

## Design of Stable Neural Control for Rate Limited Linear Systems

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**Abstract:** A novel Neural Network Controller is proposed for locally stabilizing a class of multi-input nonlinear systems consist of a linear system together with saturated and rate limited actuators. No assumption is made on the stability of the linear system. The proposed design procedure replaces earlier recursive neural network learning algorithms with a simple optimization problem. In particular, we introduced a Riccati equation as the core of the design procedure so we could establish a bridge between intelligent control methods and classical methods. The aim of the proposed two-parameter optimization procedure is to maximize the stability region of the closed loop system under the nonlinear neural feedback law. We demonstrated via simulative evaluation. Simulation experiments show that the proposed neural network controller method achieves superior performance compared to methods in literature.

**Key words:** nonlinear control; linear saturated systems; rate limit; Neural Networks.

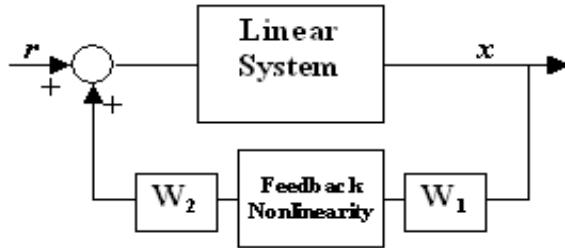
### INTRODUCTION

In recent years, the study of linear saturated systems has been drawn an increasing attention, especially the case of globally or semi-globally stabilization of null-controllable systems (1995)(1999). In contrast, the problem of stability of the systems with both amplitude and rate limits imposed on the input signals has been studied only in a few papers (Lin, Z., 1997; Shewchun, J.M., E. Feron, 1999; Stoorvogel, A., A. Saberi, 1999). Note that the problem of a linear system constraint to rate limited actuator can be translated into the problem of a linear system with position limited actuator by means of a simple technique of augmenting inputs with states.

It is well known that for unstable linear systems with saturated inputs, it is impossible to have a global or semi-global stabilization method. A familiar technique to analyze the stability of saturated linear systems is what labeled 'anti-windup technique', which has been recently generalized for exponentially unstable systems (Teel, A.R., 1999; Romanchuk, B.G., 1999). On the other hand it is easy to show that any feedback gain that stabilizes the linear system, will stabilize the constrained system in a (finite) region around origin. So, one can formulate the problem of state feedback stabilization of unstable constrained linear systems as which feedback gain leads to the greatest domain of stability. An interesting result for planar anti-stable systems is that the domain of attraction of the closed loop system can be arbitrarily close to the null controllable region, which consists of initial states that we can bring to the origin in finite time with bounded control inputs under linear state feedback (Hu, T.L., Qiu, Z. Lin, 1998). Reference (Kapoor, N., P. Daoutidis, 1997) develops a procedure to locally stabilize a single-input linear system with input constraints and also to estimate the region of closed-loop stability. Meanwhile, some authors used nonlinear feedback to stabilize linear systems (Teel, A.R., 1992; Sussmann, H.J., E.D. Sontag, Y. Yang, 1994). Indeed, it was shown that global asymptotic stability of constrained systems is not achievable by linear feedback (Fuller, A.T., 1969; Sussmann, H.J., Y. Yang, 1991). Reference (Sussmann, H.J., E.D. Sontag, Y. Yang, 1994) introduces a 'neural-network type' nonlinear rate feedback, which globally stabilizes constrained linear systems. The Reference (Hu, T., A.R. Teel, L. Zaccarian, 2006) develops a systematic Lyapunov approach to the regional stability and performance analysis of saturated systems in a general feedback configuration. They assume the system has a well-posed algebraic loop and that it is local stability. In (Corradini, M.L., G. Orlando, G. Parlangueli, 2006), the authors proposed a time-varying sliding surface for the robust stabilization of linear uncertain SISO plants with saturating actuators. An extension of the linear quadratic Gaussian control method to systems with saturating actuators and sensors was obtained in (Gokcek, C., 2006). The solution was given in terms of standard Riccati and Lyapunov equations. The state feedback gain-scheduled controller is used for linear parameter-varying systems with saturating actuators in (Montagner, V.F., et al., 2007).

In (Rouhani, M., M.B. Menhaj, 2001), We have presented an algorithm to specify the weights of a neural network controller for linear saturated systems, which have further developed to consider hystereses (Rouhani, M., M.B. Menhaj, 2002). The main objective of this paper is to further develop those papers and presents a 'Neural-Network type' nonlinear feedback that stabilizes general linear systems with the position and rate limited actuators. The rate limited actuators are studied much less than position limited actuators, although the rate limit is crucial for stability of closed loop systems. Here, we consider a multi-input linear system, which could be stable, unstable or anti-stable. A direct design procedure to specify weights of a two-layer neural network type

controller will be presented. We begin with the hypothetical problem of globally stabilization of a linear (unconstrained) system via neural network controllers. It is obvious that this problem by itself has no practical importance, but we will use the results obtained, to locally stabilize a linear system with saturated actuator. Those results has been published in (Rouhani, M., M.B. Menhaj, 2001), but repeated



**Fig. 1:** Structure of hypothetical problem of stabilization of a linear system via nonlinear feedback.

here to keep the paper self-contained. Finally, we present a new nonlinear feedback law that stabilizes a linear system with position and rate limited actuators with maximized domain of guaranteed stability. We will use a nonlinear feedback similar to the familiar linear quadratic feedback. Theoretical results are verified and compared with previous ones via some illustrative examples.

