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Solving Nonlinear Algebraic Problem Using Newton Homotopy Differential Equation

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Abstract: This paper presents an efficient algorithm for solving a nonlinear equation. In our algorithm, the nonlinear system $f(x) = 0$ is solved by a homotopy method, in which a homotopy $H(x,t) = f(x)-(1-t)f(x^0)$ is introduced and the solution path of $H(x, t) = 0$ is followed from an obvious solution (x^0, 0)to the solution (x^*,1) which we seek. An ordinary differential equation based on Newton homotopy is used for following the solution path. Our homotop algorithm is much more efficient than the conventional iterations type algorithms. Some numerical examples are given in order to demonstrate the effectiveness

Key words:

INTRODUCTION

The problem of solving systems of nonlinear equations is one of the central and important problems in various fields of science and engineering (Yamamura, 1989). In most of nonlinear problems, Newton's method is used for solving system of nonlinear equations. However, as is well known, Newton's method is not globally convergent and often fails to converge when the starting point is not close to the solution (Watson, 1987). In practical computations, it is usually difficult to guess a good starting point, and the desired solutions cannot be obtained because of the poor convergence of Newton's method.

On the other hand, the homotopy methods have been proved to be globally convergent for a general class of nonlinear equations, therefore they are useful for solving nonlinear equations whose solutions cannot be obtained by Newton's method (Yamamura, 1988). Homotopies are traditional part of topology, and have found significant application in nonlinear functional analysis and differential geometry (Kojima, 1978). In the homotopy techniques, for the given nonlinear system

$$f(x)=0 \tag{1}$$

we introduce a parameter $t \epsilon [0,1]$ and consider an equation

$$H(x,t) = tf(x) + (1-t)\, g(x) \tag{2}$$

The mapping H is called "homotopy" of (Allgower, 1994). It follows that at t=0, $H(x,0)=g(x)$, where g is an auxiliary function has obvious solution $x^{(0)}$, and at t=1, $H(x,1)=f(x)=0$ has the solution x^* which we seek. The solution x^* can be obtained by solving the homotopy equation (2) and following its path (zero curve) (Garcia, 1981). There are two fundamental types of homotopy algorithms that follow the path. One is based on piecewise-linearization such as the simplicial algorithm (Saigal, 1983). The other is by solving differential equations such as predictor-corrector algorithms (Yamamura, 1990). Because we will use the last type in our work, we explore some of them.

Yakovlov (1964) proposed to differentiate $H(x,t)=0$ respect to t, taking x as a function of t, and to seek a solution to Cauchy's problem (the differential equation obtained with initial condition $x(0)=x^{(0)}$ at the end point t=1. Todd (Todd, 1978), derived a particular differential equation by exponential homotopy and to solve it by A-stable integration techniques to find the solution of the corresponding nonlinear algebraic system. The associated-IVP of Newton homotopy can be integrated by a predictor-corrector technique to follow the path of Newton hoomotopy equation and then to find the desired solution x^* (Garcia, 1981; Watson, 1991). The corresponding path of Newton homotopy equation can be followed by numerical integration of an associated

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differential equation (Watson, 1996; Watson, 1989).

The primary purpose of this paper is to generate a practical method for finding the solution x^* to the nonlinear system $f(x)=0$ when only a poor initial approximation, x0, is known. The procedure here is (i) construct Newton homotopy $H(x,t)= f(x)-(1-t)f(x^{(0)})$ of the nonlinear system $f(x)=0$. (ii) Differentiate the Newton homotopy equation $H(x,t)=0$ by chain rule to construct linear ordinary differential equation in two unknowns x and t. (iv) Solve the obtained equation by Euler method to get the values of x, the desired solution.

2. Basic Theory:

The term *homotopy* means a continuous mapping H define on the product $R\times I$ to R, $H:R\times I\to R$ where I is the unit interval (Watson, 1992). Numerous forms of $H(x,t)$ exits, but through our work we utilize the homotopy

$$H(x,t) = f(x)-(1-t)f(x^{(0)}) \tag{3}$$

This form of the homotopy is termed the *Newton homotopy* because some of the ideas behined it comes from the work of *Sir Isaac Newton* himself (Garcia, 1981). It is particularly simple method to start. Pick an arbitrary point $x^{(0)}$. Next calculate $f(x^{(0)})$, and then let

$$g(x) = f(x) - f(x^{(0)}). \tag{4}$$

The function g, by construction, has solution $x^{(0)}$.
Given a homotopy function H: R^2 ® R, where $R^2 = R\times I$, we must now be more explicit about solutions to

