Australian Journal of Basic and Applied Sciences, 6(13): 159-166, 2012
ISSN 1991-8178

A Study of the Stability of a Non-linear Autoregressive Models

[1]Dr. Abdul Ghafoor Jasim Salim, [2]Anas Salim Youns

[1]College of Computers Sciences and Mathematics University of Mosul/ Mosul/Iraq
[2]College of Basic education University of Mosul/ Mosul/Iraq

Abstract: In this paper we suggest one of the models for non-linear autoregressive with hyperbolic triangle function. We used the local linearization approximation method to transform the model to a linear model. Also we initiated study models of low order to find the singular point and it's stability conditions and limit cycle and then we generalized the method to the model of general order of order p.

Key words: Non-linear time series model; Non-linear random vibration; Autoregressive model; Limit cycle; Singular point; Stability.

INTRODUCTION

We study a non-linear time series models because most of the phenomenon that study in a non-linear behaviour behave. In the field of discrete time non-linear time series modeling, there are many different types of a non-linear model which could be considered such as bilinear model (Priestley (1978), Rao (1977)), exponential autoregressive model (Ozaki and Oda (1977)) and threshold model (Tong (1979)) studies this models to find his statistical properties.

The linear autoregressive models of time series are stable if all the roots of the characteristic equation lies inside the unit circle.

In (1985) Ozaki proposed the method of local linearization approximation to find the stability of a non-linear exponential autoregressive models.

In (1986) Tsay R.S. studied the stability of non-linear time series. In (1988) Priestley M.B. studied the non-stability and non-linear time series. In (1990) Tong H. studied the dynamical system with stability of non-linear time series.

In this paper we study the stability of a non-linear autoregressive model with hyperbolic triangle function by using the local linear approximation method to transform the model from non-linear model to a linear model. We were initiated using the models of order one , two and three. Finally we reached to the stability of singular point and the stability condition for limit cycle of general model of order p and gave some examples to explain this method.

Basic Concepts Of Time Series:

Definition 1:

A time series is a set of observations measured sequentially through time. These measurements may be made continuously through time or be taken at a discrete set of time points (Chatfield,2000).

Definition 2:

The autoregressive model of order p is satisfies the following equation:

$$X_t+a_1X_{t-1}+a_2X_{t-2}+...+a_pX_{t-p}=Z_t$$

Where $\{Z_t\}$ is a white noise and $a_1, a_2, ..., a_p$ are real constants (Chatfield,2004).

Definition 3:

The exponential autoregressive model of order p, EXPAR(P) is defined by the following equation

$$x_t = \left(\phi_1 + \pi_1 e^{-x_{t-1}^2}\right)x_{t-1} + ... + \left(\phi_p + \pi_p e^{-X_{t-1}^2}\right)x_{t-p} + Z_t$$

Where $\{Z_t\}$ is a white noise and $\phi_1, ..., \phi_p; \pi_1, ..., \pi_p$ are the parameters of the model (Ozaki and Oda,1978).

Definition 4:

The bilinear model of order (p,q,m,s) is satisfies the equation

Corresponding Author: Dr.Abdul Ghafoor Jasim Salim, Prof.\ College of Computers Sciences and Math.\ University of Mosul \ Mosul \Iraq

$$x_t = c + \sum_{i=1}^{p} \phi_i x_{t-i} - \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \sum_{i=1}^{m}\sum_{j=1}^{s} \beta_{ij} x_{t-i}\varepsilon_{t-j} + \varepsilon_t$$

Where p , q , m and s are nonnegative, and (ε_t) is a sequence of independent identically distributed random variables and $\phi_1,......,\phi_p;\theta_1,........,\theta_q;\beta_{ij};\forall i=1,.....m,\forall j=1,......s$ are the parameters of the model. (Tsay,2010).

Definition 5:
The polynomial autoregressive model of order p is defined as the following equation

$$y(t) = p\big(y(t-1),......,y(t-p),z(t-1),.......,z(t-q)\big) + Z_t$$

Where $\{Z_t\}$ is a white noise and P(.) is a polynomial of order q (Chen and Billings,1989).

