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Simple Semi Hypergroups

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Abstract: In this paper the notion of a simple semi hypergroup which is a generalization of simple semi group from algebraic viewpoint is introduced. Some basic properties for this algebraic structure are presented and proved. Three methods for constructing new simple semi hypergroups are also presented.

Key words: semi hypergroup; hypergroups; hyperideals; simple semi hypergroups

INTRODUCTION

Hyperstructures theory was first initiated by Frederic Marty in 1934 when he defined hypergroups as a generalization of groups (Marty, F., 1934). Up to now many researchers have been studying on this field of modern algebra and its applications. The basic notions and results of the object can be found in (Corsini, P., 1993). In 2003, Corsini and Leoreanu presented numerous applications of hyperstructure theory (Corsini P. and V. Leoreanu, 2003). These applications can be used in the following areas: geometry, graphs, fuzzy sets, cryptography, automata, lattices, binary relations, codes, and artificial intelligence. This paper is an attempt to give an approach to a generalization of Rees theorem for semi group theory (Howie, J.M., 1995) to semi hypergroups theory. The plan of this paper is as follows: Section II is a brief overview of some basic notions and results on hyperstructures theory related to this research. Simple semi hypergroups and some properties of this algebraic structure are presented in Section III. In Section IV, three methods for constructing new simple semi hypergroups are presented. In this paper, it is proved that the product of two simple semi hypergroups is a simple semi hypergroup. The quotients of semi hypergroups are considered and it is shown that they are simple semi hypergroup. Section V presents a short conclusion and recommendation for further research.

Basic Notions and Preliminaries:

We recall the following definitions from (Corsini P. and V. Leoreanu, 2003). Let H be a non-empty set and $P^*(H)$ be the set of all nonempty subsets of H. An n–hyperoperation on is a map $f:H^n \to P^*(H)$ and a set H endowed with a family Γ of *hyperoperations,* is called a *hyperstructure (multivalued algebra).* If Γ is a singleton, that is $\Gamma = \{f\}$, then the hyperstructure is called *hypergroupoid*. The hyperoperation is denoted by "∘" and the image of (a, b) of H^2 is denoted by $a \circ b$ and is called the *hyperproduct* of a and b. If A and B are nonempty subsets of H then $A \circ b = \bigcup_{\substack{a \in A \\ b \in A}} a \circ b$. A *semi hypergroup* is a hypergroupoid $(H, \circ)$ such that:

$$\forall (a,b,c) \in H^3, (a \circ b) \circ c = a \circ (b \circ c).$$

A *hypergroup* is a semi hypergroup $(H, \circ)$ such that:

$$\forall a \in H, \quad H \circ a = a \circ H = H$$

For example, Let G be a group, is a normal subgroup of G. Then $(G, \circ)$ with the following hyperoperation is a hypergroup:

$$\forall (x,y) \in G^2 , \qquad x \circ y = Hxy$$

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Let (H, $\circ$) be a hypergroupoid. An element $e \in H$ is called an *identity* or *unit* if:

$$\forall a \in H, \ a \in a \circ e \cap e \circ a$$

Let (H, $\circ$) be a hypergroup endowed with at least an identity. An element $a' \in H$ is called an *inverse* of if there exists an identity $e \in H$ such that:

$$e \in a \circ a' \cap a' \circ a$$

A hyper-group H is called regular if it has at least one identity and each element has at least one inverse. Let (H, $\circ$) and (K, *) be hypergroups, and $f: H \to K$ be a map. Then:
1. f is a *homomorphism* if:

$$\forall (a,b) \in H^2, \qquad f(a \circ b) \subset f(a) * f(b)$$

2. f is a *good homomorphism* if:

$$\forall (a,b) \in H^2, \qquad f(a \circ b) = f(a) * f(b)$$

The concept of *hyperideal* in semi hypergroups theory is given as the following: A nonempty subset I of a semi hypergroup S is called *left hyperideal* if $SI \subseteq I$, a *right hyperideal* if $SI \subseteq I$, and (two sided) *hyperideal* if it is both left and right hyperideal (Davvaz, B., 2006).

Simple Semi Hypergroups:

In this section the concept of simple in the context of semi hypergroups is introduced and its properties are investigated and proved.

Definition 1:

If a semi hypergroup S with at least two elements contains an element 0 such that for all s in S, $0s=s0=\{0\}$, then 0 is said to be a *zero scalar element*(or just zero scalar) of S and S is called a *semi hypergroup with zero scalar*.

Using the terminologies of semi group theory, the following definitions for 0-simple and simple semi hypergroups are presented.

Definition 2:

A semi hypergroup without zero scalar is called *simple* if it has no proper hyperideals. A semi hypergroup with zero scalar is called *0-simple* if it has the following conditions:
1. $\{0\}$ and S are only its hyperideals.
2. $S^2 \neq \{0\}$.

