A General Approach for Applying Homotopy Perturbation Method for Differential and Systems of Differential Equations

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Abstract: Homotopy perturbation method is an efficient method for solving nonlinear equations. But there is still no general approach for applying the method. In this paper we propose some efficient rules to start the homotopy perturbation method for differential and systems of differential equations with boundary or initial conditions.

Key words: Homotopy perturbation method; Analytic solutions; Nonlinear equations; Systems of ordinary differential equations

INTRODUCTION

Homotopy perturbation method (HPM) has first come to light by He (1998) and has been accepted as an elegant and efficient tool for solving nonlinear equations (He, 1998; He, 1998; Babolian, 2009; Chowdhury, 2007; Chowdhury, 2009). In HPM an auxiliary equation, namely *homotopy equation,* is constructed using the original equation under study. The homotopy equation contains an embedding parameter (we denote it by *p)* and it uses an initial guess of the exact solution to start an iterative-like procedure to hopefully converge to the desired solution. Actually homotopy equation is a convex combination of the original equation, and another (usually closely related) equation which has the chosen initial guess *as* one of its solutions. *we* consider the equation $Au = 0$, where A is a nonlinear operator and *u* is the **unknown** function, then the homotopy equation will be as follows:

$$H(u,p) = (1-p)(Lu - Lu_0) + p(Au) = 0,$$

(1)

where *L* is an auxiliary (and mostly linear) operator, this operator is chosen beforehand based on some knowledge about the equation under study. Together with the initial guess, the operator *L,* are the two key tools of HPM. For $p = 0$, the above homotopy equation is just a quite simple equation based on the operator L, and for $p = 1$, we have the original equation. Assume that the solution of the homotopy equation could be represented as $u = u_0 + u_1 p + u_2 p^2 + \dots$. As $p \to 1$ the homotopy equation converges to the original equation, so we expect $u = u_0 + u_1 p + u_2 p^2 + \dots$ to converge to the solution of the original equation when $p \to 1$.

This is, more or less, the common terminology which is use by almost all authors for using and applying homotopy perturbation technique.

Considering the homotopy equation, one can easily find out that choosing L and u_o will determine the homotopy equation completely, so the HPM is restricted to just choosing appropriate L and u_o. In this paper we would present some general rules for suitable choosing these parameters. In section 2, after a short review on the approaches by other authors, guiding rules are presented and discussed. Section 3, applies the suggested rules to some differential equations (partial and ordinary) to verify its efficiency.

2- Some Suggestions for Applying HPM:

Studying the method we understand that, the idea is straightforward, but every one has solved it's own problem, heuristically, using some tricks. Although this show the flexibility of the method, a beginner confronts problems using it. Perusing most of the equations that have been solved by homotopy perturbation method,

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it seems that one simple way to start the homotopy perturbation method is to choose different L and u_o'S and test them to find the most accurate one.

To have a review on how authors use HPM for solving differential equations, consider the following example (Babolian, 2009).

Example 1:

Consider the time-dependent Emden-Fowler equation:

$$y_{xx} + \frac{2}{x}y_{xx} - (6 + 4x^2 - \cos(t))y = y_t$$

with the initial condition $y(x,0) = e^{x^2}$ and boundary conditions $y(0,t) = e^{\sin(t)}$ and

$$y_x(0,t) = 0$$

Let's choose $Lv = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{2}{x}\dfrac{\partial v}{\partial x}$, and $u_0 = e^{\sin(t)}$ Chowdhury and Hashim (2007). This gives the homotopy equation as follows:

$$v_{xx} + \frac{2}{x}u_x - u_{0xx} - \frac{2}{x}u_{0x} + p\left\{u_{0xx} + \frac{2}{x}u_{0x} - (6 + 4x^2 - \cos(t))v - v_t\right\} = 0.$$

Using the expansion $u = v_0 + v_1 p + v_2 p^2 + \dots$, and equating to zero the coefficients of powers of p, we have the following sub equations:

$$v_{0xx} + \frac{2}{x}v_{0x} - u_{0xx} - \frac{2}{x}u_{0x} = 0, \qquad\qquad v_0(0,t) = e^{\sin(t)}, v_{0x}(0,t) = 0,$$

$$v_{1xx} + \frac{2}{x}v_{1x} - u_{0xx} - \frac{2}{x}u_{0x} - (6 + 4x^2 - \cos(t))v_0 - v_{0t} = 0, \qquad v_1(0,t) = v_{1x}(0,t) = 0,$$

$$v_{2xx} + \frac{2}{x}v_{2x} - (6 + 4x^2 - \cos(t))v_1 - v_{1t} = 0, \qquad\qquad v_2(0,t) = v_{2x}(0,t) = 0,$$

$$\vdots$$

$$\tag{2}$$

Subsequently solving the above equations we have:

$$v_0(x,t) = e^{\sin(t)} \tag{3}$$

So, the final solution is $u = v_0 + v_1 + v_2 + \dots = e^{\sin(t)}\left\{1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \dots\right\}$, which

leads to the closed form $u(x,t) = e^{\sin(t)} + x^2$ that is the exact solution. In this equation we have so many choices for L such as:

$$\frac{2}{x}y_x - (6 + 4x^2 - \cos(t))y, \quad y_t, \quad y_t - (6 + 4x^2 - \cos(t))y, \quad y_{xx} - (6 + 4x^2 - \cos(t))y, \text{ and } y_{xx}.$$

In the same way there are different choices for u_0. Here there are two basic questions:
Why do the authors choose these L and u_0? Do other possible choices have the same results?
The answer of the second question is negative, this can be easily find out by perusing
other choices for *L*. But no body have tried to answer the first question.