Definition 6:
A singular point $x_t = f\left(x_{t-1}, x_{t-2},......,x_{t-p}\right)$ of is defined as a point which every trajectory of $x_t = f\left(x_{t-1}, x_{t-2},......,x_{t-p}\right)$ beginning sufficiently near it approaches either for $t \to \infty$ or for $t \to -\infty$. If it approaches it for $t \to \infty$ we call it stable singular point and if it approaches it for $t \to -\infty$ we call it unstable singular point.

Obviously a singular point ζ satisfies $\zeta = f(\zeta)$ (Ozaki,1982).

Definition 7:
A limit cycle of $x_t = f\left(x_{t-1}, x_{t-2},......,x_{t-p}\right)$ is defined as an isolated and closed trajectory $x_{t+1}, x_{t+2},......,x_{t+q}$ where q is a positive integer. Closed means that if initial values $\left(x_1,.....,x_p\right)$ belong to the limit cycle then $\left(x_{1+kq}, x_{2+kq},......,x_{p+kq}\right) = \left(x_1,.......,x_p\right)$ for any integer k. Isolated means that every trajectory beginning sufficiently near the limit cycle approaches either for $t \to \infty$ or for $t \to -\infty$. If it approaches it for $t \to \infty$ we call it stable limit cycle and if it approaches it for $t \to -\infty$ we call it unstable limit cycle (Ozaki,1982).

Theorem 1:
Let $\{x_t\}$ be expressed by the exponential autoregressive model of order one $x_t = \left(\phi_1 + \pi_1 e^{-x_{t-1}^2}\right)x_{t-1} + \varepsilon_t$

A limit cycle of period q $x_t, x_{t+1}, x_{t+2},......,x_{t+q}$ of the model is orbitally stable if $\left|\dfrac{\zeta_{t+q}}{\zeta_t}\right| < 1$.

For the prove see (Ozaki,1982).

Definition 8:
A non-linear autoregressive model of order p (the proposed model) is defined by

$$X_t = \sum_{i=1}^{p} [\phi_i \sinh(x_{t-1})]^i X_{t-i} + Z_t$$

Where $\{Z_t\}$ is a white noise process and $\phi_1,.......,\phi_p$ are the parameters(real constant) of the model.

Resuts:
The Stability of the Proposed Model:
In this section, we shall study the stability of a non-linear autoregressive model with hyperbolic triangle function with low order such that p=1,2,3 and then we generalized this idea to the general model of order p.

Let the non-linear model defined by the following equation

$$X_t = \sum_{i=1}^{p} [\phi_i \sinh(x_{t-1})]^i X_{t-i} + Z_t \tag{1}$$

Let p=1 in equation(1), then we have NAR(1)

$$X_t = [\phi_1 \sinh X_{t-1}]X_{t-1} + Z_t \tag{2}$$

By using the definition of singular point $\zeta = f(\zeta)$ and suppose that the white noise is not effect that is mean $Z_t = 0$.

Then $\zeta = [\phi_1 \sinh(\zeta)]\zeta$

If $(\zeta \neq 0)$ and $(\phi_1 \neq 0)$. Then $\zeta = \sinh^{-1}(\frac{1}{\phi_1})$ (3)

Or equivalently $\zeta = [\phi_1(\frac{e^{\zeta} - e^{-\zeta}}{2})]\zeta$

Then $\zeta = \ln\{\frac{1}{\phi_1}(1 \mp \sqrt{1+\phi_1^2})\}$ (3)

Therefore, the non-zero singular point exist if $\{\frac{1}{\phi_1}(1 \mp \sqrt{1+\phi_1^2})\}$ □ 0 .

