=2\cos\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}\csc\frac{\theta}{2}$$

The proof of the statements (ii) , (iii) is similar to the proof of the statements (i) hence, we omit them.

Theorem 1.2. *For every real θ and natural $m \le n$ the following Trigonometrical summation hold.*

$$\sin(m+1)\theta\sin(n+1)\theta\csc\theta=\sum_{k=0}^{m}\sin(n-m+2k+1)\theta. \tag{10}$$

So

$$U_m(x)U_n(x)=\sum_{k=0}^{m}U_{n-m+2k}(x). \tag{11}$$

Proof. Use the theorem (1.1) to obtain the result. So

$$\sum_{k=0}^{m}\sin(n-m+2k+1)\theta=\sum_{k=1}^{m+1}\sin(n-m+2k-1)\theta=\sum_{k=1}^{m+1}\sin(n-m-1)\theta\cos 2k\theta+$$

$$\sum_{k=1}^{m+1}\cos(n-m-1)\theta\sin 2k\theta=\sin(n-m-1)\theta\cos(m+2)\theta\sin(m+1)\theta\csc\theta+$$

$$\cos(n-m-1)\theta\sin(m+2)\theta\sin(m+1)\theta\csc\theta=\sin(m+1)\theta\sin(n+1)\theta\csc\theta.$$

Theorem 1.3. *For every natural m and n we have*

$$U_m(x)U_n(x)=\sum_{k=0}^{m}\sum_{k'=0}^{n}T_{|m+n-2k-2k'|}(x). \tag{12}$$

Proof. It is easy to verify that

$$\frac{\sin n\theta}{\sin\theta}=e^{i(1-n)\theta}\sum_{k=0}^{n-1}e^{2ik\theta},$$

so

$$U_m(x)U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta}\frac{\sin(m+1)\theta}{\sin\theta}=e^{-i(m+n)\theta}\sum_{k=0}^{m}\sum_{k'=0}^{n}e^{2i(k+k')\theta}=$$

$$\sum_{k=0}^{m}\sum_{k'=0}^{n}e^{i(2k+2k'-m-n)\theta}=\sum_{k=0}^{m}\sum_{k'=0}^{n}\{\cos(m+n-2k-2k')\theta-i\sin(m+n-2k-2k')\theta\}=$$

$$\sum_{k=0}^{m}\sum_{k'=0}^{n}\cos(m+n-2k-2k')\theta,$$

where, by using induction we can prove that

$$\sum_{k=0}^{m}\sum_{k'=0}^{n}\sin(m+n-2k-2k')\theta=0,$$

for briefness we omit it. So by taking $x=\cos\theta$ we obtain the result.

2- Discrete Orthogonality in The Zeros of Chebyshev Polynomials of The First Kind:

The zeros of $T_{n+1}(x)$ are as follows

$$x_k = \cos\left(\frac{(k-\frac{1}{2})\pi}{n+1}\right), \qquad k = 1,2,\cdots,n+1.$$

The following theorem based on these zeros hold.

Theorem 2.1. *Let* x_k *be the* k *th zero of* $T_{n+1}(x)$ *then for every* $i, j = 0,1\cdots n$, *the following statements hold.*

$$(i) \qquad (TT)_{ij} = \sum_{k=1}^{n+1} T_i(x_k)T_j(x_k) = \begin{cases} 0, & i \neq j, \\ \frac{n+1}{2}, & i = j \neq 0, \\ n+1, & i = j = 0. \end{cases} \tag{13}$$

$$(ii) \qquad (UU)_{ij} = \sum_{k=1}^{n+1} U_i(x_k)U_j(x_k) = \begin{cases} 0, & i+j \text{ is odd}, \\ (i+1)(n+1), & i \leq j \text{ and } i+j \text{ is even}. \end{cases} \tag{14}$$