**Main Results:**

**A preliminary Problem:**

First, we develop a stable nonlinear controller for regulating linear systems. Note that we will use method developed hereafter for a realistic problem of saturated linear systems. Consider the configuration shown in Figure 1, in which a linear system is supposed to be globally stabilized by a nonlinear controller. The linear system is described by its state space representation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (1)$$

And nonlinear feedback law by:

$$\mathbf{u} = \mathbf{W}_2 \mathbf{f}(\mathbf{W}_1 \mathbf{x}) \quad (2)$$

In which  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{W}_1$  and  $\mathbf{W}_2$  are matrices of size  $n \times n$ ,  $n \times m$ ,  $p \times n$  and  $m \times p$  respectively and  $\mathbf{x}$  is an  $n \times 1$  vector of system states.  $\mathbf{f}(\cdot)$  is any nonlinear vector function, which its elements belong to sector of  $[\beta, 1]$ ,  $0 < \beta < 1$ , as defined below

*Definition 1[20]:*

The continuous scalar function  $f(\sigma)$  is said to belong to the sector of  $[\alpha, \beta]$ , if:

$$\sigma^2 \alpha \leq \sigma f(\sigma) \leq \sigma^2 \beta, \quad \forall \sigma \in \mathbb{R} \quad (3)$$

■

The vector function  $\mathbf{f}(\cdot) = [f(\cdot) \ f(\cdot) \cdots f(\cdot)]'$  is said to belong to sector of  $[\alpha, \beta]$  if the component function  $f(\cdot)$  does so. From (2), the control signal  $\mathbf{u} = \mathbf{W}_2 \mathbf{f}(\mathbf{W}_1 \mathbf{x})$  is bounded, if the activation function  $\mathbf{f}(\cdot)$  is a bounded nonlinear function such as a sigmoid. As we seek for a global stabilizer, it is obvious that an unbounded control signal is required. However, this fact will be revised in the next sections for more realistic problems.

Substituting (1) into (2) and rearranging the terms yields:

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\mathbf{W}_2\mathbf{W}_1)\mathbf{x} + \mathbf{B}\mathbf{W}_2(\mathbf{f}(\mathbf{W}_1\mathbf{x}) - \mathbf{W}_1\mathbf{x}) \quad (4)$$

The following familiar quadratic Lyapunov function is chosen to derive the feedback controller,

$$V(\mathbf{x}) = \mathbf{x}' \mathbf{P} \mathbf{x} \quad (5)$$

where  $\mathbf{P}$  is a positive definite matrix, which will be specified later. Calculating the derivative of the Lyapunov function along the system trajectories yields:

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}' \mathbf{P} \mathbf{x} + \mathbf{x}' \mathbf{P} \dot{\mathbf{x}} \\ &= \mathbf{x}' \mathbf{A}' \mathbf{P} \mathbf{x} + \mathbf{x}' \mathbf{P} \mathbf{A} \mathbf{x} + 2\mathbf{f}(\mathbf{W}_1 \mathbf{x})' \mathbf{W}_2' \mathbf{B}' \mathbf{P} \mathbf{x}\end{aligned}\quad (6)$$

The goal is to select  $\mathbf{W}_1$  and  $\mathbf{W}_2$  matrices so that the overall system becomes globally stable. This is achievable provided that  $\mathbf{A}_f$ , defined below, is stable.

$$\begin{cases} \mathbf{B}_1 \equiv \mathbf{B} \mathbf{W}_2 \\ \mathbf{A}_f \equiv \mathbf{A} + \mathbf{B}_1 \mathbf{W}_1 \end{cases}\quad (7)$$

We aim to choose  $\mathbf{W}_1$  and  $\mathbf{W}_2$  matrices such that  $\mathbf{A}_f$  became stable (Hurwitz).  $\mathbf{A}_f$  is stable if and only if the following Lyapunov equation has unique solution for  $\mathbf{P}$ , given some  $\mathbf{Q} > \mathbf{0}$ :

$$\mathbf{A}'_f \mathbf{P} + \mathbf{P} \mathbf{A}_f = -\mathbf{Q}\quad (8)$$

Equation (6) can be rewritten as:

$$\begin{aligned}\dot{V}(\mathbf{x}) &= -\mathbf{x}' \mathbf{Q} \mathbf{x} + 2[\mathbf{f}(\mathbf{W}_1 \mathbf{x}) - \mathbf{W}_1 \mathbf{x}]' \mathbf{B}'_1 \mathbf{P} \mathbf{x} \\ &= -\mathbf{x}' \mathbf{Q} \mathbf{x} + 2 \sum_{i=1}^p \left( \frac{f(\mathbf{w}_{1i} \mathbf{x})}{\mathbf{w}_{1i} \mathbf{x}} - 1 \right) (\mathbf{x}' \mathbf{w}'_{1i}) (\mathbf{b}'_{1i} \mathbf{P} \mathbf{x})\end{aligned}\quad (9)$$