The Stability Condition For The Singular Point:

We will find the stability condition for the non-zero singular point as follows :

Put $x_s = \zeta + \zeta_s$ for s=t,t-1 , in equation (2), and suppose that the white noise is not effect that is mean $Z_t = 0$, then we have:

$$\zeta + \zeta_t = \phi_1[\sinh(\zeta + \zeta_{t-1})](\zeta + \zeta_{t-1})$$

Or $\zeta + \zeta_t = [\frac{\phi_1}{2}\left(e^{(\zeta+\zeta_{t-1})} - e^{-(\zeta+\zeta_{t-1})}\right)](\zeta + \zeta_{t-1})$ (4)

Then $\zeta_t = \phi_1[\zeta \cosh(\zeta) + \sinh(\zeta)]\zeta_{t-1}$ (5)

Since we have $\zeta = \sinh^{-1}(\frac{1}{\phi_1})$

Then $\zeta_t = [\phi_1 \sinh^{-1}(\frac{1}{\phi_1})\cosh(\sinh^{-1}(\frac{1}{\phi_1})) + 1]\zeta_{t-1}$

Or $\zeta_t = h_1\zeta_{t-1}$, where $h_1 = [\phi_1 \sinh^{-1}(\frac{1}{\phi_1})\cosh(\sinh^{-1}(\frac{1}{\phi_1})) + 1]$ (6)

Equation (6) is a first order linear autoregressive model which is stable if the root of the characteristic equation lies inside the unit circle.

i.e. if $|h_1|$ □ 1.

The Stability Condition Of The Limit Cycle:

Let the limit cycle of period q of the proposed model in the equation (2) has the form $x_t, x_{t+1}, x_{t+2}, \ldots\ldots, x_{t+q} = x_t$ .The points x_s near the limit cycle is represented as $x_s = x_s + \zeta_s$, then replaced x_t and x_{t-1} by $x_t + \zeta_t$ and $x_{t-1} + \zeta_{t-1}$ and suppose that the white noise is not effect that is mean $Z_t = 0$.

Respectively, then we have

$$x_t + \zeta_t = [\frac{\phi_1}{2}\left(e^{(x_{t-1}+\zeta_{t-1})} - e^{-(x_{t-1}+\zeta_{t-1})}\right)](x_{t-1} + \zeta_{t-1}) \tag{7}$$

Therefore

$$\zeta_t = \phi_1(\sinh x_{t-1} + x_{t-1}\cosh x_{t-1})\zeta_{t-1} \tag{8}$$

The equation (8) is a linear difference equation with a periodic coefficient, which is difficult to solve analytically.

of (8) converges to zero or not, and this can be checked by seeing whether ζ_t We want to know is whether

$\left|\frac{\zeta_{t+q}}{\xi_t}\right|$ is less than one or not (Ozaki,1985).

Let t=t+q in equation (8).

Then $\zeta_{t+q} = \phi_1(\sinh x_{t+q-1} + x_{t+q-1}\cosh x_{t+q-1})\zeta_{t+q-1}$ (9)

$$\zeta_{t+q} = \prod_{i=1}^{q}\phi_1(\sinh x_{t+i-1} + x_{t+i-1}\cosh x_{t+i-1})\zeta_t \tag{10}$$

Then we need theorem1that equation (10) is orbitally stable if $\left|\dfrac{\zeta_{t+q}}{\zeta_t}\right| < 1$.

Then $\left|\frac{\zeta_{t+q}}{\zeta_t}\right| = \left|\prod_{i=1}^{q} \phi_1 (\sinh x_{t+i-1} + x_{t+i-1} \cosh x_{t+i-1})\right| \square 1$.

For p=2 in equation(1), then we have NAR(2):

$$X_t = [\phi_1 \sinh X_{t-1}] X_{t-1} + [\phi_2^2 \sinh^2 x_{t-1}] x_{t-2} + Z_t \tag{12}$$

By using the definition of singular point $\zeta = f(\zeta)$ and suppose that the white noise is not effect that is mean $Z_t = 0$.