$$(iii) \qquad (TU)_{ij} = \sum_{k=1}^{n+1} T_i(x_k)U_j(x_k) = \begin{cases} n+1, & i \leq j \text{ and } i+j \text{ is even}, \\ 0, & i > j \text{ or } i+j \text{ is odd}. \end{cases} \tag{15}$$

$$(iv) \qquad (TV)_{ij} = \sum_{k=1}^{n+1} T_i(x_k)V_j(x_k) = \begin{cases} (-1)^{i+j}(n+1), & i \leq j, \\ 0, & i > j. \end{cases} \tag{16}$$

$$(v) \qquad (TW)_{ij} = \sum_{k=1}^{n+1} T_i(x_k)W_j(x_k) = \begin{cases} n+1, & i \leq j, \\ 0, & i > j. \end{cases} \tag{17}$$

$$(vi) \qquad (VV)_{ij} = \sum_{k=1}^{n+1} V_i(x_k)V_j(x_k) = (-1)^{i+j}(2i+1)(n+1), \quad i \leq j, \tag{18}$$

$$(vii) \qquad (WW)_{ij} = \sum_{k=1}^{n+1} W_i(x_k)W_j(x_k) = (2i+1)(n+1), \quad i \leq j, \tag{19}$$

$$(viii) \qquad (VW)_{ij} = \sum_{k=1}^{n+1} V_i(x_k)W_j(x_k) = \begin{cases} n+1, & i \leq j, \\ (-1)^{i+j}(n+1), & i > j. \end{cases} \tag{20}$$

$$(ix) \qquad (UV)_{ij} = \sum_{k=1}^{n+1} U_i(x_k)V_j(x_k) = \begin{cases} (-1)^{i+j}(i+1)(n+1), & i \leq j, \\ \frac{1}{2}(n+1)(1+(-1)^{i+j}(2j+1)), & i > j. \end{cases} \tag{21}$$

$$(x) \qquad (UW)_{ij} = \sum_{k=1}^{n+1} U_i(x_k)W_j(x_k) = \begin{cases} (i+1)(n+1), & i \leq j, \\ \frac{1}{2}(n+1)((-1)^{i+j}+2j+1), & i > j. \end{cases} \tag{22}$$

Proof. First, we prove that

$$s_i = \sum_{k=1}^{n+1} T_i(x_k) = \begin{cases} 0, & 0 < i \leq 2n+1, \\ n+1, & i = 0. \end{cases} \tag{23}$$

So, by using the relation (9) we have

$$s_i = \sum_{k=1}^{n+1} T_i(x_k) = \sum_{k=1}^{n+1} \cos i\left(\frac{(k-\frac{1}{2})\pi}{n+1}\right) = \frac{\sin\left((n+1)\left(\frac{i\pi}{n+1}\right)\right)}{2\sin(\frac{i\pi}{2n+2})} = 0,\ 0 < i \le 2n+1.$$

(i): Since $i, j = 0,1\cdots n,$ then $|i \pm j| < 2n+1$ and by using the relation (23) we have

$$(TT)_{ij} = \sum_{k=1}^{n+1} T_i(x_k)T_j(x_k) = \frac{1}{2}[T_{i+j}(x_k)+T_{|i-j|}(x_k)] = \begin{cases} 0, & i \ne j \\ \frac{n+1}{2}, & i = j \ne 0, \\ n+1, & i = j = 0. \end{cases}$$

(ii): Without loss of generality, suppose $0 < i \le j$ and use the theorem (1.2) to have

$$(UU)_{ij} = \sum_{k=1}^{n+1} U_i(x_k)U_j(x_k) = \sum_{k=1}^{n+1}\sum_{k'=0}^{i} U_{j-i+2k'}(x_k) = \sum_{k'=0}^{i}\sum_{k=1}^{n+1} U_{j-i+2k'}(x_k). \tag{24}$$

On the other hand

$$U_{2n}(x) = 2\sum_{k=0}^{n} T_{2k}(x) - 1, \qquad U_{2n-1}(x) = 2\sum_{k=0}^{n-1} T_{2k+1}(x),\ n = 0,1,2,\cdots. \tag{25}$$