In the above,  $\mathbf{w}_{1i}$  stands for the  $i$ th row of  $\mathbf{W}_1$  matrix and  $\mathbf{b}_{1i}$  denotes the  $i$ th column of  $\mathbf{B}_1$ . Note that  $\mathbf{w}_{1i} \mathbf{x}$  and  $\mathbf{b}'_{1i} \mathbf{P} \mathbf{x}$  are scalar quantities. By definition 1, we can have the following inequality constraint for the function  $f(\cdot)$ :

$$\beta < \frac{f(y)}{y} < 1, \quad \forall y \in R\quad (10)$$

Now we are at the position to introduce the first key point of our design procedure. We choose  $\mathbf{W}_1$  as

$$\mathbf{W}_1 = -\mathbf{R}^{-1} \mathbf{B}'_1 \mathbf{P}\quad (11)$$

where  $\mathbf{R} = \text{diag}\{\mathbf{r}_1, \dots, \mathbf{r}_p\} > \mathbf{0}$  is an arbitrary diagonal matrix. To make the overall system stable, we must have

$$\begin{aligned}\mathbf{x}' \mathbf{Q} \mathbf{x} - 2 \sum_{i=1}^p \left( 1 - \frac{f(\mathbf{w}_{1i} \mathbf{x})}{\mathbf{w}_{1i} \mathbf{x}} \right) * \\ (\mathbf{x}' \mathbf{P} \mathbf{b}_{1i}) r_i^{-1} (\mathbf{b}'_{1i} \mathbf{P} \mathbf{x}) > 0\end{aligned}\quad (12)$$

By (10) all terms in summation are positive; hence, the above condition is equivalent to:

$$\mathbf{x}' \mathbf{Q} \mathbf{x} - 2(1 - \beta) \sum \mathbf{x}' \mathbf{P} \mathbf{b}_{1i} r_i^{-1} \mathbf{b}'_{1i} \mathbf{P} \mathbf{x} > 0\quad (13)$$

This yields the following matrix inequality

$$\mathbf{Q} - 2(1 - \beta) \mathbf{P} \mathbf{B}_1 \mathbf{R}^{-1} \mathbf{B}'_1 \mathbf{P} > \mathbf{0}\quad (14)$$

Substituting (7) and (11) in (8), leads to the following well-known Riccati equation:

$$\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} - 2\mathbf{P}\mathbf{B}_1\mathbf{R}^{-1}\mathbf{B}_1'\mathbf{P} = -\mathbf{Q} \quad (15)$$

This equation, has unique solution for  $\mathbf{P}$ , if the pair  $(\mathbf{A}, \mathbf{B}_1)$  is completely controllable. For  $\mathbf{A}_f$  to be stable, this solution has to satisfy (14). Now, the second key point is noted. By adding up the term  $2(1-\beta)\mathbf{P}\mathbf{B}_1\mathbf{R}^{-1}\mathbf{B}_1'\mathbf{P}$  to both sides of (15), we will have:

$$\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} - 2\beta\mathbf{P}\mathbf{B}_1\mathbf{R}^{-1}\mathbf{B}_1'\mathbf{P} = -\mathbf{Q}_1 \quad (16)$$

With

$$\mathbf{Q}_1 \equiv \mathbf{Q} - 2(1-\beta)\mathbf{P}\mathbf{B}_1\mathbf{R}^{-1}\mathbf{B}_1'\mathbf{P} > \mathbf{0} \quad (17)$$

Again (16) is indeed a Riccati equation, which is equivalent to (15) and by choosing  $\mathbf{Q}_1 > \mathbf{0}$  we can easily satisfy the inequality constraint (14). Therefore, we could begin by choosing a positive definite  $\mathbf{Q}_1$ , solve (16) for  $\mathbf{P}$  and finally set  $\mathbf{Q}$  as

$$\mathbf{Q} = \mathbf{Q}_1 + 2(1-\beta)\mathbf{P}\mathbf{B}_1\mathbf{R}^{-1}\mathbf{B}_1'\mathbf{P} \quad (18)$$

The Riccati equation (16) yields a unique solution, if the pair  $(\mathbf{A}, \mathbf{B}_1)$  is completely controllable (note that, this is a sufficient condition).

So we may conclude the design method as follows:

- Choose  $\mathbf{W}_2$  so that  $(\mathbf{A}, \mathbf{B}_1)$  is completely controllable. Obviously, this requires the pair  $(\mathbf{A}, \mathbf{B})$  to be completely controllable.
- Solve Riccati equation (16) for  $\mathbf{P}$ , given  $\mathbf{Q}_1 > \mathbf{0}$  and diagonal matrix  $\mathbf{R} > \mathbf{0}$ .
- Set  $\mathbf{W}_1$  by (11).

By this procedure, we are sure the resulting nonlinear feedback controller will be able to globally stabilize the linear system.

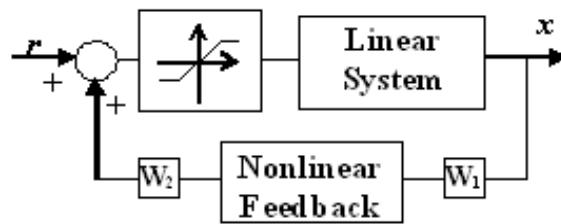
#### Linear Saturated Systems:

Now, we extend this result to the cases in which we have actuators with limited operating range. This is a common problem with real systems because every physical device could be imposed to bounded inputs. Figure 2 shows the new configuration. As one may easily see the overall plant represents a special kind of nonlinear systems.