Then $\zeta = [\phi_1 \sinh(\zeta)]\zeta + [\phi_2^2 \sinh^2(\zeta)]\zeta$

The singular points of the model in equation(12) are

$$\zeta = \sinh^{-1}\left(\tfrac{-\phi_1 \mp \sqrt{\phi_1^2 + 4\phi_2^2}}{2\phi_2^2}\right) \tag{13}$$

The Stability Condition For The Singular Point:

We will find the stability condition for the non-zero singular points of equation(12) as follows :

Put $x_s = \zeta + \zeta_s$ for all s=t,t-1,t-2 , and suppose that the white noise is not effect that is mean $Z_t = 0$, then we have:

$$\begin{aligned} \zeta + \zeta_t &= [\tfrac{\phi_1}{2}\left(e^{(\zeta+\zeta_{t-1})} - e^{-(\zeta+\zeta_{t-1})}\right)](\zeta + \zeta_{t-1}) \\ &+ [\tfrac{\phi_2^2}{4}\left((e^{(\zeta+\zeta_{t-1})} - e^{-(\zeta+\zeta_{t-1})})(e^{(\zeta+\zeta_{t-1})} - e^{-(\zeta+\zeta_{t-1})})\right)](\zeta + \zeta_{t-2}) \end{aligned} \tag{14}$$

Then

$$\zeta_t = [\phi_1 \zeta \cosh(\zeta) + \phi_1 \sinh(\zeta) + 2\phi_2^2 \zeta \sinh(\zeta)\cosh(\zeta)]\zeta_{t-1} + [\phi_2^2 \sinh^2(\zeta)]\zeta_{t-2}$$

Or

$$\zeta_t = h_1 \zeta_{t-1} + h_2 \zeta_{t-2} \tag{15}$$

Where

$h_1 = [\phi_1 \zeta \cosh(\zeta) + \phi_1 \sinh(\zeta) + 2\phi_2^2 \zeta \sinh(\zeta)\cosh(\zeta)]$,

$h_2 = [\phi_2^2 \sinh^2(\zeta)]$

Then equation (15) is a linear model of order two which have characteristic equation of the form,

$v^2 - h_1 v - h_2 = 0$

Then $h_1 = (\lambda_1 + \lambda_2), h_2 = -\lambda_1 \lambda_2$

Where λ_1, λ_2 are the roots of the characteristic equation of the model.

The stability condition is that $|\lambda_i| \square 1$; for all i=1,2 .

The Stability Condition Of The Limit Cycle:

Let the limit cycle of period q of the proposed model in equation (12) have the form $x_t, x_{t+1}, x_{t+2}, ..., x_{t+q} = x_t$ .The points x_s near the limit cycle is represented as $x_s = x_s + \zeta_s$, then replaced x_t, x_{t-1} and x_{t-2} by $x_t + \zeta_t$, $x_{t-1} + \zeta_{t-1}$ and $x_{t-2} + \zeta_{t-2}$ and suppose that the white noise is not effect that is mean $Z_t = 0$, then we have

$$\begin{aligned} x_t + \zeta_t &= [\tfrac{\phi_1}{2}\left(e^{(x_{t-1}+\zeta_{t-1})} - e^{-(x_{t-1}+\zeta_{t-1})}\right)](x_{t-1} + \zeta_{t-1}) \\ &+ [\tfrac{\phi_2^2}{4}\left((e^{(x_{t-1}+\zeta_{t-1})} - e^{-(x_{t-1}+\zeta_{t-1})})(e^{(x_{t-1}+\zeta_{t-1})} - e^{-(x_{t-1}+\zeta_{t-1})})\right)](x_{t-2} + \zeta_{t-2}) \end{aligned} \tag{16}$$

Then

$$\zeta_t = [\phi_1 x_{t-1}\cosh(x_{t-1}) + \phi_1 \sinh(x_{t-1}) + 2\phi_2^2 \sinh(x_{t-1})\cosh(x_{t-1})x_{t-2}]\zeta_{t-1} + [\phi_2^2 \sinh^2(x_{t-1})]\zeta_{t-2} \qquad (17)$$

The equation (17) is a linear difference equation with a periodic coefficient, which is difficult to solve analytically.

