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Fuzzy Fredholm-volterra Integral Equations and Existance and Uniqueness of Solution of Them

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Abstract: In this paper,we will introduce a kind of fuzzy Volterra-Fredholm integral equation of the second kind. We investigate existance and uniqueness of solution of them.

Key words: component; Fuzzy integral equations; Fuzzy solution; Fuzzy valued functions

INTRODUCTION

The fuzzy differential and integral equations are important part of the fuzzy analysis theory and they have the important value of theory and application in control theory.

Seikkala (1987) has defined the fuzzy derivative which is the generalization of the Hukuhara derivative (Puri, 1983), the fuzzy integral which is the same as that of (Duboise, 1982), and by means of the extension principle of Zadeh, showed that the fuzzy initial value problem $x'(t) = f(t, x(t)), x(0) = x_0$ has a unique fuzzy solution when *f* satisfies the generalized Lipschitz condition which guarantees a unique solution of the deterministic initial value problem. Park and *et al.* (2000) have studied the Cauchy problem of fuzzy differential equation, charcterized those subsets of fuzzy sets in which the peano theorem is valid. Park and *et al.* (1999, 2000) have considered the existence of solution of fuzzy integral equation in Banach space and (Subrahmaniam 1994) have proved the existence of solution of fuzzy functional equations.

Park and Jeong (2000) have studied existence of solution of fuzzy integral equations of the form

$$x(t) = f(t) + \int_0^t f(t, s, x(s))\,ds, \quad 0 \le t$$

where *f* and *x* are fuzzy functions.

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$$x(t) = f(t) + \int_0^t f(t, s, x(s))\,ds + \int_0^a g(t, s, x(s))\,ds \quad 0 \le t \le a$$

But in this paper, we study the existence and uniqueness of the solution of fuzzy Volterra-Fredholm integral equation of the form

$$x(t) = f(t) + \int_0^t k(t,s)\, f(s, x(s))\,ds + \int_0^a h(t,s) g(s, x(s))\,ds$$

where $0 \le t \le a$ and $x(t)$ is an unknown fuzzy set-valued mapping and two kernels $k(t, s)$ and $h(t, s)$ are determined fuzzy set-valued mappings.

This paper is organized as following:

In Section 2, the basic concept of fuzzy number operation is brought. In Section 3, existence and uniqueness of solution of fuzzy Volterra-Fredholm integral equation of the second kind is investigated and finally, conclusion is drawn in section 5.

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Preliminaries:

Let P(ℜ) denote the family of all nonempty compact convex subsets of ℜ and define the addition and scalar multiplication in P(ℜ) as usual. Let *A* and *B* be two nonempty bounded subsets of ℜ. The distance between *A* and *B* is defined by the Hausdorff metric,

$$d(A,B)=\max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\sup_{b\in B}\inf_{a\in A}\|a-b\|\},$$

where $\|.\|$ denotes the usual Euclidean norm in ℜ. Then it is clear that (P(ℜ), *d*) becomes a metric space.

Let *I=[0, a]* ⊂ℜ be a closed and bounded interval and denote

$E=\{u:\Re\to[0,1]\,|\,u$ satisfies properties below

where

- *u* is normal, i.e., there exists an $x_0\in\Re$ such that $u(x_0)=1$,
- *u* is fuzzy convex,
- *u* upper semicontinuous,
- $[u]^0=cl\{x\in\Re\,|\,u(x)>0\}$ is compact.

For *0* ≤ *a* ≤ 1 denote $[u]^\alpha=\{x\in\Re\,|\,u(x)\geq\alpha\}$. Then α-level set $[u]^\alpha\in P(\Re)$ for all *0* ≤ *a* ≤ 1.

The set *E* is named set of all fuzzy real numbers. Obviously ℜ ⊂ *E*.