Nonlinear saturated inputs can be modeled as:

$$\mathbf{u} = \bar{u} \text{Sat}(\mathbf{W}_2 \mathbf{f}(\mathbf{W}_1 \mathbf{x})) \quad (19)$$

Where  $\text{Sat}(\cdot)$  is defined below:



**Fig. 2:** A linear system with bounded actuator subject to nonlinear feedback.

$$Sat(y) = \begin{cases} +1 & , y > 1 \\ y & , |y| < 1 \\ -1 & , y < -1 \end{cases} \quad (20)$$

Again, note that by imposing limited input signals to a linear system, only local stability could be achieved. So, we can relax the unboundedness property imposed on nonlinear feedback law discussed in the previous development and employ any squashing non-linearity, e.g. hyperbolic tangent. This implies that both  $f(\cdot)$  and  $Sat(\cdot)$  belong to the section  $[0,1]$ . That is, in addition to equations (1) and (2), we have:

$$\|\mathbf{u}\|_\infty < \bar{u} \quad (21)$$

$$-1 \leq f(\sigma) \leq 1 \quad (22)$$

At this point before further proceeding, it might be helpful if we could justify why we came up with a nonlinear controller to make the system given in Figure 2 stable. Because one may raise the question: how about to augment the control nonlinearity into the plant and stick with a linear controller. Our answer is that the proposed controller structure allows us to improve the transient response of the closed loop system by employing different nonlinearities for  $f(\cdot)$ .

Due to local stability, system states must be restricted to:

$$\|\mathbf{x}\|_\infty < \bar{x} \quad (23)$$

Choosing the same Lyapunov function as in (5), for a stable overall system, it is necessary to have:

$$\begin{aligned} \dot{V} = & -\mathbf{x}' \mathbf{Q} \mathbf{x} + \\ & 2\bar{u} \sum_{i=1}^m \sum_{j=1}^p \mathbf{x}' \mathbf{w}'_{1j} w_{2ij} \mathbf{b}'_i \mathbf{P} \mathbf{x} * \\ & \left( 1 - \frac{Sat(w_{2ij}f(\mathbf{w}_{1j}\mathbf{x}))}{w_{2ij}f(\mathbf{w}_{1j}\mathbf{x})} \cdot \frac{w_{2ij}f(\mathbf{w}_{1j}\mathbf{x})}{w_{2ij}\mathbf{w}_{1j}\mathbf{x}} \right) < 0 \end{aligned} \quad (24)$$

For an arbitrary diagonal  $m \times m$  matrix  $\mathbf{W}_1$ , select  $\mathbf{W}_1$  as:

$$\mathbf{W}_1 = -\mathbf{P} \mathbf{B}_1 \mathbf{R}^{-1} \quad (25)$$

Here, we set  $\mathbf{B}_1 = \bar{u} \mathbf{B} \mathbf{W}_2$ . In the light of Eqs. (20) and (23), we will have:

$$\begin{cases} \beta_2 < \frac{Sat(y)}{y} < 1 \\ \beta_1 < \frac{f(y)}{y} < 1 \end{cases} \quad (26)$$

From (24) and (26), we finally came up with the following inequality:

$$\mathbf{Q} - 2(1 - \beta_2 \beta_1) \mathbf{P} \mathbf{B}_1 \mathbf{R}^{-1} \mathbf{B}'_1 \mathbf{P} > \mathbf{0} \quad (27)$$

In this case, instead of (16) we must solve the following Riccati equation for  $\mathbf{P}$ :

$$\mathbf{A}' \mathbf{P} + \mathbf{P} \mathbf{A} - 2\beta_1 \beta_2 \mathbf{P} \mathbf{B}_1 \mathbf{R}^{-1} \mathbf{B}'_1 \mathbf{P} = -\mathbf{Q}_1 \quad (28)$$

For a given  $\hat{\mathbf{W}}_2$ , we begin by setting:

$$\mathbf{W}_2 = \frac{1}{\beta_2 \|\hat{\mathbf{w}}_2\|_\infty} \hat{\mathbf{W}}_2 \quad (29)$$

One can easily check that (29) will satisfy (26). If we choose some values for  $\beta_1$  and  $\beta_2$  and continue the design procedure by solving (28) and obtaining  $\mathbf{W}_1$  from (25), we obtain the following upper limit for  $\bar{x}$ :

$$\bar{x} < \frac{\bar{y}}{\|\mathbf{W}_1\|_\infty} \quad (30)$$

where  $\bar{y}$  is the solution of equation (31):

$$\frac{f(\bar{y})}{\bar{y}} = \beta_1 \quad (31)$$

The best choice for  $\beta_1$  and  $\beta_2$ , which give the largest domain of stability, could be found by every optimization method. Then, the guaranteed domain of stability can be easily determined using the largest contour of the form  $\mathbf{x}'\mathbf{P}\mathbf{x} = k$  ( $k$  is a constant) constrained to  $\|\mathbf{x}\|_\infty < \bar{x}$ .