We can find out some points from the previous example, or any other equation that has been solved by HPM:

There is no general way to find the best *L* and to start the homotopy perturbation method.

Moreover all of the authors that used HPM for solving PDE's and ODE's, apply the initial/boundary conditions to the first element (v_0) of the final solution ($u = v_0 + v_1 + ...$), and vanish these conditions for other elements $(v_i, I = 1,2...)$. For instance consider the above example, here the boundary conditions for the main

solution is, $y(0,t) = e^{\sin(t)}$ and $y_x(0,t)$. In the sub equations (2), the authors assigned these conditions as follows,

$$v_0(0,t) = e^{\sin(t)}, \quad v_{0x}(0,t) = 0,$$

and

$$v_i(0,t) = 0 \quad , v_{ix}(0,t) = 0 \,, \, i = 1,2,... \;.$$

Here we present some suggestions for applying the homotopy perturbation method to differential and system of differential equations.

Consider the differential equation. *A(u)-f(r)=0* with the appropriate initial/boundary conditions, in almost all cases the homotopy equation is constructed as follows:

$$H(v,p) = (1-p)(Lu - Lu_0) + p(A(u) - f(r)) = 0$$

By expanding $u = v_0 + v_1 p + v_2 p^2 + ...,$ and equating to zero the coefficients of powers of p, we

have the sub equations:

$$Lv_0 = Lu_0,$$

$$Lv_1 = F_1(u_0, v_0),$$

$$Lv_2 = F_2(v_0, v_1),$$

$$Lv_3 = F_3(v_0, v_1, v_2),$$

$$\vdots$$

Applying the initial/boundary conditions to v_0 and vanishing them to other elements, is the straightforward way to solve sub equations in HPM. Moreover one obvious solution for the first sub equation is $v_0 = u_0$.

We have to choose *L* and u_0 based on our experience. Moreover, even if we approximate the solution, i.e.

$$u \simeq v_0 + v_1 + ... + v_n,$$ still initial/boundary conditions are valid for the approximate solution.

Our suggestion is to construct the homotopy equation as:

$$H(v,p) = (1-p)Lu + p\left(A(u) - f(r)\right) = 0.$$

Here we have the following sub equations:

$$Lv_0 = 0, \quad Lv_1 = F_1(u_0), \quad Lu_2 = F_2(v_0, v_1), ...$$

So we do not choose initial guess $v_0 = u_0$ we find v_0 by solving the first equation. Moreover from the above discussion it's obvious that homotopy equation is constructed based on L. So if we choose L the homotopy equation will be identified. The following rules are suggested for choosing L.

Rule 1:

L must depends on the main equation, but not necessarily a part of that. The approximate solution of the equation is u $\simeq v_0 + v_1 + ... v_n$, that v_i's are the solutions of the above sub equations. These sub equations are completely based on L, so it is natural to choose L, based on the main equation.

Rule 2:

L must be easy-to-solve.
As we know in HPM we replace a nonlinear equation with infinite number of linear equations. So the method is useful if those sub equations be easy-to-solve.

Rule 3:

It is better to choose L based on initial/boundary conditions of the main equation.

Rule 4:

Sometimes we get better results by assigning initial/boundary conditions not in classic way(applying them to and vanishing them for other elements). We will discuss these rules with details by some examples in next section.
In homotopy equation, for $p = 1$. we have

$$H(v,1) = Lv + N(v) - f(r) = A(v) - f(r) = 0,$$

so it seems that for any choice of L the final result will be the same. But this is not true, we can see from different examples that L, has an essential role in convergence of the solution of HPM to the main solution.
As we mentioned before HPM has been used for solving systems of equations. Studying previous works on systems of equations, we could find out that constructing homotopy equations in a system of equations are just the same as for single equations. Consider the following system of differential equations:

$$A_1(u_1, ..., u_n) - f_1(r) = 0,$$

$$A_2(u_1, ..., u_n) - f_2(r) = 0,$$

$$\vdots$$

$$A_n(u_1, ..., u_n) - f_n(r) = 0,$$

(4)

with suitable initial/bounary conditions. In classic view homotopy equations are constructed as follows:

$(1-p) L_1 u_1 + p(A_1 (u_1 ,... , u_n)- f_1 (r)) = 0,$

$(1-p) L_2 u_2 + p(A_2 (u_1 ,... , u_n)- f_2 (r)) = 0,$

$\vdots$

$(1-p) L_n u_n + p(A_n (u_1 ,... , u_n)- f_n (r)) = 0,$ 　　　　　　(5)

where $Ai = Li + Ni$, for $i = 1, 2, \ldots, n$.

