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A Generalized Trapezoid Rule of Error Inequalities

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Abstract: This paper deals A new generalization of ostrowskis integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means.

Key words: trapezoid, Error bounded, inequality, Quadrature rule.

INTRODUCTION

In this work a number of authors have considered error inequalities for some known and some new quadrature rules. Sometimes they have considered generalizations of these rules. For example, the well-known trapezoid quadrature rule is considered in (N. Ujević, 2004; P. Cerone, S.S. Dragomir, 2000) and some generalizations are given in (P. Cerone, S.S. Dragomir, 2000) and (Lj. Dedić, M. Matić, J. Pečarić 2001). In (P. Cerone, S.S. Dragomir, 2000) we can find.

$$\int_a^b f(t)dt = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{2^{k+1}(k+1)!}[f^{(k)}(a) + (-1)^k f^{(k)}(b)]$$
$$+\frac{(-1)^n}{n!}\int_a^b \left(t - \frac{a+b}{2}\right)^n f^{(n)}(t)dt. \tag{1}$$

For n=1 we get the trapezoid rule.

$$\int_a^b f(t)dt = \frac{b-a}{2}[f(a)+f(b)] - \int_a^b \left(t-\frac{a+b}{2}\right) f'(t)dt. \tag{2}$$

Original Results:

Theorem 1:

Let $f: [a,b] \to R$ be a function such that $f^{(n-1)}$ is absolutely continuous. Then

$$\int_a^b f(x)dx = \frac{f(a)+f(b)}{2}(b-a) - \sum_{i=1}^{m} \frac{2i(b-a)^{2i+1}}{2^{2i}(2i+1)!} f^{(2i)}\left(\frac{a+b}{2}\right) + R(f), \tag{3}$$

where $m = \left[\frac{n-1}{2}\right]$, the integer part of $(n-1)/2$,

$$R(f) = (-1)^n \int_a^b S_n(t) f^{(n)}(t)dt \tag{4}$$

And

$$S_n(t) = \begin{cases} \dfrac{(t-a)^{n-1}}{n!}\left[t + \dfrac{(n-2)a - nb}{2}\right], & t \in [a, \dfrac{a+b}{2}] \\ \dfrac{(t-b)^{n-1}}{n!}\left[t + \dfrac{(n-2)b - na}{2}\right], & t \in (\dfrac{a+b}{2}, b] \end{cases} \tag{5}$$

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Proof:

We briefly sketch the proof. First we note that

$$S_1(t) = t - \frac{a+b}{2}$$

$$S_2(t) = \frac{1}{2}(t-a)(t-b) \tag{6}$$

are Peano kernels for the trapezoid quadrature rule, that is, we have

$$\int_a^b S_2(t) f''(t)dt = -\int_a^b S_1(t) f'(t)dt = -\frac{f(a)+f(b)}{2}(b-a) + \int_a^b f(t)dt. \tag{7}$$

Now it is not difficult to prove that (3) holds, for example, by induction.

Corollary 2:

If we introduce the notations

$$H_n(t) = \frac{(t-a)^{n-1}}{n!}\left[t + \frac{(n-2)a - nb}{2}\right] \tag{8}$$

$$T_n(t) = \frac{(t-b)^{n-1}}{n!}\left[t + \frac{(n-2)b - na}{2}\right] \tag{9}$$

Then we see that H_n and T_n form Appell sequences of polynomials, that is

$$H'_n(t) = H_{n-1}(t), \qquad T'_n(t) = T_{n-1}(t), \qquad H_\circ(t) = T_\circ(t) = 1.$$

Thus we can also use integration by parts to prove that (3) holds.

Theorem 3:

The Peano kernels $S_n(t), n > 1$, satisfy

$$\int_a^b S_n(t)dt = 0, \qquad if\ n\ is\ odd, \tag{10}$$

$$\int_a^b |S_n(t)|dt = \frac{n(b-a)^{n+1}}{2^n(n+1)!}, \tag{11}$$

$$\max_{t\in[a,b]} |S_n(t)| = \frac{(n-1)(b-a)^n}{2^n n!}. \tag{12}$$

Proof:

A simple calculation gives

$$\int_a^b S_n(t)dt = -\frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}[1-(-1)^{n+1}]. \tag{13}$$

From the above relation we see that (10) holds, since $1-(-1)^{n+1}=0$ if n is odd.