**Remark:**

Note that the optimum domain of stability depends on  $\mathbf{R}$ ,  $\mathbf{Q}_1$  and  $\mathbf{W}_2$ . To study the effect of each of these parameters, one can rewrite (28) as:

$$\begin{aligned} & \mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} - \\ & 2\beta_1\beta_2\bar{u}^2\mathbf{P}\mathbf{B} (\mathbf{W}_2'\mathbf{R}^{-1}\mathbf{W}_2) \quad \mathbf{B}'\mathbf{P} = -\mathbf{Q}_1 \end{aligned} \quad (32)$$

Consider  $\mathbf{W}_2'\mathbf{R}^{-1}\mathbf{W}_2$  as a single parameter and also note that  $\beta'$ s are free parameters which must be tuned during optimization. Therefore, recalling from the theory of linear quadratic optimal control, the absolute value of  $\mathbf{W}_2'\mathbf{R}^{-1}\mathbf{W}_2$  is of no importance, but its relative values of entries, which could be interpreted as the relative importance of each of inputs, play much more important role. One can also interpret  $\mathbf{Q}_1$  in the same way.

The following algorithm summarizes the solution method:

**Algorithm 1:**

Inputs:  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\bar{u}$ .

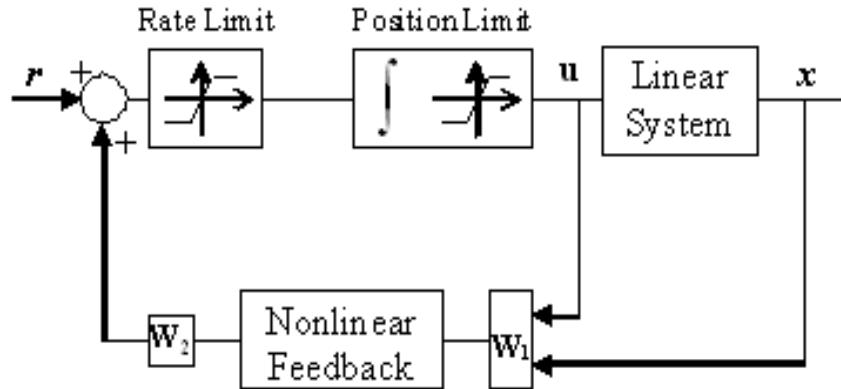
Free parameters chosen by the designer:  $\mathbf{R}$ ,  $\mathbf{Q}_1$ ,  $\hat{\mathbf{W}}_2$ ,  $\beta_{10}$  and  $\beta_{20}$ .

1. Set  $\beta_1 = \beta_{10}$  and  $\beta_2 = \beta_{20}$ .
2. Set  $\mathbf{W}_2 = (\frac{1}{\beta_2 \|\hat{\mathbf{w}}_2\|_\infty}) \hat{\mathbf{W}}_2$ ,  $\mathbf{B}_1 = \bar{u} \mathbf{B} \mathbf{W}_2$ .
3. Solve Riccati equation (28) for  $\mathbf{P}$ . Set  $\mathbf{W}_1 = -\mathbf{R}^{-1} \mathbf{B}_1' \mathbf{P}$ .
4. Calculate  $\bar{x}$  from (30) and (31). Find guaranteed region of stability as the greatest contour of the form  $\mathbf{x}'\mathbf{P}\mathbf{x} = k$  contained in  $\|\mathbf{x}\|_\infty < \bar{x}$ .
5. Determine  $\beta'$ s such that  $\bar{x}$  is maximized.

■

**Linear Systems With Position And Rate Limits:**

Consider the linear system (1) together with the following constraints on the input signals:



**Fig. 3:** A linear system with rate and position limited actuators subjected to nonlinear feedback.

$$\begin{cases} |u_i| \leq \bar{u}_i \\ |\dot{u}_i| \leq \dot{\bar{u}}_i \end{cases} \quad i = 1, \dots, m \quad (33)$$

The objective of this section is to develop a design procedure that specifies  $\mathbf{W}_1$  and  $\mathbf{W}_2$ , such that the overall system has greatest domain of attraction (see figure 3).

The first step is to augment the state of plant with the control as:

$$\mathbf{x}_a = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}, \quad \mathbf{A}_a = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_a = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_m \end{bmatrix} \quad (34)$$

This leads to the following augmented system,

$$\dot{\mathbf{x}}_a = \mathbf{A}_a \mathbf{x}_a + \mathbf{B}_a \dot{\mathbf{u}} \quad (35)$$

The control rate now appears as the system input and the original control is part of states of the system. So if there was no constraint on the amplitude of control signal and the only constraint was rate limit, we could employ directly the proposed design procedure of last section, Algorithm 1, for the augmented system. But when both rate and amplitude limits are to be satisfied, we need to modify this Algorithm.

Before proceeding, it is helpful to normalize constraints (33) to  $\pm 1$ . This is accomplished by means of an input scaling and a state transformation as:

$$\begin{aligned} \hat{u}_i &= \frac{\dot{u}_i}{\dot{\bar{u}}_i}, \quad \mathbf{x}_{aT} = \mathbf{T} \mathbf{x}_a, \\ \mathbf{A}_{aT} &= \mathbf{T} \mathbf{A}_a \mathbf{T}^{-1}, \\ \mathbf{B}_{aT} &= \begin{bmatrix} \mathbf{0} & & \\ \bar{u}_1 / \bar{u}_1 & \dots & 0 \\ \vdots & \ddots & \\ 0 & & \bar{u}_m / \bar{u}_m \end{bmatrix} \end{aligned} \quad (36)$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} & & \\ & 1/\bar{u}_1 & & \\ \mathbf{0} & & \ddots & \\ & & & 1/\bar{u}_m \end{bmatrix} \quad (37)$$

In the following, we will drop the subscript T and simply assume that all rate and amplitude limits are scaled to  $\pm 1$ .