If $i+j$ is odd then for $k' = 0,\cdots,i$, $i+j-2k'$ is odd. Therefore, from the relations (23),(25) we have $\sum_{k=1}^{n+1} U_i(x_k)U_j(x_k) = 0.$

if $i+j = 2m$ is even then use the relation (25) to have

$$\sum_{k=1}^{n+1} U_i(x_k)U_j(x_k) = \sum_{k=1}^{n+1}\sum_{k'=0}^{i} U_{2(m+k')}(x_k) = 2\sum_{k=1}^{n+1}\sum_{k'=0}^{i}\sum_{k''=0}^{m+k'} T_{2k''}(x_k) - \sum_{k=1}^{n+1}\sum_{k'=0}^{i} 1 =$$

$$2(i+1)(n+1) - (i+1)(n+1) = (i+1)(n+1).$$

When $i = 0$ or $j = 0$, by using the relation (25) the result obtains.

(iii): Since $2T_i(x) = U_i(x) - U_{i-2}(x)$ then

$$(TU)_{ij} = \sum_{k=1}^{n+1} T_i(x_k)U_j(x_k) = \frac{1}{2}\sum_{k=1}^{n+1} U_i(x_k)U_j(x_k) - \frac{1}{2}\sum_{k=1}^{n+1} U_{i-2}(x_k)U_j(x_k).$$

Use the relation (14) to obtain the result. So, if $i+j$ is odd then $(TU)_{ij} = 0.$ If $i \le j$ and $i+j$ is even then

$$(TU)_{ij} = \frac{1}{2}\big((i+1)(n+1) - (i-2+1)(n+1)\big) = n+1. \tag{26}$$

If $j \le i-2$ and $i+j$ is even then

$$(TU)_{ij} = \frac{1}{2}\big((j+1)(n+1) - (j+1)(n+1)\big) = 0. \tag{27}$$

If $j = i-1$ then $i+j$ is odd and in this case $(TU)_{ij} = 0.$ So, for $i > j$ we have $(TU)_{ij} = 0.$

(iv): Since $V_i(x) = U_i(x) - U_{i-1}(x)$ then use the relation (15) to obtain the result.

(v): Since $W_i(x) = U_i(x) + U_{i-1}(x)$ then use the relation (15) to obtain the result.

(vi): Since

$$V_i(x)V_j(x) = U_i(x)U_j(x) + U_{i-1}(x)U_{j-1}(x) - U_i(x)U_{j-1}(x) - U_{i-1}(x)U_j(x) \tag{28}$$

then use the relation (14) to obtain the result. So, if $i \le j$ and $i+j$ is even then

$$(VV)_{ij} = \sum_{k=1}^{n+1} U_i(x_k)U_j(x_k) + \sum_{k=1}^{n+1} U_{i-1}(x_k)U_{j-1}(x_k) - 0 = (i+1)(n+1) + i(n+1) = (2i+1)(n+1). \quad (29)$$

if $i \le j$ and $i + j$ is odd then we obtain

$$(VV)_{ij} = -(2i+1)(n+1). \quad (30)$$

(vii): Since

$$W_i(x)W_j(x) = U_i(x)U_j(x) + U_{i-1}(x)U_{j-1}(x) + U_i(x)U_{j-1}(x) + U_{i-1}(x)U_j(x) \quad (31)$$

then use the relation (14) to obtain the result.

The proof of the statement $(viii)$ is similar to the statements $(iii),(iv),(v)$ hence, we omit it.