We have

$$\int_a^b |S_n(t)|dt = \int_a^{\frac{a-b}{2}} |H_n(t)|dt + \int_{\frac{a+b}{2}}^b |T_n(t)|dt$$

$$= \frac{n(b-a)^{n+1}}{2^n(n+1)!}. \tag{14}$$

Finally, we have

$$\max_{t\in[a,b]} |S_n(t)| = \max\{\max_{t\in[a,\frac{a+b}{2}]} |H_n(t)|, \quad \max_{t\in[\frac{a+b}{2},b]} |T_n(t)|\}$$

$$= \max\left\{\left|H_n\left(\frac{a+b}{2}\right)\right|, \left|T_n\left(\frac{a+b}{2}\right)\right|\right\}$$

$$= \frac{(n-1)(b-a)^n}{2^n n!} \tag{15}$$

We introduce the notations

$$I = \int_a^b f(t)dt,$$

$$F = \frac{f(a)+f(b)}{2}(b-a) - \sum_{i=1}^{m} \frac{2i(b-a)^{2i+1}}{2^{2i}(2i+1)!} f^{(2i)}\left(\frac{a+b}{2}\right). \tag{16}$$

Theorem 4:

Let $f : [a,b] \to R$ be a function such that $f^{(n-1)}, n > 1$, is absolutely continuous and there exist real numbers γ_n, Γ_n such that $\gamma_n \le f^{(n)}(t) \le \Gamma_n, t \in [a,b]$. Then

$$|I-F| \le \frac{\Gamma_n - \gamma_n}{(n+1)!}\frac{n}{2^{n+1}}(b-a)^{n+1} \qquad if\ n\ is\ odd \tag{17}$$

And

$$|I-F| \le \frac{(b-a)^{n+1}\, n}{2^n(n+1)!}\left\|f^{(n)}\right\|_\infty \qquad if\ n\ is\ even. \tag{18}$$

Proof:

Let n be odd. From (4) and (10) we get

$$R(f) = (-1)^n \int_a^b S_n(t) f^{(n)}(t)dt = (-1)^n \int_a^b S_n(t)\left[f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2}\right]dt$$

Such that we have

$$|R(f)| = |I - F| \leq \max_{t\in[a,b]} \left| f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \int_a^b |S_n(t)|dt. \tag{19}$$

We also have

$$\max_{t\in[a,b]} \left| f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \leq \frac{\Gamma_n - \gamma_n}{2}. \tag{20}$$

From (19), (20) and (7) we get

$$|I - F| \leq \frac{\Gamma_n - \gamma_n}{(n+1)!} \frac{n}{2^{n+1}} (b-a)^{n+1}.$$

Let n be even. Then we have

$$|R(f)| = |I - F| \leq \int_a^b |S_n(t)|dt \, \|f^{(n)}\|_\infty = \frac{(b-a)^{n+1} n}{2^n (n+1)!} \|f^{(n)}\|_\infty. \tag{21}$$

Theorem 5;

Let $f : [a,b] \to R$ be a function such that $f^{(n-1)}, n > 1$, is absolutely continuous and let n be odd. If there exists a real number γ_n such that $\gamma_n \leq f^{(n)}(t), t \in [a,b]$ then

$$|I - F| \leq (G_n - \gamma_n) \frac{(n-1)(b-a)^{n+1}}{2^n n!}, \tag{22}$$

Where

$$G_n = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}$$

If there exists a real number Γ_n such that $f^{(n)}(t) \leq \Gamma_n, t \in [a,b]$ then

$$|I - F| \leq (\Gamma_n - G_n) \frac{(n-1)(b-a)^{n+1}}{2^n n!}. \tag{23}$$

Proof:

we have

$$|R(f)| = |I - F| = \left| \int_a^b (f^{(n)}(t) - \gamma_n) S_n(t) dt \right|, \tag{24}$$

