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Numerical Solution to a Functional Integral Equations Using the Legendre-spectral Method

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Abstract: In this paper, an application of Legendre-spectral method is applied to solve functional integral equations. The Legendre Gauss points are used as collocation nodes and Lagrange scheme is employed to interpolate the quantities needed. Using this approach a generalized functional integral equation both linear and nonlinear could be considered. The method is appropriate and so easy adaption for this class of integral equations. The results revealing that method is very effective, simple and accurate.

Key words: Functional integral equations; Legendre–spectral method; Lagrange interpolation; Gauss quadrature points and weights.

INTRODUCTION

Nonlinear functional-integral equations have been studied in the vehicular traffic, the biology, theory of optimal control and economics, etc. (Argyros, I.K., 1985). There are various cases of functional integral equations in literature, (Argyros, I.K., 1985; Deimling, K., 1985; Banas, J., B. Rzepka, 2003; Xiaoling Hu, Jurang Yan, 2006; Dhage, B.C., 2006; Maleknejad, K., 2008) and etc. In above-mentioned references the existence solution, analytical behavior, stability and applications of functional integral equations are investigated. But there are few methods for numerically solving this problems in literature. Some numerical scheme presented for very special cases of this equations, (Babolian, E., 2008; Abbasbandy, S., 2007).
A functional integral equation could be stated as:

$$x(t) = g(t, x(t)) + f\left(t, \int_0^t u(t,s,x(s))ds, \int_0^\alpha u(t,s,x(s))ds, x(a(t))\right) \qquad 0 \le t \le \alpha, \tag{1.1}$$

where α is a finite constant. In fact the functional integral equation of the above form contains a lot of special types of functional integral equations. For example, differential equations with transformed argument or differential equations of neutral type can be transformed to equations of the type (1.1). It worth to point out that there is a class of functional integral equations which defines on $t \in [0,\infty)$ (Banas, J., I.J. Cabrera, 2007; Olszowy, L., 2008).

This type is not our purpose in this work.

In this paper we will solve numerically (1.1) using a Legendre spectral scheme which presented recently by Tang *et al*. for Volterra type integral equation. We suppose that the equation (1.1) has a unique solution. The class of solution methods based on orthogonal polynomials have become known as spectral methods. Spectral methods are implemented in various ways, but the method presented in Tang *et al*. is so appropriate for Eq. (1.1). We extend it to solve this functional integral equation. The remainder of the paper is organized as follows: In Section 2, the Legendre - spectral method is presented. In Section 3, numerical results for some problems, are investigated and the corresponding tables and figures are presented. Finally in Section 4 the report ends with a brief conclusion.

2 Legendre-spectral Method:

Recently Tang *et al* presented a promising Legendre-spectral method for solving Volterra integral

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equations. Their methods based on the Legendre Gauss collocation points and Lagrange interpolation method. They proved that the numerical errors in the infinity norm will decay exponentially. We present this approach to solve of a functional integral equations (1.1).

Without loss of generality, assume that interval [-1, 1] is replaced by interval [0, a]. Set the collocation points as the set of N Legendre-Gauss, or Gauss-Radua, or Gauss-Lobatto points $\{t_j\}_{j=1}^{N}$. Assume that Eq. (1.1) holds at t_j on [-1, 1]:

$$x(t_j) = g(t_j, x(t_j)) + f\left(t_j, \int_{-1}^{t} u(t_j, s, x(s))ds, \int_{-1}^{1} u(t_j, s, x(s))ds, x(a(t_j))\right) \quad j = 1, 2, \ldots, N \tag{2.1}$$

The main difficulty to obtaining high rate of accuracy is to compute the integral term in (2.1). In fact for small values of t_j , there is a little information available for x(s). To overcome this difficulty the integral interval $[-1, t_j]$ is transferred to a fix interval [-1, 1]. We first make the following simple linear transformation:

$$s(t,\theta) = \frac{t+1}{2}\theta + \frac{t-1}{2}, \qquad -1 \le \theta \le 1 \tag{2.2}$$

Then (2.1) takes the form:

$$x(t_j) = g(t_j, x(t_j)) + f\left(t_j, \frac{t_j+1}{2}\int_{-1}^{1} u(t_j, s(t_j,\theta))x(s(t_j,\theta))d\theta, \int_{-1}^{1} u(t_j, \theta, x(\theta))d\theta, x(a(t_j))\right) \tag{2.3}$$
$$j = 1, 2, \ldots, N$$