We want to know is whether ζ_t of (17) converges to zero or not, and this can be checked by seeing whether $\left|\frac{\zeta_{t+q}}{\xi_t}\right|$ is less than one or not (Ozaki,1985).

Let t=t+q in equation (17).

$$\zeta_{t+q} = [\phi_1 x_{t+q-1}\cosh(x_{t+q-1}) + \phi_1 \sinh(x_{t+q-1}) + 2\phi_2^2 \sinh(x_{t+q-1})\cosh(x_{t+q-1})x_{t+q-2}]\zeta_{t+q-1} + [\phi_2^2 \sinh^2(x_{t+q-1})]\zeta_{t+q-2} \qquad (18)$$

$$\zeta_{t+q} = \left(\begin{array}{l} \prod_{i=1}^{q}[\phi_1 x_{t+i-1}\cosh(x_{t+i-1}) + \phi_1 \sinh(x_{t+i-1}) + 2\phi_2^2 \sinh(x_{t+i-1})\cosh(x_{t+i-1})x_{t+i-2}] \\ + \prod_{i=1}^{q-1}[\phi_2^2 \sinh^2(x_{t+i-1})] \end{array} \right)\zeta_t \qquad (19)$$

Then we need theorem1that equation (19) is orbitally stable if $\left|\frac{\zeta_{t+q}}{\zeta_t}\right| < 1$.

Therefore

$$\left|\frac{\zeta_{t+q}}{\zeta_t}\right| = \left|\left(\begin{array}{l} \prod_{i=1}^{q}[\phi_1 x_{t+i-1}\cosh(x_{t+i-1}) + \phi_1 \sinh(x_{t+i-1}) + 2\phi_2^2 \sinh(x_{t+i-1})\cosh(x_{t+i-1})x_{t+i-2}] \\ + \prod_{i=1}^{q-1}[\phi_2^2 \sinh^2(x_{t+i-1})] \end{array} \right)\right| \square 1 \qquad (20)$$

For p=3 in equation(1), then we have:

$$X_t = [\phi_1 \sinh X_{t-1}]X_{t-1} + [\phi_2^2 \sinh^2 x_{t-1}]x_{t-2} + [\phi_3^3 \sinh^3 x_{t-1}]x_{t-3} + Z_t \qquad (21)$$

Then $\zeta = [\phi_1 \sinh(\zeta)]\zeta + [\phi_2^2 \sinh^2(\zeta)]\zeta + [\phi_3^3 \sinh^3(\zeta)]\zeta$.

Then, we divided above equation on singular point $\zeta \neq 0$.

Therefore, we get equation of three order and by using reference (Al-Azzawi,*2012*), we get that

Let a, b and c are real constants such that $a = \frac{\phi_2^2}{\phi_3^3}, b = \frac{\phi_1}{\phi_3^3}, c = -\frac{1}{\phi_3^3}$

$q = c - \frac{1}{3}ab + \frac{2}{27}a^3$

$\Delta = c^2 + \frac{4}{27}b^3 - \frac{2}{3}abc - \frac{1}{27}a^2b^2 + \frac{4}{27}a^3c$

Case one if $\Delta = 0$

Then we gets three real roots and we find it by

$x_1 = -2\sqrt[3]{\frac{q}{2}} - \frac{a}{3}, x_2 = x_3 = \sqrt[3]{\frac{q}{2}} - \frac{a}{3}$

Case two if $\Delta \, \square \, 0$

Then we gets three different real roots and we find it by

$$x_{k+1} = \sqrt[6]{16(q^2 - \Delta)}\cos\frac{\cos^{-1}\frac{-q}{\sqrt{q^2-\Delta}} + 2\pi k}{3} - \frac{a}{3}, k = 0,1,2$$