Proposition 1:

Semi hypergroup S is *0-simple* if and only if $SaS=S$ for all $a \neq 0$ in. This means if and only if for every $a, b \in S - \{0\}$ there exist x, y in S such that $b \in xay$.

Proof:

First, suppose that S is a 0-simple semi hypergroup. It is easy to see that S^2 is a hyperideal of S. By Definition 2, $S^2 \neq \{0\}$. Thus $S^2=S$ and it follows that $S^3=S$. Now consider an element $a \in S$ that is not a zero scalar element. It is clear that SaS is a hyperideal of S. In the case of $SaS=0$ the set $I=\{s \in S / SsS=\{0\}\}$ is a nonempty subset of S since $a \in I$. If x is an element of SI then there exist elements such as s in S and i in I such that $x \in si$, and hence

$$SxS \subseteq SsiS \subseteq SiS = \{0\}$$

This implies $SxS=\{0\}$ and $x \in I$. In a similar way, it can be shown that IS is also a subset of I. It follows that I is a hyperideal of S, hence $I=S$. Thus for all s in S, $SsS=\{0\}$, that is $S^3=\{0\}$, which is a contradiction to $S^3=S$. Therefore $SaS=S$ for every $a \neq 0$ in S.

Conversely, if $SaS=S$ for all $a\neq 0$ in S then S^2 is not equal to $\{0\}$ ($S^2=\{0\}$ means $SaS=\{0\}$ for all $a\neq 0$ in). Now, suppose that a is an element of S that is not a zero scalar and I is a hyperideal of S containing a. Then $S = SaS \subseteq SIS \subseteq I$ or S=1.

Thus S is 0-simple.
Proposition 1 leads to the following corollary.

Corollary: A semi hypergroup S is simple if and only if for all a in S, $S=SaS$.

Proposition 2:
Every hypergroup is a simple semi hypergroup.

Proof:
If H is a hypergroup, then for all a in, $aH = Ha = H$. Hence, $H = aH \subseteq H$ and so $HH \subseteq H$. On the other hand, $aH = H$ thus $HaH = HH = H$ that is H is simple.

Constructing New Simple Semi Hypergroups:
This section is concerned with some methods to construct new simple semi hypergroups. Let (S, *) and (T, #) be two semi hypergroups. It has been proved that the Cartesian product of these two semi hypergroups is a semi hypergroup with the following hyperoperation (Corsini, P., 1993):

$$(s_1,t_1) \otimes (s_2,t_2) = s_1 * s_1 \times t_1 \# t_2$$

Theorem 1:
Let (S, *) and (T, #) be two simple semi hypergroups. Then the product $S\times T$ with the above hyperoperation is a simple semi hypergroup.

Proof:
Suppose that (a, b) is an arbitrary element of $S\times T$ that is not a zero scalar. It is clear that $SaS=S$ and $TaT=T$, and so $S\times T=SaS\times TbT$. Now, consider $z=(x,y)$ as an element of $S\times T$. It follows that there exists (c, d) in $Sa\times Tb$ and (s, t) in $S\times T$ such that:

$$(x,y) \in cs \times dt = (c,d) \otimes (s,t)$$

There exists also (s', t') in $S\times T$ such that $(c, d) \in (s', t')\,(a, b)$, and so

$$(x,y) \in (c,d) \otimes (s,t) \subseteq S \times T \ (a,b)(s,t) \subseteq$$

$$S \times T \ (a,b)\, S \times T$$

Hence $S\times T \subseteq S\times T\ (a, b)S\times T$. It is clear that

$S\times T\ (a, b)S\times T \subseteq S\times T$,

and so $S\times T = S\times T\ (a, b)S\times T$.

That is $S\times T$ is simple.
Let (H, $\circ$) be a hypergroupoid and ρ be an equivalence relation on H. We say that ρ is *regular* on the right if the following implication holds:

$$a\rho b \Longrightarrow \forall u \in H, \forall x \in a \circ u, \exists y \in b \circ u : x\rho y$$
and

$$\forall y' \in b \circ u, \exists x' \in a \circ u : x'\rho y'$$

Similarly, the regularity on the left can be defined. The equivalence relation ρ is said to be regular if it is regular on the right and on the left (Corsini, P., 1993). New simple semi hypergroups by using the next proposition can also be constructed.