Now by assumption $u_i = u_{i0} + u_{i1} p + u_{i2} p^2 + \ldots$, for $i = 0, 1, 2, \ldots$, and equating to zero the coefficients of powers of p. in the above equations will be find step by step. In the first step u_{i0}'s will be identified, and so on.

It is worth mentioning that in the proposed way, we solve equations in each step from the first one to n-th equation. It means that in any step k, previous elements have been determined, for example in the third equation, u_{1k}, u_{2k} are obtained before, so we can use them in this step. This fact guides us to propose the following system of homotopy equations:

$(1\text{-}p) L_1 u_1 + p(A_1 (u_1, \ldots, u_n)\text{-} f_1 (r)) = 0,$

$(1\text{-}p) L_2 (u_1, u_2) + p(A_2 (u_1, \ldots, u_n)\text{-} f_2 (r)) = 0,$

$\vdots$

$(1\text{-}p) L_n (u_1, u_n) + p(A_n (u_1, \ldots, u_n)\text{-} f_n (r)) = 0,$ \hfill (6)

In any step k, we solve the equations of (6) sequentially. Solving the first equation we get u_{1k}, so in the next equation $(1\text{-}p) L_2 (u_1, u_2) + p(A_2 (u_1, \ldots, u_n)\text{-} f_2 (r)) = 0$, the only unknown is that can be evaluate by solving a linear equation, and so on. In the above system for homutopy perturbation method we use the property of HPM and system of equations in the same time.

3- Examples:
3.1 Reaction-diffusion Equation:

The one-dimentional time-dependent reaction-diffusion equation is of the following form (Sami Bataineh, 2007):

$$w_t (x,t) = Dw_{xx} (x, t) + q(x, t)w (x, t), \qquad (x, t) \in \mathbb{R}^2$$

where $w_t = \dfrac{\partial w}{\partial t}, w_{xx} = \dfrac{\partial^2 w}{\partial x^2}$ and is the concentration, $q(x, t)$ is the reaction parameter, and $D > 0$ is the diffusion coefficient.

The initial and boundary conditions are

$w(x.0) = g(x), \quad w(0,t) = f_0(t),$

$w_x (0, t) = f_1 (t), \qquad x, t \in \mathbb{R}$

Reaction-diffusion equations describe a wide variety of nonlinear systems in physics, chemistry, and engineering (Button, 1998; Cantrell, 2003; Grindrod, 1996; Smoller, 1994). Here we solve some special cases for $q(x,t) ==$ *const.* and $q(x, t) = 2t$.

3.1.1 q(x,t)=const:

Let

$q(x,t) = -1, \quad D = 1, \quad g(x) = e^{-x} + x$

$f_0 (t) = 1, \quad f_1 (t) = e^{-t} - 1,$

so the equation will be $w_t = w_{xx} - w$, with the conditions $w(x, 0) = e^{-x} + x$, $w (0, t) = 1$, and $w_t (0,t) = e^{-t} - 1$, for $x, t \in$ R. In this case the equation is linear and we can choose any part of the equation as L except the part that contains w_t and w_{xx} or w and w_{xx} simultaneously, because in these cases L is not easy to solve. So by Rule 2, we have the following choices for L:

$w_t, -w_{xx}, w_t + w.$

In this equation we have initial and boundary conditions. If we use the initial condition, then by Rule 3, we have to choose $w_t + w$ *or* w_t as L. Moreover since $w_t + w$ in comparison with , is more dependent on the main equation, by Rule 1, L must be $w_t + w$. On the other hand, with the boundary conditions, we have only one choice, $-w_{xx}$. In the following we solve the homotopy equation in the above mentioned cases.

Case 1:

Let $L = w_t + w$, with the init iril condition $w\ (x,\ 0) = e^{-x} + x$. Here the homotopy equation will be as followes:

$$H(w,p) = (1 - p)(w_t + w) + p(w_t + w - w_{xx}) = 0,$$

or,

$$H(w,p) = w_t + w - pw_{xx} = 0$$

Let:

$$w = w_0 + w_1 p + w_2 p^2 + \dots \ . \tag{7}$$

be the solution of equation (6). Now by the assumptions $w_2\ (x,\ 0) = e^{-x} + x$ and $w_1\ (x,\ 0) = w_2\ (x,\ 0) = \dots = 0$ the initial condition for the final solution ($w = w_0 + w_1 + w_2 + \dots$) holds. Using (6) along with (7), one has:

$$(w_{0t} + w_{1t}\ p + w_{2t}\ p^2 + \dots) + (w_0 + w_1\ p + w_2\ p^2 + \dots) - p\ (w_{0xx} + w_{1xx}\ p + w_{2xx}\ p^2 + \dots) = 0 \tag{8}$$

where $w_{it} = \dfrac{\partial w_i}{\partial t}, and\ w_{ixx} = \dfrac{\partial^2 w_i}{\partial x^2}$.