Case three if $\Delta \, \square \, 0$

Then we gets one real root and two complex conjugate roots and we find it by

$$x_1 = \sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}} - \tfrac{a}{3}$$
$$x_2 = -\tfrac{1}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right) - \tfrac{a}{3} + i\tfrac{\sqrt{3}}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right)$$
$$x_3 = -\tfrac{1}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right) - \tfrac{a}{3} - i\tfrac{\sqrt{3}}{2}\left(\sqrt[3]{\tfrac{-q-\sqrt{\Delta}}{2}} - \sqrt[3]{\tfrac{-q+\sqrt{\Delta}}{2}}\right)$$

The singular points of the model in equation (21) are that

$$\zeta = \sinh^{-1}(x_i), \forall i = 1,2,3 \tag{22}$$

The Stability Condition For The Singular Point:

We will find the stability condition for the non-zero singular points of equation (21) as follows :

Put $x_s = \zeta + \zeta_s$ for all s=t,t-1,t-2,t-3 , and $Z_t = 0$, then we have:

$$\begin{aligned}\zeta + \zeta_t &= [\tfrac{\phi_1}{2}\left(e^{\zeta+\zeta_{t-1}} - e^{-\zeta-\zeta_{t-1}}\right)](\zeta + \zeta_{t-1}) \\ &+ [\tfrac{\phi_2^2}{4}\left((e^{\zeta+\zeta_{t-1}} - e^{-\zeta-\zeta_{t-1}})(e^{\zeta+\zeta_{t-1}} - e^{-\zeta-\zeta_{t-1}})\right)](\zeta + \zeta_{t-2}) \\ &+ [\tfrac{\phi_3^3}{8}\left((e^{\zeta+\zeta_{t-1}} - e^{-\zeta-\zeta_{t-1}})(e^{\zeta+\zeta_{t-1}} - e^{-\zeta-\zeta_{t-1}})(e^{\zeta+\zeta_{t-1}} - e^{-\zeta-\zeta_{t-1}})\right)](\zeta + \zeta_{t-3})\end{aligned} \tag{23}$$

Then

$$\begin{aligned}\zeta_t &= [\phi_1\zeta\cosh(\zeta) + \phi_1\sinh(\zeta) + 2\phi_2^2\zeta\sinh(\zeta)\cosh(\zeta) + 3\phi_3^3\zeta\sinh^2(\zeta)\cosh\zeta]\zeta_{t-1} + \\ &[\phi_2^2\sinh^2(\zeta)]\zeta_{t-2} + [\phi_3^3\sinh^3(\zeta)]\zeta_{t-3}\end{aligned} \tag{24}$$

Or

$\zeta_t = h_1\zeta_{t-1} + h_2\zeta_{t-2} + h_3\zeta_{t-3}$ is a linear model of order three.

Where

$$h_1 = [\phi_1\zeta\cosh(\zeta) + \phi_1\sinh(\zeta) + 2\phi_2^2\zeta\sinh(\zeta)\cosh(\zeta) + 3\phi_3^3\zeta\sinh^2(\zeta)\cosh\zeta]$$
$$h_2 = [\phi_2^2\sinh^2(\zeta)]; h_3 = [\phi_3^3\sinh^3(\zeta)]$$

The characteristic equation of linear model is $v^3 - h_1v^2 - h_2v - h_3 = 0$

Then

$$h_1 = \lambda_1 + \lambda_2 + \lambda_3, h_2 = -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3), h_3 = \lambda_1\lambda_2\lambda_3$$

Where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the characteristic equation of the model.

The stability condition is that $\forall i = 1,2,3, |\lambda_i| \square 1$.

The Stability Condition Of The Limit Cycle:

Let the limit cycle of period q of the proposed model (21) has the form $x_t, x_{t+1}, x_{t+2}, \ldots\ldots, x_{t+q} = x_t$. The points x_s near the limit cycle is represented as $x_s = x_s + \zeta_s$, then replaced x_t, x_{t-1}, x_{t-2} and x_{t-3} by $x_t + \zeta_t$, $x_{t-1} + \zeta_{t-1}$, $x_{t-2} + \zeta_{t-2}$ and $x_{t-3} + \zeta_{t-3}$ and suppose that the white noise is not effect that is mean $Z_t = 0$, then we have

$$\begin{aligned}x_t + \zeta_t &= [\tfrac{\phi_1}{2}\left(e^{x_{t-1}+\zeta_{t-1}} - e^{-x_{t-1}-\zeta_{t-1}}\right)](x_{t-1} + \zeta_{t-1}) + \\ &[\tfrac{\phi_2^2}{4}\left((e^{x_{t-1}+\zeta_{t-1}} - e^{-x_{t-1}-\zeta_{t-1}})(e^{x_{t-1}+\zeta_{t-1}} - e^{-x_{t-1}-\zeta_{t-1}})\right)](x_{t-2} + \zeta_{t-2}) \\ &+ [\tfrac{\phi_3^3}{8}\left(e^{x_{t-1}+\zeta_{t-1}} - e^{-x_{t-1}-\zeta_{t-1}}\right)^3](x_{t-3} + \zeta_{t-3})\end{aligned} \tag{25}$$

Then

$$\begin{aligned}\zeta_t &= [\phi_1x_{t-1}\cosh x_{t-1} + \phi_1\sinh x_{t-1} + 2\phi_2^2x_{t-2}\sinh x_{t-1}\cosh x_{t-1} + 3\phi_3^3x_{t-3}\sinh^2 x_{t-1}\cosh x_{t-1}]\zeta_{t-1} + \\ &[\phi_2^2\sinh^2(x_{t-1})]\zeta_{t-2} + [\phi_3^3\sinh^3(x_{t-1})]\zeta_{t-3}\end{aligned} \tag{26}$$

The equation (26) is a linear difference equation with a periodic coefficient, which is difficult to solve analytically.

We want to know is whether ζ_t of (26) converges to zero or not, and this can be checked by seeing whether $\left|\frac{\zeta_{t+q}}{\xi_t}\right|$ is less than one or not (Ozaki,1985).

Put t=t+q in equation (26).

$$\begin{aligned}\zeta_{t+q} &= [\phi_1 x_{t+q-1}\cosh x_{t+q-1} + \phi_1 \sinh x_{t+q-1} + 2\phi_2^2 x_{t+q-2}\sinh x_{t+q-1}\cosh x_{t+q-1} + \\ &3\phi_3^3 x_{t+q-3}\sinh^2 x_{t+q-1}\cosh x_{t+q-1}]\zeta_{t+q-1} + \\ &[\phi_2^2\sinh^2 x_{t+q-1}]\zeta_{t+q-2} + [\phi_3^3\sinh^3 x_{t+q-1}]\zeta_{t+q-3}\end{aligned} \tag{27}$$

$$\begin{aligned}\zeta_{t+q} &= \prod_{i=1}^{q}\begin{matrix}[\phi_1 x_{t+i-1}\cosh x_{t+i-1} + \phi_1\sinh x_{t+i-1} + 2\phi_2^2 x_{t+i-2}\sinh x_{t+i-1}\cosh x_{t+i-1} + \\ 3\phi_3^3 x_{t+i-3}\sinh^2 x_{t+i-1}\cosh x_{t+i-1}]\zeta_t\end{matrix} \\ &+ \prod_{i=2}^{q}[\phi_2^2\sinh^2 x_{t+i-1}]\zeta_t + \prod_{i=3}^{q}[\phi_3^3\sinh^3 x_{t+i-1}]\zeta_t\end{aligned} \tag{28}$$

Then by using theorem1, the model of equation (21) is orbitally stable if $\left|\frac{\zeta_{t+q}}{\zeta_t}\right| < 1$.