Definition 1:

An arbitrary fuzzy number *u* in the parametric form is represented by an ordered pair of functions $(\underline{u},\overline{u})$ which satisfy the following requirements:

- $\overline{u}:\alpha\to\overline{u}^\alpha\in\Re$ is a bounded left-continuous non-decreasing function over [0, 1],
- $\underline{u}:\alpha\to\underline{u}^\alpha\in\Re$ is a bounded left-continuous non-increasing function over [0, 1],
- $\underline{u}^\alpha\leq\overline{u}^\alpha$, $0\leq\alpha\leq1$.

Let $D:E\times E\to\Re+\cup\{0\}$ be defined by

$$D(u,v)=\sup_{0\leq\alpha\leq1}d([u]^\alpha,[v]^\alpha),$$

where *d* is the Hausdorff metric defined in (P(ℜ) *d*). Then *D* is a metric on *E*. Further, (*E, D*) is a complete metric space (Duboise, 1982; Seikala, 1987).

Definition 2:

A mapping $x:I\to E$ is bounded, if there exists r > 0 such that

$$D(x(t),\tilde{0})<r\quad\forall t\in I.$$

Also, we can be proved

- $D(u+w,v+w)=D(u,v)$ for every $u,v,w\in E$,
- $D(u\tilde{*}\,v,\tilde{0})=D(u,\tilde{0})D(v,\tilde{0})$ for every $u,v,w\in E$,

where the fuzzy multiplication is besbed on the extension principle that can be proved by α-cuts of fuzzy numbers *u, v*.

- $D(\lambda u, \lambda v) = |\lambda| D(u,v)$ for every $u, v \in E$ and $\lambda \in \Re$,
- $D(u+v, w+z) \le D(u,w) + D(v,z)$ for $u, v, w,$ and $z \in E$.

Definition 3:

A mapping $F: I \to E$ is strongly measurable if for all $\alpha \in [0,1]$ the set valued map $F_\alpha : I \to P(\Re)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when P(ℜ) the topology induced by the Hausdorff metric.

Definition 4:

A mapping $F: I \to E$ is said to be integrably bounded if there is an integrable function *h* such that $\|x\| \le h(t)$ for every $x \in F_0(t)$.

Definition 5;.

The integral of a fuzzy mapping $F:[0,1] \to E$ in defined levelwise by

$[\int_I F(t)dt]^\alpha = \int_I F_\alpha(t)dt = \{\int_I f(t)dt \mid f: I \to \Re$ is a measurable selection for $F_\alpha\}$

for all $\alpha \in [0,1]$.

It was proved by Puri and Relescu (1983) that a strongly measurable and integrably bounded mapping $F: I \to E$ is integrable (i.e.,$\int_I F(t)dt \in E$).

We recall some integrability properties for the fuzzy set-valued mappings (Kaleva, 1990).

Definition 6:

Let $f: \Re \to E$ be a fuzzy valued function. If for arbitrary fixed $t_0 \in \Re$. and $\varepsilon > 0$, a $\delta > 0$ such that

$|t - t_0| < \delta \Rightarrow D(f(t), f(t_0)) < \varepsilon,$

f is said to be continuous.

Theorem 1:

If $F: I \to E$ is continuous then it is integrable.

Theorem 2:

Let $F, G: I \to E$ be integrable and $\lambda \in \Re$. Then

- $\int_I (F(t) + G(t))dt = \int_I F(t)dt + \int_I G(t)dt,$
- $\int_I \lambda F(t)dt = \lambda \int_I F(t)dt,$
- $D(F,G)$ is integrable,

- $D(\int_I F(t)dt, \int_I G(t)dt) \leq \int_I D(F(t), G(t))dt,$

Existence Theorem
We consider the fuzzy Volterra-Fredholm integral equation

$$\begin{aligned} x(t) = f(t) + \int_0^t k(t,s)\, f(s, x(s))\, ds \\ + \int_0^a h(t,s) g(s, x(s))\, ds,\ 0 \leq t \leq a \end{aligned} \tag{1}$$

where $f:[0,a] \to E$ and $k:\Delta \to E, h:\Delta \to E,$ where $\Delta = (t,s): 0 \leq s \leq t \leq a$ and $g:[0,a]\times E \to E$ and $q:[0,a]\times E \to E$ are continuous.