We note that rate limit acts as a saturated input for augmented system (35) and the position limit represents indeed a bound on the state space. This suggests us employment of the Algorithm 1 with some modifications ensuring the boundedness of the input signal within its limits, is good enough to handle this problem.

As one can easily observe, by applying Algorithm 1 to a specified problem, we will get a bound on state space on which the overall system guaranteed to be stable. So what we need to do here is to compare this bound on the augmented state  $\mathbf{x}_a$  with the position limit on the input signal. In step 4 of the Algorithm 1, one needs to

calculate a bound  $\bar{x}_a$  on the state variables  $\mathbf{x}_a$  such that  $\frac{f(\mathbf{w}_{1i}\mathbf{x}_a)}{\mathbf{w}_{1i}\mathbf{x}_a} > \beta_1$  for all  $\mathbf{x}_a \in \|\mathbf{x}_a\| < \bar{x}_a$ . If

this bound is less than the normalized position limit, i.e.  $\bar{x}_a < 1$ , we have to choose the same bound for plant states:  $\|\mathbf{x}\| < \bar{x}_a$ . But for  $\frac{\bar{y}}{\|\mathbf{W}_1\|_\infty} > 1$ , we may say that still we are allowed to increase  $\|\mathbf{x}_a\|$  without losing the stability of the close-loop system. The following algorithm summarizes the results:

**Algorithm 2:**

Inputs:  $\mathbf{A}$  and  $\mathbf{B}$

Free parameters chosen by the designer:  $\mathbf{R}$ ,  $\mathbf{Q}_1$ ,  $\hat{\mathbf{W}}_2$ ,  $\beta_{10}$  and  $\beta_{20}$

1. Augment inputs with state variables and normalize the rate and position limits to 1.
2. Set  $\beta_1 = \beta_{10}$  and  $\beta_2 = \beta_{20}$ .
3. Set  $\mathbf{W}_2 = (\frac{1}{\beta_2 \|\hat{\mathbf{W}}_2\|}) \hat{\mathbf{W}}_2$ ,  $\mathbf{B}_1 = \bar{u} \mathbf{B} \mathbf{W}_2$ .
4. Solve Riccati equation (28) for  $\mathbf{P}$ . Set  $\mathbf{W}_1 = -\mathbf{R}^{-1} \mathbf{B}' \mathbf{P}$ .
5. Calculate  $\bar{x}_a$  from (30) and (31). If  $\bar{x}_a < 1$ , set  $\bar{x} = \bar{x}_a$ . Otherwise calculate  $\bar{x} = \frac{\bar{y} - \|\mathbf{W}_{1u}\|_\infty}{\|\mathbf{W}_{1x}\|}$ , where  $\mathbf{W}_{1x}$  and  $\mathbf{W}_{1u}$  are respectively partitions of  $\mathbf{W}_1$  associated with states and inputs.
6. Let:
- 7.

$$\mathbf{P} = \left[ \begin{array}{c|c} \mathbf{P}_x & \mathbf{P}_{xu} \\ \hline \mathbf{P}'_{xu} & \mathbf{P}_u \end{array} \right]$$

Where  $\mathbf{P}_x$  and  $\mathbf{P}_u$  are respectively  $n \times n$  and  $m \times m$  matrices. Find guaranteed region of stability as the greatest contour of the form  $\mathbf{x}' \mathbf{P}_x \mathbf{x} = k$  ( $k$  is a constant) contained in  $\|\mathbf{x}\|_\infty < \bar{x}$ .

8. Determine  $\beta'$ s such that  $\bar{x}$ , which is a function of  $\beta'$ s is maximized.

■

Note that as  $\beta_1$  and  $\beta_2$  appear as multiplication to  $\mathbf{R}$  in (28) the absolute value of  $\mathbf{R}$  is of no practical importance. For problems with more than one input, the relative values of  $\mathbf{R}$  and  $\mathbf{Q}_1$  matrices may be interpreted as similar with LQR problem for linear behavior of the system. The same may be said for  $\hat{\mathbf{W}}_2$ , as the actual value of  $\mathbf{W}_2$  depends on  $\beta_2$  which will be set by the procedure.

As mentioned before the weight matrices  $\mathbf{W}_1$  and  $\mathbf{W}_2$  are of size  $p \times n$  and  $m \times p$  respectively. The number of neurons has to be not less than the number of inputs  $m \leq p$  for pair  $(\mathbf{A}, \mathbf{B}_1)$  to remain controllable. For linear behavior of the system, it is clear that more neurons makes no sense as  $\mathbf{W}_2 \times \mathbf{W}_1$  is always  $m \times n$  matrix. The authors believe that the activation function and the number of neurons may affect the transient performance of the system as example 3 below demonstrates it. However, our simulations do not show significant differences in performance when the number of neurons further increased.