Using an N-point Gauss quadrature rule related to the Legendre weights $\{w_j\}$ in [-1, 1] gives:

$$x(t_j) = g(t_j, x(t_j)) + f\left(t_j, \frac{t_j+1}{2}\sum_{K=1}^{N} u(t_j, s(t_j,\theta_k))x(s(t_j,\theta_k))w_k, \sum_{k=1}^{N} u(t_j, \theta_k, x(\theta_k))w_k, x(a(t_j))\right) \tag{2.4}$$
$$j = 1, 2, \ldots, N$$

where the set $\{\theta_k\}_{k=1}^{N}$ coincide with the collocation points $\{t_j\}_{j=1}^{N}$. We now need to represent $x(s(t_j,\theta_k))$ and $x(\alpha(t_j))$ in terms of x_p for j = 1, 2, ...,N. To this end, we expand them using Lagrange interpolation polynomials as

$$x(\sigma) \approx \sum_{p=1}^{N} x_p l_p(\sigma), \tag{2.5}$$

where l_p is the p-th Lagrange basis function. Combining Eq. (2.4) and (2.5) yields:

$$x(t_j) = g(t_j, x(t_j)) + f\left(t_j, \frac{t_j+1}{2}\sum_{k=1}^{N} u\left(t_j, s(t_j,\theta_k), \sum_{p=1}^{N} x_p l_p(s(t_j,\theta_k))\right) w_k, \sum_{k=1}^{N} u(t_j,\theta_k,x_k) w_k, \sum_{p=1}^{N} x_p l_p(\alpha(t_j))\right) \quad (2.6)$$

$$j = 1,2,\ldots,N$$

Eq. (2.6) can then be solved by some methods suitable for solving the non-linear systems. Note that if we consider the linear functional integral equation, it leads to a linear system of equations and could be solved by a proper method such as Gauss elimination of LU decomposition methods.

3 Numerical Results:

In this section, the method is applied to some numerical examples. All computations are performed by the Matlab 7.1 software package. The numerical scheme (2.6) leads to a non-linear system for $\{x_j\}_{j=1}^{N}$, and a proper solver for the non-linear system should be used. To solve it, we use the robust routine fsolve from the optimization toolbox of Matlab. fsolve should be provided with an initial guess as a starting point. For different starting points we observed same convergence point with more or less iterations.

3.1 Example 1:

Let us now consider the following Fredholm functional integral equation of the second kind (Babolian, E., 2008):

$$x(t) + e^{-s}x(\alpha(t)) + \int_{-1}^{1} e^{t-s}x(s)ds = g(s), \quad (3.1)$$

Where the exact solution is

$$x(t) = e^{t}$$

As reported in (Babolian, E., 2008), we consider four different options for $\alpha(t)$ as

$$\text{case1}: \alpha(t) = 0.8t, \qquad \text{case2}: \alpha(t) = t^2/2,$$
$$\text{case3}: \alpha(t) = te^{-t}, \qquad \text{case4}: \alpha(t) = \sin(t).$$

and with appropriate right hand side. The numerical results are shown in Table 1 and Fig. 1 in terms of absolute errors for different numbers of N from 2 up to 15. Results show the excellent convergence of method. Note that in this example, the final system is linear and it is solved by a numerical methods for linear algebraic systems.

3.2 Example 2:

Consider the following nonlinear Fredholm-Volterra integral equation (which is an special case of Eq. (1.1)) (Minggen Cui, Hong Du, 2006):

$$x(t) = g(t) + \int_0^t F_1(t,s)N_1(x(s))ds + \int_0^t F_2(t,s)N_2(x(s)) \qquad 0 \le t \le 1, \quad (3.2)$$

Where

$$F_1(t,s) = \sin(t-s), \qquad F_2(t,s) = t-s,$$
$$N_1(x(t)) = \cos(x(t)), \qquad N_2(x(t)) = 1 + x(t)^2.$$

Table 1: Error estimates for Example 1.

N	case 1	case 2	case 3	case 4
2	0.29×10^{-1}	0.59×10^{0}	0.16×10^{0}	0.78×10^{-2}
3	0.22×10^{-1}	0.34×10^{0}	0.11×10^{0}	0.12×10^{-1}
4	0.39×10^{-2}	0.66×10^{-1}	0.26×10^{-1}	0.24×10^{-2}
5	0.64×10^{-3}	0.94×10^{-1}	0.24×10^{-2}	0.44×10^{-3}
6	0.74×10^{-4}	0.31×10^{-2}	0.86×10^{-3}	0.55×10^{-4}
7	0.62×10^{-5}	0.15×10^{-3}	0.18×10^{-3}	0.51×10^{-5}
8	0.34×10^{-6}	0.23×10^{-4}	0.11×10^{-4}	0.35×10^{-6}
9	0.16×10^{-7}	0.25×10^{-6}	0.16×10^{-5}	0.17×10^{-7}
10	0.10×10^{-8}	0.47×10^{-7}	0.30×10^{-6}	0.63×10^{-9}
11	0.94×10^{-10}	0.16×10^{-8}	0.17×10^{-7}	0.39×10^{-10}
12	0.53×10^{-11}	0.33×10^{-10}	0.11×10^{-8}	0.21×10^{-11}
13	0.15×10^{-12}	0.26×10^{-11}	0.18×10^{-9}	0.14×10^{-12}
14	0.65×10^{-14}	0.16×10^{-12}	0.90×10^{-12}	0.65×10^{-14}
15	0.35×10^{-14}	0.12×10^{-11}	0.22×10^{-10}	0.10×10^{-15}