Theorem:

Let a , L_1 and L_2 be positive numbers. Assume that Eq.(1) satisfies the following conditions:

- $f:[0,a] \to E$ is continuous and bounded.
- $k:\Delta \to E, h:\Delta \to E$ are continuous where $\Delta = (t,s): 0 \leq s \leq t \leq a$ and there exist $M_1 > 0$ and $M_2 > 0$ such that

$$\int_0^t D(k(t,s), \tilde{0})ds \leq M_1, \quad \int_0^t D(h(t,s), \tilde{0})ds \leq M_2$$

- $g:[0,a]\times E \to E, q:[0,a]\times E \to E$ are continuous and satisfy the Lipschitz condition, i.e.,

$$\begin{aligned} &D(g(t,x(t)), g(t,y(t)) \leq L_1 D(x(t), y(t)), \\ &D(q(t,x(t)), q(t,y(t)) \leq L_2 D(x(t), y(t)), \quad 0 \leq t \leq a, \end{aligned} \tag{2}$$

where $L_1 < M_1^{-1}, L_2 < M_2^{-1}$ and $x, y:[0,a] \to E$.

- $g(t,\tilde{0}), q(t,\tilde{0})$ are bounded on $[0,a]$.

then there exists a unique solution $x(t)$ of Eq.(1) on $[0,a]$ and the successive iterations

$$\begin{aligned} &x_0(t) = f(t) \\ &x_{n+1}(t) = f(t) + \int_0^t k(t,s) g(s, x_n(s))ds \\ &\qquad + \int_0^a h(t,s) q(s, x_n(s))ds, \quad (n = 0, \ldots) \end{aligned} \tag{3}$$

are uniformly convergent to x(t) on $[0,a]$.

Proof:

It is easy to see that all $x_n(t)$ are bounded on $[0,a]$. Indeed $x_0 = f(t)$ is bounded by hypothesis. Assume that $x_{n-1}(t)$ is bounded, we have

$$
\begin{aligned}
&D(x_n(t),\tilde{0}) \\
&\quad \le D(f(t),\tilde{0}) + D(\int_0^t k(t,s)g(s,x_{n-1}(s))ds,\tilde{0}) \\
&\quad + D(\int_0^a h(t,s)q(s,x_n(s))ds,\tilde{0})ds \\
&\quad \le D(f(t),\tilde{0}) + \int_0^t D(k(t,s),\tilde{0})D(g(s,x_{n-1}(s)),\tilde{0})ds \\
&\quad + \int_0^a D(h(t,s),\tilde{0})D(q(s,x(s)),\tilde{0})ds \le D(f(t),\tilde{0}) \\
&\quad + (\sup_{0\le t\le a} D(g(t,x_{n-1}(t)),\tilde{0}))\int_0^t D(k(t,s),\tilde{0})ds \\
&\quad + D(q(\xi,x_n(\xi)),\tilde{0}))\int_0^t D(h(t,s),\tilde{0})ds
\end{aligned} \tag{4}
$$

where $\xi \in [0,a]$. Taking every assumptions into account

$$
\begin{aligned}
&D(g(t,x_{n-1}(t)),\tilde{0}) \\
&\quad \le D(g(t,x_{n-1}(t)),g(t,\tilde{0})) + D(g(t,\tilde{0}),\tilde{0}) \\
&\quad \le L_1 D(x_{n-1}(t),\tilde{0}) + D(g(t,\tilde{0}),\tilde{0}), \\
&\quad D(q(t,x_{n-1}(t)),\tilde{0}) \\
&\quad \le D(q(t,x_{n-1}(t)),q(t,\tilde{0})) + D(q(t,\tilde{0}),\tilde{0}) \\
&\quad \le L_2 D(x_{n-1}(t),\tilde{0}) + D(q(t,\tilde{0}),\tilde{0}),
\end{aligned} \tag{5}
$$