Equating to zero the coefficients of the powers of p in (8), we have the following sub equations:

$w_{0t} + w_0 = 0$ $\qquad\qquad\qquad\qquad w_0\ (x,0) = e^{-x} + x,$
$w_{1t} + w_1 - w_{0xx} = 0$, $\qquad\qquad w_1\ (x,0) = 0$,
$w_{2t} + w_2 - w_{1xx} = 0$, $\qquad\qquad w_2\ (x,0) = 0$,
$\vdots$

which imply that:

$$w_0(x,t) = e^{-t}e^{-x} + e^{-t}x$$

$$w_1(x,t) = te^{-t}e^{-x}$$

$$w_2(x,t) = \frac{t^2}{2}e^{-t}e^{-x}$$

$\vdots$

It can be shown by induction that., $w_n(x,t) = \dfrac{t^n}{n!}e^{-t}e^{-x}$. Consequently:

$$w = w_0 + w_1 + w_2 + \dots$$

$$= e^{-t}x + e^{-t}e^{-x}\left(1 + t + \frac{t^2}{2} + \dots\right)$$

$$= e^{-t}x + e^{-t}e^{-x}e^{t}$$

$$= e^{-t}x + e^{-x}$$

which is the exact solution of the equation.

Case 2:

Let $w = -w_{xx}$ boundary conditions, $w(0,t) = 1$, $w_x(0, t) = e^{-t} -1$, so the homotopy equation will be as follows:

$$H(w, p) = (1-p) (-w_{xx}) + p (w_t + w - w_{xx}) = 0$$

or

$$H(w, p) = -w_{xx} + p (w_t + w) = 0 \tag{9}$$

with the boundary conditions

Let, $w = w_0 + w_1 p + w_2 p_2 + ... \tag{10}$

so by the assumptions $w_0(0,t) = 1$, $w_1(0,t) = w_2(0,t) = ...$ and $w_{0x}(0,t) = e^{-t} - 1$, $w_{1x}(0,t) = w_{2x}(0,t) = ... = 0$, the boundary conditions for w holds. By (9) and (10), we have:

$$- (w_{0xx} + w_{1xx} p + w_{2xx} p^2 + ...) + p ((w_{0t} + w_{1t} p + w_{2t} p^2 + ...) + (w_0 + w_1 p + w_2 p^2 + ...)) = 0 \tag{11}$$

with the above boundary conditions.
Equating to zero the coefficients of the powers of *p* in (11), we have the following sub equations:

$- w_{0xx} = 0$ $\qquad\qquad$ $w_0(0,t) = 1$, $w_{0x}(0,t) = e^{-t} -1$
$- w_{1xx} + w_{0t} + w_0 = 0$ $\qquad$ $w_1(0,t) = 0$, $w_{1x}(0,t) = 0$
$- w_{2xx} + w_{1t} + w_1 = 0$ $\qquad$ $w_2(0,t) = 0$, $w_{2x}(0,t) = 0$
$\vdots$

which imply that,

$$w_0 = 1 + x(e^{-t} - 1), \quad w_1 = \frac{x^2}{2} - \frac{x^3}{3!}, \quad w_2 = \frac{x^4}{4!} - \frac{x^5}{5!}, ...$$

one can show by induction that, $w_n = \frac{x^{2n}}{(2n)!} - \frac{x^{2n+1}}{(2n+1)!}$.

So: $w = w_0 + w_1 + w_2 + ... = 1 + xe^{-t} - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} = xe^{-t} + e^{-x}$

which is the exact solution.

3.1.2 $q(x,t)=2t$:

Let
$q(x,t)= 2t$, $D=1$, $g(x)=e^x$, $f_0(t) = e^{t+t2}$), $f_1(t) = e^{t+t2}$)

So the equation will be $w_t = w_{xx} + 2tw$. with the conditions $w(x,0) = e^x$, $w(0,t) = e^{t+t2}$), $w_x(0,t) = e^{t+t2}$), for $x, t \in \mathbb{R}$

In this case the equation is again linear. So we have four choices for an easy-to-solve *L* (rule 2):

$-w_{xx}$, w_t, $-2tw$, $w_t - 2tw$,

If we use the initial condition, we have to choose w_t or $w_t - 2tw$, as *L* (rule 3). Moreover since $w_t - 2tw$, in comparison with w_t is more dependent on the main equation, by rule 1, *L* must be $w_t - 2tw$.

On the other hand, with the boundary conditions, we have only one choice for *L*, $-w_{xx}$. In the following we solve the homotopy equation in the above mentioned cases.

Case 1:

Let $L = -w_{xx}$ with the conditions, $w(0,t) = w_x(0,t) =$. Here the homotopy equation is:

$H(w,p) = (1-p) (- w_{xx}) + p(w_t - 2tw- w_{xx}) = 0.$

Or

$H(w,p) = - w_{xx} + p (w_t - 2tw) = 0$ (12)

Let

$w = w_0 + w_1 p + w_2 p^2 + \dots ,$ (13)

And $w_o (0,t) = w_{ox} (0,t) = e^{t+t2}$, $w_1 (0,t) = w_2 (0,t) = \dots = 0$, $w_{1x} (0,t) = w_{2x} (0,t) = \dots = 0$.
Using (12) along with (13) one has:

$- (w_{oxx} + w_{1xx} p + w_{2xx} p^2 + \dots) + p((w_{ot} + w_{1t} + w_{2t} p^2 + \dots) -2t(w_o + w_1 p + w_2 p^2 + \dots)) = 0$ (14)

with the above boundary conditions.
    Equating to zero the coefficients of tlie powers of p in (14), we have the following sub equations:

$- w_{oxx} = 0,$ $w_o (0,t) = e^{t+t2})$, $w_{ox} (0,t) = e^{t+t2})$,
$-w_{1xx} + w_{ot} - 2t w_o = 0,$ $w_1 (0,t) = 0$, $w_{1x} (0,t) = 0,$
$-w_{2xx} + w_{1t} - 2t w_1 = 0,$ $w_2 (0,t) = 0$, $w_{2x} (0,t) = 0,$
$\vdots$

which imply that:

$$w_0 = e^{t+t^2} + e^{t+t^2} x, w_1 = e^{t+t^2} + e^{t+t^2} \left(\frac{x^2}{2!} + \frac{x^3}{3!} \right), w_1 = e^{t+t^2} + e^{t+t^2} \left(\frac{x^4}{4!} + \frac{x^5}{5!} \right), \dots,$$

It can be easily shown that, $w_n = e^{t+t^2} \left(\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} \right)$. So the final solution will be:
$$w = w_0 + w_1 + w_2 + \dots$$
$$= e^{t-t^2} \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)$$
$$= e^{t-t^2} e^x$$

which is the exact solution.

Case 2:
    Let $L = w_t - 2tw$, with the condition $w(x,0) = e^x$. In this case the homotopy equation will be:

$H9w,p) = (1-p) (w_t-2tw) + p(w_t - 2tw - w_{xx}) = 0$

or

$w_t - 2tw - p w_{xx} = 0.$ (15)

Let

$w = w_0 + w_1 p + w_2 p^2 + \dots$ (16)

And $w_0 (x,0) = e^x$, $w_1 (x,0) = w_2 (x,0) = \dots = 0$.
Using (16) along with (15) one has:

$(w_{0t} + w_{1t} p + w_{2t} p^2 + \dots) -2t(w_0 + w_1 p + w_2 p^2 + \dots) -p(w_{0xx} + w_{1xx} p + w_{2xx} p^2 + \dots) = 0$ (17)

Equating to zero the coefficients of the powers of p in (17), we have the following sub equations:

$w_{0t} - 2t\, w_0 = 0$ $w_0\,(x,0) = e^x,$
$w_{1t} - 2t\, w_1 - w_{0xx} = 0$ $w_1\,(x,0) = 0,$
$w_{2t} - 2t\, w_2 - w_{1xx} = 0$ $w_2\,(x,0) = 0,$
$\vdots$

solving the above equations we have:

$w_0 = e^{t^2} e^x,$

$w_1 = e^{t^2} e^x t,$

$w_2 = e^{t^2} e^x \dfrac{t^2}{2},$

$\vdots$

by induction we have: $w_n = t^{t^2} e^x \dfrac{t^n}{n!}$
So:

$w = w_0 + w_1 + w_2 + \ldots$

$= e^x e^{t^2} \left(1 + t + \dfrac{t^2}{2} + \ldots\right)$

$= e^x e^{t - t^2}$

That is the exact solution.

3.2 Helmholtz equation:
Helmholtz equation is of the form

$$\nabla^2 u + f(x,y)u = g(x,y) \tag{18}$$

With the initial/boundary conditions:

$u(0,y) = \psi 1(y), u_x\,(0,y) = \psi_2\,(y),\ u(x,0) = \psi_3\,(x),\ u_y\,(x,0) = \psi_4\,(x)$

$f(x,y),\ g(x,y),\ \psi_1\,(y),\ \psi_2\,(y),\ \psi_3\,(x)$ and $\psi_4\,(x)$, are known functions (Momani, 2006).
A special case of Helmholtz equation is of the form

$$u_{xx} + u_{yy} = u \tag{19}$$

with the conditions $u(0,y) = y$ and $u_x\,(0,y) + \cosh\,(y).$

Here we have three choices for easy_to_solve L (rule 2): u, u_{xx} , u_{yy}.
Up to the conditions (rule 3) we have just one choice $Lu = u_{xx}$.
So the homotopy equation is as follows:

$H(u.p) = (1-p)\,(u_{xx}) + p(u_{xx} + u_{yy} - u) = 0$

Or

$$H(u.p) = u_{xx} + p(u_{yy} - u) = 0 \tag{20}$$

with the conditions $u(0,y) = y$ and $u_x\,(0,y) = y + \cosh\,(y).$

Let:

$$u = u_0 + u_1\, p + u_2\, p^2 + \ldots \tag{21}$$

with the conditions $u_0(0,y) = y$. $u_{0x}(0,y) = y + cosh(y)$, $u_1(0,y) = u_2(0,y) = ... = 0$ and $u_{1x}(0,y) = u_{2x} x(0,y) = ... = 0$.