**Simulations:**

To illustrate the ability and performance of presented method, we perform some test simulations.

*Example 1:* By this example, we wish to have a comparison study of our methods and the ones given in Reference (Kapoor, N., P. Daoutidis, 1997). Consider the same single input system as in Reference(Kapoor, N., P. Daoutidis, 1997):

$$\mathbf{A} = \begin{bmatrix} -0.5 & 1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{u} = 1$$

For a linear feedback of  $u = 2x_1 - 31x_2$ , it is found that for all initial values belong to  $\mathbf{Z}_0 = \{\mathbf{x} : -0.4 \leq x_1 \leq 0.4, -0.2 \leq x_2 \leq 0.2\}$ , the stability of the closed loop system is guaranteed.

We Applied the Algorithm 1 to this problem, with  $\mathbf{R} = 1$ ,  $\mathbf{Q}_1 = I_{2 \times 2}$ ,  $\hat{\mathbf{W}}_2 = I_{2 \times 2}$ ,  $\beta_{10} = 1$  and  $\beta_{20} = 1$ .

The new domain of guaranteed stability is as the ellipsoid shown in figure 4. The rectangular region in this figure shows the stability domain obtained in Reference (Kapoor, N., P. Daoutidis, 1997). Figure 5 represents the results of a simulation where the system is stabilized with the proposed nonlinear feedback. As easily observed, the controller presented in Reference 9 fails to stabilize the system while our results could successfully make the overall system stable.

**Example 2:**

In the above example, we considered a linear system with position limited actuators. In this example we will consider the following saturated and rate limited multi-input linear system:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix},$$

$$\bar{\mathbf{u}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \dot{\bar{\mathbf{u}}} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

Note that the linear system is anti-stable with both eigenvalues, i.e. +1 and +2, in the open right half-plane. Three cases are considered: both amplitude and rate limits are effective, only amplitude limits on actuators are effective or, the third case, only rate limits are effective. In the first case, we employed Algorithm 2 to normalized augmented system. For the later cases, we applied Algorithm 1 to normalized system and to normalized augmented system, respectively. In all cases, we choose  $\mathbf{R} = I_{2 \times 2}$ ,  $\mathbf{Q}_1 = I_{2 \times 2}$ ,  $\hat{\mathbf{W}}_2 = I_{2 \times 2}$ ,  $\beta_{10} = 1$  and  $\beta_{20} = 1$ . Figures 6 and 7 show the results. As one may expect, the best result in term of the smallest overshoot and settling time is obtained when only amplitude limit is active. From Figure 7, it is seen that the input to the augmented system (at the right column) is highly oscillatory, while the actual input (at the left column) is very smooth and bounded. The bounds on the input (first and second rows, left column) and the bounds on the rate of change of inputs (first and third rows, right column) are visible in this figure, too.

**Example 3:**

The objective of this example is to show the effect of different nonlinear functions,  $f(\cdot)$ , used in the feedback loop on the transient response of the closed loop system. Consider the system given in example 2 when only amplitude limits on actuators are active. We employed three nonlinear functions in feedback loop: saturation function  $Sat(\mathbf{x})$  as defined above, sigmoidal function  $Sigm(\mathbf{x})$

and an arbitrarily chosen piece-wise linear function  $f_3(\mathbf{x})$ :

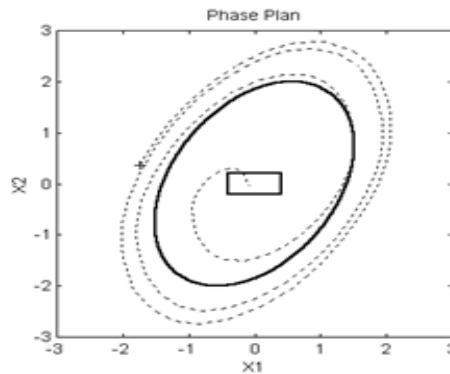
$$Sigm(x) = \frac{1-e^{-2x}}{1+e^{-2x}}$$

$$f_3(x) = \begin{cases} x & |x| < 0.1 \\ sign(x)(|x|/11 + 0.0909) & 0.1 < |x| < 10 \\ sign(x) & 10 < |x| \end{cases}$$

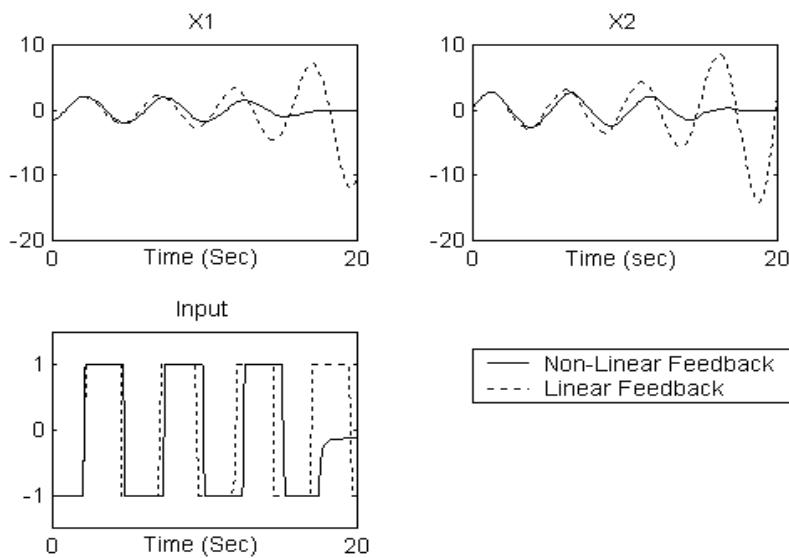
Figure 8 demonstrates a simulation result for these nonlinear functions. As it is seen from this figure, the fastest response is obtained by means of sigmoidal function, while saturation function gives the least overshoot.