We obtain that $x_n(t)$ is bounded. Thus, $x_n(t)$ is a sequence of bounded functions on $[0,a]$. Next we prove that $x_n(t)$ are continuous on $[0,a]$. For $0 \le t_1 \le t_2 \le a$, we have

$$
\begin{aligned}
&D(x_n(t_1),x_n(t_2)) \\
&\le D(f(t_1),f(t_2)) \\
&+ D(\int_0^{t_1} k(t_1,s)g(s,x_{n-1}(s))ds, \int_0^{t_2} k(t_2,s)g(s,x_{n-1}(s))ds) \\
&+ D(\int_0^a h(t_1,s)q(s,x_{n-1}(s))ds, \int_0^a h(t_2,s)q(s,x_{n-1}(s))ds) \\
&\le D(f(t_1),f(t_2)) \\
&+ \int_0^{t_1} D(k(t_1,s)g(s,x_{n-1}(s),\tilde{0})D(k(t_2,s)g(s,x_{n-1}(s),\tilde{0})ds \\
&+ \int_{t_1}^{t_2} D(k(t_2,s),\tilde{0})D(g(s,x_{n-1}(s)),\tilde{0})ds \\
&+ \int_0^a D(q(s,x_{n-1}(s),\tilde{0})\int_0^a D(h(t_1,s),h(t_2,s))ds
\end{aligned}
$$

then, we obtain

$$D(x_n(t_1), x_n(t_2)) \le D(f(t_1), f(t_2))$$
$$+\sup_{0\le t\le a} D(g(s, x_{n-1}(s), \tilde{0}))\int_0^{t_1} D(k(t_1,s), k(t_2,s))ds$$
$$+\sup_{0\le t\le a} D(g(s, x_{n-1}(s), \tilde{0}))\int_{t_1}^{t_2} D(k(t_2,s), \tilde{0})ds$$
$$+D(q(\xi, x_{n-1}(\xi), \tilde{0}))\int_0^{a} D(h(t_1,s), h(t_2,s))ds$$

By hypotheses and (5), we have

$D(x_n(t_1), x_n(t_2)) \to 0$ as $t_1 \to t_2$.

Thus the sequence $x_n(t)$ is continuous on $[0,a]$. Relation (2) and its analogue corresponding to n + 1 will give for $n\geq 1$:

$$D(x_{n+1}(t), x_n(t))$$
$$\le \int_0^{t} D(k(t,s),\tilde{0})D(g(s, x_{n-1}(s)), g(s, x_n(s)))ds$$
$$+\int_0^{a} D(h(t,s),\tilde{0})D(q(s, x_{n-1}(s))q(s, x_n(s)))ds$$
$$\le (\sup_{0\le t\le a} D(g(t, x_{n-1}(t)), g(t, x_n(t))))\int_0^{t} D(k(t,s),\tilde{0})ds$$
$$+D(q(\xi, x_n(\xi)), q(\xi, x_{n-1}(\xi))))\int_0^{a} D(h(t,s),\tilde{0})ds$$
$$\le M_1L_1\sup_{0\le t\le a} D(x_n(t), x_{n-1}(t)) + M_2L_2D(x_n(\xi), x_{n-1}(\xi))$$

Thus we get

$$\begin{aligned}&\sup_{0\le t\le a} D(x_{n+1}(t), x_n(t))\\ &\le M_1L_1\sup_{0\le t\le a} D(x_n(t), x_{n-1}(t)) + M_2L_2D(x_n(\xi), x_{n-1}(\xi))\end{aligned} \tag{6}$$