Using (21) along with (20), one has:

$(u_{0xx} + u_{1xx} p + u_{2xx} p^2 + ...) + p((u_{0yy} + u_{1yy} p + u_{2yy} p^2 + ...) - (u_0 + u_1 p + u_2 p^2 + ...)) = 0$,

with the proposed conditions for u_i, $i=0,1,2....$.

Equating to zero the coefficients of the powers of p in (22), we have the following sub equations:

$u_{0xx} = 0$ $u_0(0,y) = y$, $u_{0x}(0,y) = y + cosh(y)$

$u_{1xx} + u_{0yy} - u_0 = 0$ $u_1(0,y) = 0$, $u_{1x}(0,y) = 0$

$u_{2xx} + u_{1xx} - u_1 = 0$ $u_2(0,y) = 0$, $u_{2x}(0,y) = 0$

$\vdots$

by solving these equations we have

$$w_0 = xy + x\cosh(y) + y$$

$$w_1 = y\left(\frac{x^2}{2} + \frac{x^3}{3!}\right)$$

$$w_2 = y\left(\frac{x^4}{4!} + \frac{x^5}{5!}\right)$$

$\vdots$

and by induction we have, $w_n = y\left(\dfrac{x^{2n}}{(2n)!} + \dfrac{x^{2n+1}}{(2n+1)!}\right)$.

So the final solution is:

$$w = w_0 + w_1 + w_2 + ... = x\cosh(y) + y + yx + y\left(\frac{x^2}{2} + \frac{x^3}{3!}\right) + y\left(\frac{x^4}{4!} + \frac{x^5}{5!}\right)$$

$$= x\cosh(y) + ye_x$$

Which is the exact solution.

3.3 RLW Equation:

Cconsider the RLW equation which reads (Ganji, 2006):

$$ut - uxx + \left(\frac{u^2}{2}\right)x = 0 \qquad x \in \mathbb{R}, t > 0$$

with the initial condition

Here for an easy to solve *L* we have two choices (rule 2),

$$u_t, \left(\frac{u^2}{2}\right)x$$

but $\left(\frac{u^2}{2}\right)x$, is not a right choice for the initial condition (rule 3). So we have only one choice, $Lu = u_t$. The homotopy equation in this case is $\left(\left(\frac{u^2}{2}\right)x = uu_x\right)$:

$H(u,p) = (1-p) u_t + p(u_t - u_{xxt} + uu_x) = 0$

or

$H(u,p) = u_t + p(uu_x - u_{xxt}) = 0$ (24)

Let: $u = u_0 + u_1 p + u_2 p^2 + ...$ (25)

and, $u_0 (x,0) = x$, $u_1(x,0) = u_2(x,0) = ... = 0$.
By (24) and (25) one can easily see that:

$(u_{0t} + u_{1t} p + u_{2t} p^2 + ...) + p[(u_0 + u_1 p + u_2 p^2 + ...)$
$(u_{0x} + u_{1x} p + u_{2x} p^2 + ...) - (u_{0xxxt} + u_{1xxxt} p + u_{2xxxt} p^2 + ...)] = 0.$ (26)

Equating to zero the coefficients of the powers of p in (26), we have the following sub equations:

$u_{0t} = 0$ $u_0 (x,0) = x,$
$u_{1t} + u_0 u_{0x} - u_{0xxt} = 0$ $u_1 (x,0) = 0,$
$u_{2t} + u_1 u_{0x} + u_0 u_{1x} - u_{1xxt} = 0$ $u_2 (x,0) = 0,$
$u_{3t} + (u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}) - u_{2xxt} = 0$ $u_3 (x,0) = 0,$
$\vdots$

solving the above equations we have:

$u_0 = x,$
$u_1 = - xt = x(-t),$
$u_2 = xt^2 = x(-t)^2,$
$u_3 = -xt^3 = x(-t)^3,$
$\vdots$

by induction we find out that, $u_n = x(-t)^n$.
So the final solution is:

$$u = u_0 + u_1 + u_2 + u_3 + ... = x(1 + (-t) + (-t)^2 + (-t)^3 + ...) = \frac{x}{1+t}$$

which is the exact solution.

3.4 Laplace Equation:

     Consider the two-dimensional Laplace equation (Sommerfeld, 1949), $u_{xx} + u_{yy} = 0$, subject to the boundary conditions of,

$u(0,y) = 0$, $u(\pi,y) = sinh(\pi) cos(y)$, $u(x,0) = sinh(x)$, $u(x,\pi) = -sinh(x)$.