(ix): Use the relation (14) to obtain that

$$(UV)_{ij} = \sum_{k=1}^{n+1} U_i(x_k)V_j(x_k) = \begin{cases} (-1)^{i+j}(i+1)(n+1), & i \le j, \\ -j(n+1), & i > j \text{ and } i+j \text{ is odd.} \\ (j+1)(n+1), & i > j \text{ and } i+j \text{ is even.} \end{cases} \quad (32)$$

So, for $i > j$ from the above equation we have

$$(UV)_{ij} = (-1)^{i+j}(n+1)j + \frac{1}{2}(n+1)(1+(-1)^{i+j}) = (-1)^{i+j}(n+1)(j+\frac{1}{2}) + \frac{n+1}{2} =$$

$$\frac{1}{2}(n+1)(1+(-1)^{i+j}(2j+1)).$$

(x): Similar to the proof of the statement (ix) we obtain the result.

The following theorem is an immediate result of the above theorem.

Theorem 2.2. *Let* x_k *be the* k*th zero of* $T_{n+1}(x)$ *then for every* $i, j = 0,1\cdots n$, *the following statements hold.*

$$(i) \quad (VV1)_{ij} = \sum_{k=1}^{n+1}(1+x_k)V_i(x_k)V_j(x_k) = \begin{cases} 0, & i \ne j, \\ n+1, & i = j. \end{cases} \quad (33)$$

$$(ii) \quad (VV2)_{ij} = \sum_{k=1}^{n+1}(1+x_k)^2 V_i(x_k)V_j(x_k) = \begin{cases} 0, & |i-j| \ne 0,1, \\ n+1, & i = j \ne 0, \\ \frac{3}{2}(n+1), & i = j = 0, \\ \frac{1}{2}(n+1), & |i-j| = 1. \end{cases} \quad (34)$$

$$(i) \quad (WW1)_{ij} = \sum_{k=1}^{n+1}(1-x_k)W_i(x_k)W_j(x_k) = \begin{cases} 0, & i \ne j, \\ n+1, & i = j. \end{cases} \quad (35)$$

$$(ii) \quad (WW2)_{ij} = \sum_{k=1}^{n+1}(1-x_k)^2 W_i(x_k)W_j(x_k) = \begin{cases} 0, & |i-j| \ne 0,1, \\ n+1, & i = j \ne 0, \\ \frac{3}{2}(n+1), & i = j = 0, \\ -\frac{1}{2}(n+1), & |i-j| = 1. \end{cases} \quad (36)$$

Proof. Since

$$(1+x)V_n(x) = T_n(x) + T_{n+1}(x), \quad n = 0,1,\cdots,$$

$$(1-x)W_n(x) = T_n(x) - T_{n+1}(x), \quad n = 0,1,\cdots.$$

then use the relations (13),(16),(17) to obtain the results.

3- Some Applications:

The above theorems can be applied in the areas of mathematics where the inner product (6) is defined. Also interpolation in the zeros of Chebyshev polynomials is an interesting subject that has a close relation to the discrete orthogonality of Chebyshev polynomials (Mason, J.C., D.C. Handscomb, 2003; Rivlin, T.J., 1974). One may use the given inner products in the theorems (2.1) to find the numerical solution of Fredholm integral equation of the second kind. This makes the cost of operations and running time of the computer programmes to be low.

Consider the following integral equation

$$u(x) + \lambda \int_{-1}^{1} k(x,y)u(y)dy = f(x), \tag{37}$$

where $u(x)$ is unknown function and $k(x,y), f(x) \in L^2[-1,1]$ are known functions and λ is a nonzero constant. We use the Bubnov-Galerkin method to find the solution (Kythe, P.K., P. Puri, 2002). In this method choose two complete orthogonal systems of functions $\{\phi_n(x)\}$, $\{\psi_n(x)\}$ and suppose

$$u(x) \simeq u_n(x) = \sum_{i=0}^{n} c_i \phi_i(x), \tag{38}$$

$$k(x,y) \simeq k_{MN}(x,y) = \sum_{p=0}^{M}\sum_{q=0}^{N} A_{pq}\psi_p(x)\psi_q(y), \tag{39}$$

define the residual function

$$R_n(x) = u_n(x) + \lambda \int_{-1}^{1} k_{MN}(x,y)u_n(y)dy - f(x). \tag{40}$$

The coefficients c_i are obtain from the conditions that

$$< R_n(x), \psi_j(x) >= 0, \quad j = 0,\cdots,n, \tag{41}$$

where the inner product is defined by the relation (6) in the set of the zeros of $T_{n+1}(x)$. The relation (41) leads to the following linear system of the equations.