For n=0, we have

$$\begin{aligned}&D(x_1(t), x_0(t))\\ &= D(\int_0^{t} k(t,s)g(s, f(s))ds + \int_0^{a} h(t,s)q(s, f(s))ds, \tilde{0})\\ &\le \int_0^{t} D(k(t,s)g(s, f(s)),\tilde{0})ds + \int_0^{a} D(h(t,s)q(s, f(s)),\tilde{0})ds\\ &\le \int_0^{t} D(k(t,s),\tilde{0})D(g(s, f(s)),\tilde{0})ds\\ &+\int_0^{a} D(h(t,s),\tilde{0})D(q(s, f(s)),\tilde{0})ds\\ &\le \sup_{0\le t\le a} D(g(t, f(t)),\tilde{0})\int_0^{t} D(k(t,s),\tilde{0})ds\\ &+D(q(\xi, f(\xi)),\tilde{0})\int_0^{a} D(h(t,s),\tilde{0})ds\end{aligned} \tag{7}$$

So, we obtain

$$\sup_{0\le t\le a} D(x_1(t), x_0(t)) \le M_1N_1 + M_2N_2,$$

where

$$N_1 = \sup_{0\le t\le a} D(g(t, f(t)), \tilde{0}) \quad , \quad N_2 = D(q(\xi, f(\xi)), \tilde{0})$$

Moreover, from (6), we derive

$$\sup_{0\le t\le a} D(x_{n+1}(t), x_n(t)) \le L_1^n M_1^{n+1} N_1 + L_2^n M_2^{n+1} N_2 \tag{8}$$

which shows that the series $\sum_{n=1}^{\infty} D(x_n(t), x_{n-1}(t))$ is dominated, uniformly on $[0, a]$, by the series

$$M_1N_1\sum_{n=0}^{\infty}(L_1M_1)^n + M_2N_2\sum_{n=0}^{\infty}(L_2M_2)^n.$$

But (2) guarantees the convergence of the last series, implying the uniform convergence of the sequence $x_n(t)$. If we denote $x(t) = lim_{n\to\infty} x_n(t)$, then *x(t)* satisfies (1). It is obviously continuous on [0, *a*] and bounded.

To prove the uniqueness, let *y(t)* be a continuous solution of (1) on [0, *a*]. Then

$$y(t) = f(t) + \int_0^t k(t,s)g(s, y(s))ds + \int_0^a h(t,s)q(s, y(s))ds \tag{9}$$

From (2) and (9), we obtain for $n \ge 1$.

$$\begin{aligned}
&D(y(t), x_n(t)) \\
&\le \int_0^t D(k(s,t),\tilde{0}) D(g(s, y(s)), g(s, x_{n-1}(t)))ds \\
&+ \int_0^a D(h(s,t),\tilde{0}) D(q(s, y(s)), q(s, x_{n-1}(t)))ds \\
&\le \sup_{0\le t\le a} D(g(t, y(t)), g(t, x_{n-1}(t))) \int_0^t D(k(t,s),\tilde{0})ds \\
&+ D(q(\xi, y(\xi)), q(\xi, x_{n-1}(\xi))) \int_0^a D(h(t,s),\tilde{0})ds \\
&\le L_1M_1 \sup_{0\le t\le a} D(y(\xi), x_{n-1}(\xi)) + L_2M_2 D(y(\xi), x_{n-1}(\xi))
\end{aligned}$$

Since $L_1M_1 < 1, L_2M_2 < 1$

$$lim_{n\to\infty} x_n(t) = y(t) = x(t), \quad 0 \le t \le a,$$

which ends the proof of theorem.

Conclusion:

In this paper, we proved the existence and uniqueness of solution of fuzzy Volterra-Fredholm integral equation. Here, we use fuzzy kernels to obtain such solutions. For future research, we will prove existence and uniqueness of solution of Volterra- Fredholm integro-differential equations.

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