Let $Lu = u_{xx}$, so the homolopy euation will be as follows:

$H(v,p) = (1-p)(v_{xx}) + p(u_{xx} + u_{yy}) = 0,$

By the assumption $u = u_0 + u_1 p + u_2 p^2 + ...,$ we have the following sub equations:

$v_{0xx} = 0$ $v_0 (0,y) = 0$, $v_0 (\pi,y) = sinh (\pi) cos (y),$
$v_{1xx} = - v_{0yy}$ $v_1 (0,y) = 0$, $v_1 (\pi,y) = 0,$
$v_{2xx} = - v_{0yy}$ $v_2 (0,y) = 0$, $v_2 (\pi,y) = 0,$
$\vdots$ (27)

the solution of the first equation is $\frac{sinh(\pi)}{\pi}$ $cos(y)x$. But Sadighi and Ganji in (2007) used the approximation

$sinh(\pi) \simeq \pi$ and set $v_0 = xcos(y)$ which is not acceptable. It seems that there is no reason to use such an approximation. Now we assign the boundary conditions in another way. By the Maclauran expansion of *sinh(x)*

at $x=\pi$, we have $sinh(\pi) = \pi + \frac{\pi^3}{3!} + ...,$ here we assign the boundary conditions to sub equations in

homotopy perturbation method as follows:

$$v_{0xx} = 0 \qquad\qquad v_0(0,y) = 0, v_0(\pi, y) = \pi\cos(y)$$

$$v_{1xx} = -v_{0yy} \qquad\qquad v_1(0,y) = 0, v_1(\pi, y) = \frac{\pi^3}{3!}\cos(y)$$

$$v_{2xx} = -v_{1yy} \qquad\qquad v_2(0,y) = 0, v_2(\pi, y) = \frac{\pi^5}{5!}\cos(y) \tag{28}$$

$$\vdots$$

The solutions of the above equations are:

$$v_0(x,y) = x\cos(y)$$

$$v_1(x,y) = \frac{x^3}{3!}\cos(y)$$

$$v_2(x,y) = \frac{x^5}{5!}\cos(y) \tag{29}$$

$$\vdots$$

so the final solution will be $u = v_0 + v_1 + v_2 + ... = \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + ... \right)\cos(y) = \sinh(x)\cos(y)$, which is the exact solution of the equation.

Consider the two-dimensional Laplace equation, $u_{xx} + u_{yy} = 0$, subject to the boundary conditions of,

$u(0,y) = sin(y)$, $u(\pi,y) = cosh(\pi) sin(y)$. $u(x,0) = 0$, $u(x,\pi) = 0$.

Let $Lu = u_{xx}$, so the homotopy equation will be as follows:

$H(v,p) = (1-p) (u_{xx}) + p(u_{xx} + u_{yy}) = 0,$

By the assumption $u = v_0 + v_0 p + v_2 p^2 + ...,$ we have the following sub equations in classic view:

$$v_{0xx} = 0 \qquad\qquad v_0\ (0,y) = sin(y),\ v_0\ (\pi,\ y) = cosh(\pi)\ sin(y),$$
$$v_{1xx} = -\ v_{0yy} \qquad\qquad v_1\ (0,y) = 0,\ v_1\ (\pi,y) = 0,$$
$$v_{2xx} = -\ v_{1yy} \qquad\qquad v_2\ (0,y) = 0,\ v_2\ (\pi,y) = 0,$$
$$\vdots \tag{31}$$

the solution of the first equation is $\frac{\cosh(\pi)-1}{\pi}$, But Sadighi and Ganji in (2007) used the approximation $cosh(\pi) \approx 1$ and set, $v_0 = sin(y),$ which is not acceptable. Now we assign the boundary conditions in another way. By the Maclauran series of $cosh(x)$ at $x = \pi$ we have $cosh(\pi) = 1 + \frac{\pi^2}{2!} + ...,$ here we assign the boundary conditions to sub equations in homotopy perturbation method as follows:

$$v_{0xx} = 0, \qquad\qquad v_0\ (0,y) = sin(y),\ v_0\ (\pi,y) = sin(y),$$

$$v_{1xx} = -\ v_{0yy}\ , \qquad\qquad v_1\ (0,y) = 0,\ v_1\ (\pi,y) = \frac{\pi^2}{2!}\ sin(y),$$

$$v_{2xx} = -\ v_{2yy}\ , \qquad\qquad v_2\ (0,y) = 0,\ v_2\ (\pi,y) = \frac{\pi^4}{4!}\ sin(y),$$

$$\vdots \tag{32}$$

The solutions of the above equations are:

$$v_0(x,y) = \sin(y)$$

$$v_1(x,y) = \frac{x^2}{2!}\sin(y)$$

$$v_2(x,y) = \frac{x^4}{4!}\sin(y) \tag{33}$$

$$\vdots$$

So the final solution will be:

$$u = v_0 + v_1 + v_2 + \ldots = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) \sin(y) = \cosh(x) \sin(y),$$

which is the exact solution of the equation.

3.5 System of ODEs:
Consider the following system of equation (Saadatmandi, 2009):

$$u''(x) + (2x-1) u'(x) + \cos(\pi, x) v'(x) = f_1(x),$$
$$v''(x) + xu(x) = f_2(x). \qquad 0 \leq x \leq 1, \tag{35}$$

with the following boundary conditions.