**Conclusion:**

In this paper, a novel nonlinear feedback design method is presented to stabilize multivariable linear systems with both position and rate limited actuators. Stability of the closed loop system was guaranteed and the domain of attraction was estimated as well. No assumption is made on the stability of linear system, so that the theory could be applied to any stable, unstable or anti stable system. Some illustrative examples have been given. The results showed excellent performance and flexibility of the proposed design procedure in comparison with the recently cited methods in literature.



**Fig. 4:** Phase plane of system in example 1. Doted line shows a trajectory of the closed loop system under nonlinear state feedback.



**Fig. 5:** State profiles and inputs for system of example 1.

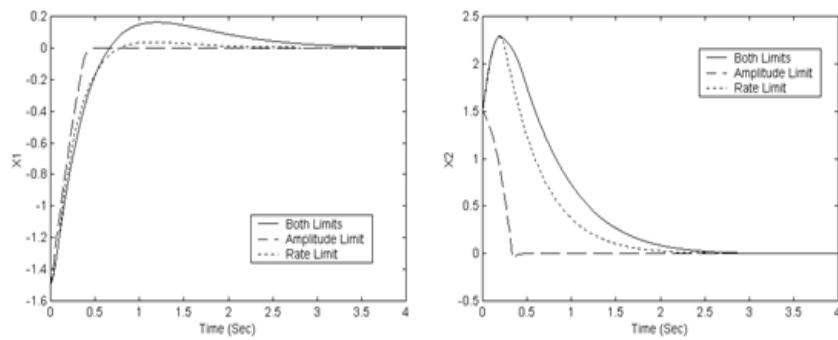


Fig. 6: State profiles of system in example 2.

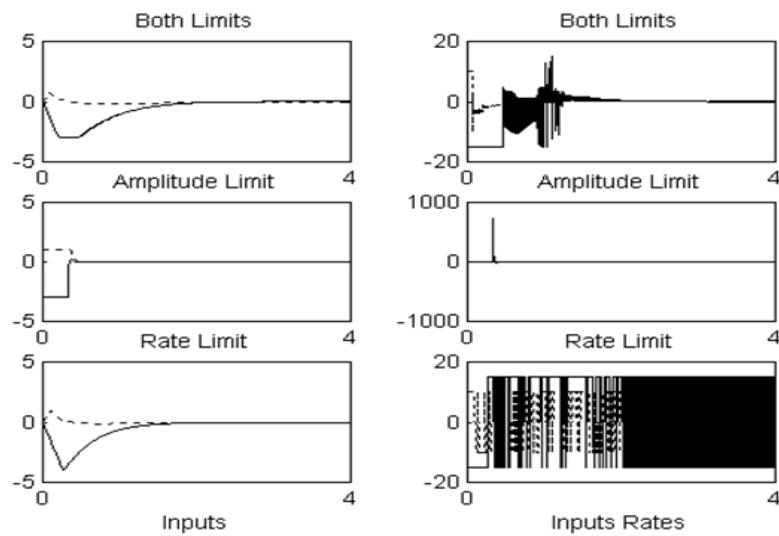


Fig. 7: Inputs(left) and their rates of change (right) for system in example 2.

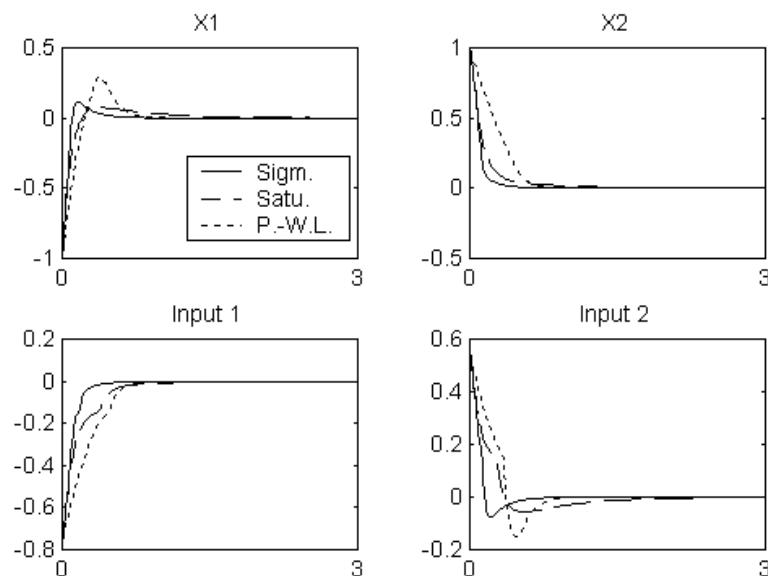


Fig. 8: State and inputs for system of example 3.

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