Proposition 3:

Let $(H, \circ)$ be a simple semi hypergroup and ρ a regular equivalence relation on H. Then H/ρ is a simple semi hypergroup with respect to the following hyperoperation:

$$\forall(\bar{x},\bar{y}) \in (H/\rho)^2,\ \ \bar{x} \otimes \bar{y} = \{\bar{z}\ \mid z \in x \circ y\}$$

Proof:

Suppose that and $\bar{a}$ and $\bar{b}$ are two arbitrary elements of H/ρ. There exist elements a and b in H such that $\bar{a}$ and $\bar{b}$ are respectively the images of a and b in H/ρ with respect to the canonical projection. Since H is simple, there exist elements x and y in H such that $b \in ray$. Thus $\bar{b} \in \overline{xay}$ and hence $\bar{b} \in \overline{xay}$ (in this case canonical projection is a good homomorphism), that is H/ρ is simple.

The following theorem is an approach to a generalization of Rees theorem in semi group theory.

Theorem 2:

Let H be a regular hypergroup, and I, ∧ be nonempty sets. Let $P=(p_{\lambda i})$ be a ∧ I regular matrix (in the sense that it has no row or column that consists entirely of zeros) with entries from H. Then $S=I\times H\times\wedge$ with respect to the following hyperoperation is a simple semi hypergroup:

$$(i,a,\lambda)(j,b,\mu) = \{(i,t,\mu)/\ t\ \in a\,p_{\lambda i}\ b\}$$

Proof:

In a direct verification, the associativity of the hyperoperation can be proved. Let (i, a, λ), (j, b, μ) and (k, c, Ψ) be arbitrary elements of S and z is an element of the following set:

$$(i,a,\lambda)\,[(j,b,\mu)(k,c,\psi)]$$

$$\bigcup_{t\in bp_{\mu k}c} \{(i,x,\quad \psi)/\ x \in ap_{\lambda j}t\}$$

There exists t' in $bp_{\mu k}c$ and x' in $ap_{\mu k}t$ such that $z=(i, x', \Psi)$.

This means there exists v in $p_{\mu k}c$ and u in $ap_{\lambda j}$ such that $t' \in bv$ and $x' \in ut'$ and so $x' \in ubv$. It follows that there exists s in $ap_{\lambda j}b$ such that $x' \in sp_{uk}c$. Thus

$$z = (i,x',\psi) \in \bigcup_{s\in ap_{\lambda j}b} \{(i,y,\quad \psi)\mid\ y \in sp_{\mu k}c\} =$$

$$[(i,a,\lambda)\ (j,b,\mu)]\,(k,c,\psi)$$

In a similar way, it can be shown that every element of $[(i, a, \lambda)\ (j, b, \mu)]\ (k, c, \Psi)$ is an element of the set $(i, a, \lambda)\ [(j, b, \mu)\ (k, c, \Psi)]$. Therefore, S is a semi hypergroup.

To verify that S is simple, suppose that (i, a, λ) and (j, b, μ) are two elements of S that are not zero scalar. Since H is a hypergroup, thus there exist elements x and y in H such that $b \in ray$, and due to the regularity of H, there exists an identity element e in H such that $b \in xeaey$, and so $b \in xp_{vi}^{-1}p_{vi}ap_{\lambda k}{}^{-1}y$ (by the regularity of matrix P the elements v in $\wedge$ and k in I can be chosen such that $p_{\lambda k}$ and p_{vi} are not zero scalar).

It follows that there exists t in $xp_{vi}^{-1}p_{vi}a$ such that $b \in tp_{\lambda k}p_{\lambda k}{}^{-1}y$ hence

$$(j, b, \mu) \in \{(j, e, v), e \in xp_{vi}{}^{-1}\}\,(i, a, \lambda)\,\{(k, f, \mu), f \in p_{\lambda k}{}^{-1}y\}$$

$$= \bigcup_{t \in xp_{vi}{}^{-1}p_{vi}a} \{(j, s, \mu),\ s \in tp_{\lambda k}p_{\lambda k}{}^{-1}y\}$$

Thus S is simple.

Conclusions:

This paper extends and improves the notion of simple semi group to simple semi hypergroup. Using the results of this paper, completely simple semi hypergroup can be considered. By the proof of a generalization of Rees theorem in semi group theory, completely simple semi hypergroups can be characterized.

REFERENCES

Corsini, P., 1993. Prolegomena of Hypergroup Theory (second ed.), Aviani Editor.

Corsini P. and V. Leoreanu, 2003. Applications of Hyperstructures Theory, Advanced in Mathematics, Kluwer Academic Publishers.

Davvaz, B., 2006. Intuitionistic Hyperideal of semi hypergroups, Bull. Malays. Math. Sci. Soc., 2(29): 203-207.

Howie, J.M., 1995. Fundamentals of Semigroups Theory, Oxford University Press.

Marty, F., 1934. Sur uni generalization de la notion de group, in: 8th Congress Math. Scandenaves, Stockholm, pp: 45-49.