Since (6) holds. Then we have

$$\left| \int_a^b (f^{(n)}(t) - \gamma_n) S_n(t) dt \right| \leq \max_{t\in[a,b]} |S_n(t)| \int_a^b (f^{(n)}(t) - \gamma_n) dt$$

$$= \frac{(n-1)(b-a)^n}{2^n n!} [f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma_n (b-a)]$$

$$= \frac{(n-1)(b-a)^{n+1}}{2^n n!} (G_n - \gamma_n). \tag{25}$$

In a similar way we can prove that (14) holds.

Corollary 6:

Note that we can apply the estimations (17) and (18) only if $f^{(n)}$ is bounded. On the other hand, we can apply the estimation (22) if $f^{(n)}$ is unbounded above and we can apply the estimation (23) if $f^{(n)}$ is unbounded below.

Example:

If we consider the integral (special function) $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt$ and apply the summation formula (3) to this integral. We get the summation formula $\mathrm{Si}(x) = F(x) + R(x)$, where

$$F(x) = \frac{x}{2}\left(1 + \frac{\sin x}{x}\right) - \sum_{i=1}^{m} \frac{2ix^{2i+1}}{2^{2i}(2i+1)!} f^{(2i)}\left(\frac{x}{2}\right) \tag{26}$$

And $f(t) = (\sin t)/t$. We calculate the derivatives $f^{(j)}(t)$ as follows. We have

$$(g(t)h(t))^{(j)} = \sum_{k=0}^{j} \binom{j}{k} g^{(k)}(t) h^{(j-k)}(t).$$

If we choose $g(t) = \sin t$ and $h(t) = 1/t$ then we get

$$f^{(j)}\left(\frac{x}{2}\right) = \sum_{i=0}^{\left\lfloor \frac{j-1}{2} \right\rfloor} \binom{j}{2i+1} (-1)^{j-i+1} \frac{(j-2i-1)!\, 2^{j-2i}}{x^{j-2i}} \cos\frac{x}{2}$$
$$+ \sum_{i=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \binom{j}{2i} (-1)^{j-i} \frac{(j-2i)!\, 2^{j-2i+1}}{x^{j-2i+1}} \sin\frac{x}{2}. \tag{27}$$

We now compare the summation formula (26) with the known compound formula (for the trapezoid rule),

$$\int_0^x f(t)dt = \frac{h}{2}[f(0) + F(x)] + h\sum_{i=1}^{n-1} f(x_i) + R(x). \tag{28}$$

Where $x_i = ih, h = x/r$

Let us choose $x = 1$. The "exact" value is $\mathrm{Si}(1) = 0.946083070367$.

If we choose $m = 2$ in (26) and $n = 100$ in (28) then we get $\mathrm{Si}(1) \approx 0.946080675618$ and $\mathrm{Si}(1) \approx 0.94608056025$, respectively.

If we choose $m = 3$ in (26) and $n = 8200$ in (28) then we get $\mathrm{Si}(1) \approx 0.946083078954$ and $\mathrm{Si}(1) \approx 0.946083069999$, respectively.

If we choose $m = 4$ in (26) and $n = 32000$ in (28) then we get $\mathrm{Si}(1) \approx 0.946083070347$ and $\mathrm{Si}(1) \approx 0.946083070342$, respectively.

All calculations are done in double precision arithmetic. The first approximate results (derived from 26) are obtained much faster than the second approximate results (derived from 28). The same is valid if we use some quadrature rule of higher order, for example Simpson's rule. This is a consequence of the fact that we have to calculate the function $\sin t$ many times when we apply the compound formula and we have only to calculate $\sin(x/2)$ and $\cos(x/2)$ when we apply the summation formula.

Similar summation formulas can be obtained for the integrals (special functions) $\int_0^x [(e^t - 1)/t]dt$, $\int_0^x [(\cos t - 1)/t]dt$, $\int_0^x \exp(-t^2)dt$, *etc*.

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