Therefore

$$\left|\frac{\zeta_{t+q}}{\zeta_t}\right| = \left|\begin{matrix}\prod_{i=1}^{q}\begin{matrix}[\phi_1 x_{t+i-1}\cosh x_{t+i-1} + \phi_1\sinh x_{t+i-1} + \\ 2\phi_2^2 x_{t+i-2}\sinh x_{t+i-1}\cosh x_{t+i-1} + \\ 3\phi_3^3 x_{t+i-3}\sinh^2 x_{t+i-1}\cosh x_{t+i-1}]\end{matrix} \\ + \prod_{i=2}^{q}[\phi_2^2\sinh^2 x_{t+i-1}] + \prod_{i=3}^{q}[\phi_3^3\sinh^3 x_{t+i-1}]\end{matrix}\right| \square 1 \tag{29}$$

Discussion:

Examples:

In this section we gives tow examples to explain how to find the singular points of the proposed model of order one, two and the conditions of stability of singular points.

Example(1):

If $\phi_1 = 0.5$ **,** then the model in equation (2) is $X_t = [0.5\sinh X_{t-1}]X_{t-1} + Z_t$,

Therefore $\zeta = \sinh^{-1}(\frac{1}{0.5}) = \sinh^{-1}(2) = 1.444$

Apply equation (6) we have that $\zeta_t = 2.614\zeta_{t-1}$.

Then, the singular point of the example is not stable because the root of the characteristic equation lies outside the unit circle.

Example(2):

$X_t = [-2.222\sinh X_{t-1}]X_{t-1} + [0.444^2\sinh^2 x_{t-1}]x_{t-2} + Z_t$

Then either $\zeta_2 = -0.4208$ Or $\zeta_1 = 3.1531$

If $\zeta_1 = 3.1531$

Then apply equation (15) we have that $\zeta_t = 62.188\zeta_{t-1} + 26.91\zeta_{t-2}$

The characteristic equation of linear model is $v^2 - 62.188v - 26.91 = 0$

Then $\lambda_1 = 62.6, \lambda_2 = -0.43$ are the roots of the characteristic equation of the model.

Then, the singular point of the example is not stable because one of the roots of the characteristic equation lies outside the unit circle.

If $\zeta_2 = -0.4208$

Then apply equation (15) we have that $\zeta_t = 2.06\zeta_{t-1} + 0.037\zeta_{t-2}$

The characteristic equation of linear model is $v^2 - 2.06v - 0.037 = 0$

Then $\lambda_1 = 2.0778, \lambda_2 = -0.0178$ are the roots of the characteristic equation of the model.

Then, the singular point of the example is not stable because one of the roots of the characteristic equation lies outside the unit circle.

Conclusions:

For The General Order Model, I.E.:

Let

$$X_t = \sum_{i=1}^{p} [\phi_i \sinh(x_{t-1})]^i X_{t-i} + Z_t$$

Then, the absolute values of the characteristic roots of

$$v^p - h_1 v^{p-1} - h_2 v^{p-2} - h_3 v^{p-3} - .... - h_p = 0$$

are all less than one, where

$$h_1 = \phi_1[\zeta \cosh(\zeta) + \sinh(\zeta)] + 2\phi_2^2 \zeta \sinh(\zeta)\cos(\zeta) +$$

$$3\phi_3^3 \zeta \sinh^2(\zeta)\cos(\zeta) + ... + p\phi_p^p \zeta \sinh^{p-1}(\zeta)\cosh(\zeta)$$

$$h_i = \phi_i^i \sinh^i(\zeta); \forall i = 2,3,.........................p-1, p$$

Theorem 2:

A limit cycle of period q , X_{t+1} ,…….., X_{t+q} of the model in equation (1) is orbitally stable when all the eigenvalues of the matrix,

$A = A_q \, . \, A_{q-1}$ ……………………………… A_1

have absolute value less than one, where

$$A_1 = \begin{pmatrix} \phi_1(\sinh x_{t+i-1} + x_{t+i-1}\cosh x_{t+i-1}) + 2\phi_2^2 x_{t+i-2}\sinh x_{t+i-1}\cosh x_{t+i-1} + \sum_{j=3}^{p} j\phi_j^j x_{t+i-j}\sinh^{j-1} x_{t+i-1}\cosh x_{t+i-1} & . & .... & ... & \phi_p^p \sinh^p x_{t+i-1} \\ 1 & 0 & .... & 0 & 0 \\ 0 & 1 & .... & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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