$$\sum_{k=1}^{n+1}\psi_j(x_k)\left\{\sum_{i=0}^{n} c_i\phi_i(x_k) + \lambda\sum_{i=0}^{n}\sum_{p=0}^{M}\sum_{q=0}^{N} c_i A_{pq}\psi_p(x_k)\int_{-1}^{1}\psi_q(y)\phi_i(y)dy\right\} =$$

$$\sum_{k=1}^{n+1}\psi_i(x_k)f(x_k), \quad i,j = 0,1,\cdots,n.$$

So

$$\sum_{i=0}^{n} c_i\left\{\sum_{k=1}^{n+1}\phi_i(x_k)\psi_j(x_k) + \lambda\sum_{p=0}^{M}\sum_{q=0}^{N} A_{pq}B_{iq}\sum_{k=1}^{n+1}\psi_p(x_k)\psi_j(x_k)\right\} =$$

$$\sum_{k=1}^{n+1}\psi_i(x_k)f(x_k), \quad i,j = 0,1,\cdots,n.$$

where

$$B_{iq} = \int_{-1}^{1}\psi_q(y)\phi_i(y)dy,, \quad i = 0,1,\cdots,n, q = 0,1,\cdots N.$$

One may choose

$$\phi_i \in \{T_i\},\{U_i\},\{V_i\},\{W_i\}, \quad i = 0,\cdots,n,$$

$$\psi_j \in \{T_j\},\{U_j\},\{V_j\},\{W_j\}, \quad j = 0,\cdots,n,$$

to obtain the solution. So we have many choices to compute the inner product (41). Some detailed methods for solving the Eq. (37) can be found in (Delves, L.M., J.L. Mohammed, 1985; Kythe, P.K., P. Puri, 2002).

ACKNOWLEDGMENTS

This work has been funded and supported by Islamic Azad University-Karaj Branch, and the authors are thankful to it.

REFERENCES

Abramowitz, M., I.A. Stegun, 1973. Handbook of Mathematical Functions, Dover, New York,

Abramowitz, M., I.A. Stegun, 1984. Pocketbook of Mathematical Functions, Verlag Harri Deutsch, Germany,

Arfken, G.B., H.J. Weber, 2001. Mathematical Methods for Physicists, 5nd ed., Academic Press, New York,

Brychkov, Y.A., 2008. Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas, CRC Press, Moscow, Russia.

Delves, L.M., J.L. Mohammed, 1985. Computational Methods For Integral Equations, Cambridge University Press, Cambridge.

Gradshteyn, I.S., I.M. Ryzhik, 2007. Table of Integrals, Series and Products, 7nd ed., Academic Press, Oxford.

Jolley, L.B.W., M.A. (cantab.), M.I.E.E., Summation, 1961. of series, Second Revised Edition, Dover Publications, INC, New York.

Kythe, P.K., P. Puri, 2002. Computational Methods for Linear Integral Equations, University of New Orlans, New Orlans, USA,

Mason, J.C., D.C. Handscomb, 2003. Chebyshev Polynomials, CRC Press, New York,

Rivlin, T.J., 1974. The Chebyshev Polynomials, John Wiley, New York.

Rivlin, T.J., 1990. Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory, 2nd ed, John Wiley, New York,

Snyder, M.A., 1966. Chebyshev Methods in Numerical Approximation, Prentice–Hall, Englewood Cliffs, NJ,

Zygmund, A., 1959. Trigonometric Series, 2nd ed., CUP, Cambridge.