$$H(x, t) = 0. \tag{5}$$

In particular, define

$$H^{-1} = \{(x,t)/H(x,t)=0\} \tag{6}$$

as the set of all solutions (x,t)ÎR^2 to the system $H(x,t)=0$. The general situation, however, is much more complex, as H^{-1} could be rather arbitrary. The solution points (x,t) that satisfy homotopy $H(x,t)= 0$ could be all over the place and in no particular configuration. Still, H^{-1}, by definition, includes all of them. (Kojima, 1978).
Understand well that in H^{-1} both x and t vary. However, we also must denote the solution for t fixed. Let

$$H^{-1}(t) = \{x/ H(x,t) = 0\}. \tag{7}$$

Now $H^{-1}(0)$ consists of all the start points $x(0)$, or, equivalently, all solutions to

$$H(x,0) = g(x) = 0. \tag{8}$$

Similarly, $H^{-1}(1)$ consists of all the points $x^* = x(1)$ which solve

$$H(x,1) = f(x) = 0. \tag{9}$$

Carefully observe that H^{-1} has points (x,t) Î R while $H^{-1}(t)$ has points x in R. Intuitively, H^{-1} describes all solutions (x,t) where t is allowed to change. However, $H^{-1}(t)$ specifies only the x solutions for t fixed.

***Theorem 1**. (**Implicit Function Theorem**):*

Let H: $R^2\to R$ be continuously differentiable, $(\bar{x}, \bar{t})$ Î H^{-1} and $JH_x(\bar{x},\bar{t})$ be invertible. Then in a neighborhood of $(\bar{x}, \bar{t})$ all points (x,t) that satisfy $H(x,t) = 0$ are on a single continuously differentiable path through $(\bar{x}, \bar{t})$ (Yakovlev, 1964).

The term *globally convergent* means that we can easily obtain a starting point which leads to the solution (Yamamura, 1991).

3. The Homotopy Differential Equations:

Section 2 introduced the concept of following a path from an easily start point to a solution. Under quite moderate assumptions the path was shown to exist and be well behaved, so that it could be easily followed. To derive the homotopy differential equation, we can follow the foloowing mathematical steps:

$$H\,(x,t) = f\,(x)-(1-t)\,f\,(x^{(0)}) \qquad (10)$$

$$H\,(x,t) = 0 \qquad (11)$$

By chain rule, we obtained

$$\frac{\partial H}{\partial x}\frac{dx}{dt}+\frac{\partial H}{\partial t}=0 \qquad (12)$$

$$\frac{dx}{dt}=\frac{-\frac{\partial H}{\partial t}}{\frac{\partial H}{\partial x}}=\frac{-H_t}{H_x}=f\,(x,t) \qquad (13)$$

This equation is a first order differential equation named as *Newton Homotopy Differential Equation* (NHDE). This equation can be solved numerically using Euler's method to find x for parameter $t \in [0,1]$. The numerical solution can be formulated as in below;

$$x^{(k)} = x^{(k-1)} + h\,f\,(x,\,t);\ h = \mathrm{r}t \qquad (14)$$

$$x^{(k)} = x^{(k-1)} + h\,(-\frac{H_t}{H_x});\ k = 0,\,1,\,2,\,\ldots \qquad (15)$$

4. Homotopy Algorithm:

In summary, we summarize the algorithm steps as follows:

1. Write the given system in the form $f\,(x)=0$.
2. Write the Newton homotopy $H\,(x,t)= f\,(x)-(1-t)\,f\,(x^{(0)})$ for the given system.
3. Write NHDE
4. Choose h.
5. Solve NHDE using Euler's formula
6. Put the results in a table

5. Numerical Example:

This section gives some numerical examples in order to demonstrate the effectiveness of the proposed algorithm. All the computation was implemented on PC computer by MATLAB computer language version 5.3.

Example:

6.Conclusion:

In this paper a new version of homotopy algorithms has been proposed which is more efficient than previous algorithms, because the start point can be choosed arbitrary. The proposed algorithm is simple can be easily programmed using recursive functions.

Finding one or all solutions of nonlinear algebraic equations is extremely only. Therefore the objective of this paper has been to develop a clever heuristic that seems to work well in practice. Our algorithm has been tested successfully by using HOMMAT software in MATLAB version 5.3.

However, perhaps it is difficult to get results very close to the exact roots by the proposed algorithm and HOMMAT package. It is left as future problems to develop an efficient algorithm that can find approximate solution which closes to the exact solution. This algorithm can be used to solve many problems in since and engineering field as resistive nonlinear networks.

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