$$u(0) = u(1) = 0, \qquad\qquad V(0) = v(1) = 0,$$

where $f_1(x) = -\pi^2 \sin(\pi x) + (2x-1) \cos(\pi x) + (2x-1) \cos(\pi x)$, and $f_2(x) = 2 + x\sin(\pi x)$. In this case we set $L_1 u = u'' + \pi^2 \sin(\pi x)$, and $L_2(u,v) = v'' + xu - 2 - x\sin(\pi x)$, So the system of homotopy equations will be as follows:

$$u'' + \pi^2 \sin(\pi x) = p \{(2x-1) \cos(\pi x) + (2x-1) \cos(\pi x) - (2x-1)u' - \cos(\pi x)v'\}$$
$$v'' + xu - 2 - x\sin(\pi x) = 0, \tag{36}$$

so by the assumption $u = u_0 + u_1 p + \ldots$ *and* $v = v_0 + v_1 p + \ldots$, the sub systems of the above homotopy system will be as follows:
first system:

$$u_0'' + \pi^2 \sin(\pi x) = 0, \qquad\qquad u_0(0) = u_0(1) = 0,$$
$$v_0'' + xu_0 - 2 - x\sin(\pi x) = 0, \qquad\qquad v_0(0) = v_0(1) = 0,$$

solving the first equation of the first system we have, $u_0 = \sin(\pi x)$, so the next equation will be, $v_0'' + x\sin(\pi x) - x\sin(\pi x) - 2 = 0$, with the boundary conditions, $u_0(0) = v_0(1) = 0$.
So the solution is $v_0(x) = x^2 - x$.
second system:

$$u_1'' = p \{(2x-1) \pi \cos(\pi x) + (2x-1) \cos(\pi x) - (2x-1) u_0' - \cos(\pi x) v_0'\}, \quad u_1(0) = u_1(1) = 0,$$
$$v_1'' + xu_1 = 0 \qquad v_1(0) = u_1(1) = 0,$$

the solution of the above system is, $u_1 = 0$, $v_1 = 0$.
Continuing this approach we have $u_i = v_i = 0$, $i \geq 1$.
So the final solution is $u = u_0 + u_1 + \ldots = \sin(\pi x)$ $v = v_0 + v_1 + \ldots = x^2 - x$. These are the exact solutions of the above systems.

Example 3.1: Homogeneous 2-by-2 stiff system of linear ODE'S (Adomian, 1988; Sami Bataineh, 2007) is as follows:

$$u' + k(1+\xi) u + k(\xi-1) v = 0 \qquad\qquad u(0) = 1,$$

$$v' + k(1+\xi) v + k(\xi-1) u = 0 \qquad\qquad v(0) = 3, \tag{37}$$

k and ξ are constants.

Let $L_1(u) = u' + k(1+\xi) u$ and $L_2(v) = v' + k(1+\xi)v$. so the homotopy system will be:

$$u' + k(1+\xi) u = pk(1-\xi)v, \qquad\qquad u(0) = 1,$$
$$v' + k(1+\xi) v = pk(1-\xi)u, \qquad\qquad v(0) = 3, \tag{38}$$

By the assumption $u = u_0 + u_1 p + ...$ and $v = v_0 + v_1 p + ...,$ we have the following Sub systems,

first system:

$$u_0' + k(1+\xi) u_0 = 0, \qquad\qquad u_0 (0) = 1$$
$$v_0' + k(1+\xi) v_0 = 0, \qquad\qquad v_0 (0) = 3 \qquad\qquad (39)$$

second system:

$$u_1' + k (1+\xi) u_1 = k(1-\xi) v_0, \qquad\qquad u_1 (0) = 0,$$
$$v_1' + k (1+\xi) v_1 = k(1-\xi) u_0, \qquad\qquad v_1 (0) = 0, \qquad\qquad (40)$$

the solution of this system is, $u_1(t) = 3k(1-\xi)te^{-k(1+\xi)t}$, and $v_1(t) = k(1-\xi)te^{-k(1+\xi)t}$ continuing this approach we

have $\quad u_2(t) = \dfrac{[k(1-\xi)]^2 t^2}{2} e^{-k(1+\xi)t}$ and $v_2(t) = \dfrac{[k(1-\xi)]^2 t^2}{2} e^{-k(1+\xi)t}$. We can show by induction that:

$u_n(t) = \dfrac{3[k(1-\xi)]^n t^n}{n!} e^{-k(1+\xi)t}$, for odd n, and $u_n(t) = \dfrac{[k(1-\xi)]^n t^n}{n!} e^{-k(1+\xi)t}$, for even n, and $v_n(t) = \dfrac{3[k(1-\xi)]^n t^n}{n!} e^{-k(1+\xi)t}$

for even n and $v_n(t) = \dfrac{[k(1-\xi)]^n t^n}{n!} e^{-k(1+\xi)t}$ for an odd n. The final solutions will be:

$$u = u_0 + u_1 + u_2 + u_3 + ... =$$

$$e^{-k(1+\xi)t}\left\{1 + k(1-\xi)t + \frac{[k(1-\xi)t^2]}{2!} + ...\right\} + 2e^{-k(1+\xi)t}\left\{1 + k(1-\xi)t + \frac{[k(1-\xi)t^3]}{3!} + ...\right\} =$$

$2e^{-2k\xi t} - e^{-2kt}$. In the same way $v = 2e^{-2k\xi t} - e^{-2kt}$

From the examples we find out that the proposed approach can be useful for constructing the homotopy equation. Of course we do not achieve the exact solution for any equation, but we have still an easy approach for constructing the homotopy equation.

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