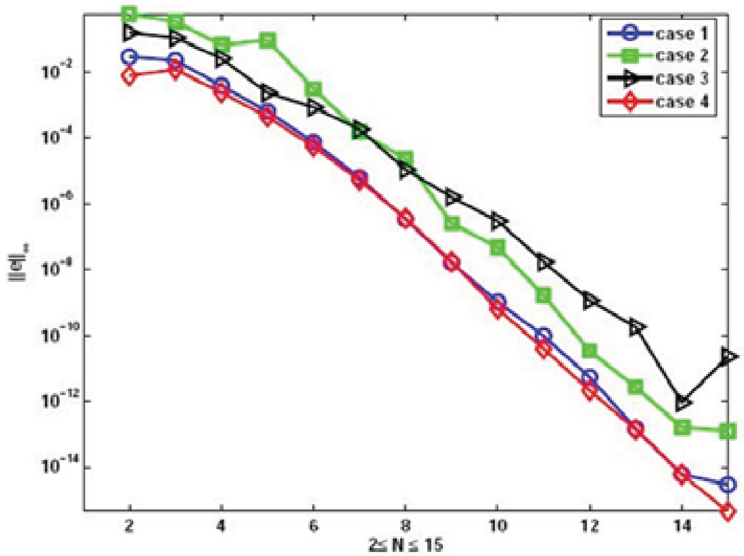

Fig. 1: Error estimate for cases 1, 2, 3 and 4: Example 1.

And

$$g(t) = \frac{19}{12} - \frac{7}{3}t - \frac{1}{4}\cos(t-1) - \frac{1}{2}\sin(t-1) + \frac{1}{4}\cos(t+1).$$

Then $x(t) = 1 - t$ is the exact solution. The calculations are performed using N = 5, i.e. 5 Legendre-Gauss points are chosen in interval [0, 1] and then the estimation of error at uniform mesh is carried out using Lagrange interpolation at these five collocation points. Note that for this problem, the procedure leads to a nonlinear system of equations for unknowns $x_j = x(t_j)$. `o start the routine fsolve we use the initial guess [0, 0, . . . , 0]. In Table 2, the numerical results are compared with the best results given in (Minggen Cui, Hong Du, 2006) in terms of absolute error at some selected points.

Table 2: Maximum absolute errors for different N.

t	Exact	Absolute errors of Ref. [12]	$\|e_\infty\|$ of this paper
0.08	0.92	1.18148×10^{-4}	1.06294×10^{-11}
0.16	0.84	1.44021×10^{-4}	6.64244×10^{-12}
0.24	0.76	1.65987×10^{-4}	5.32363×10^{-12}
0.32	0.68	1.84716×10^{-4}	3.84476×10^{-12}
0.40	0.60	2.00596×10^{-4}	6.90670×10^{-13}
0.48	0.52	2.14138×10^{-4}	4.34258×10^{-12}
0.56	0.44	2.25485×10^{-4}	1.01471×10^{-11}
0.64	0.36	2.36015×10^{-4}	1.43028×10^{-11}
0.72	0.28	2.45419×10^{-4}	1.30782×10^{-11}
0.80	0.20	2.52873×10^{-4}	1.42963×10^{-12}
0.88	0.12	2.63029×10^{-4}	4.26998×10^{-11}
0.96	0.04	2.75999×10^{-4}	7.98729×10^{-11}

3.3 Example 3:

In this example we consider the following nonlinear Volterra functional integral equation:

$$x(t) + 2x(2t)e^{\left(\int_0^t \frac{2e^s x(s)}{1+x(s)^2} ds - 2t\right)} = 1 + e^t + e^{2t}, \qquad 0 \le t \le 1. \tag{3.3}$$

and the exact solution is

$$x(t) = e^t.$$

For this problem, the procedure leads to a nonlinear system of equations too. As before the initial guess [0,0, . . . , 0] is utilized. Numerical results are presented in Table 3 and Fig. 2 in terms of absolute errors for N = 2 up to 15. The results confirm the ability and accuracy of the scheme.

Table 3: Maximum absolute errors for different N

N	2	3	4	5	6	7	8
$\|e_\infty\|$	0.58×10^{0}	0.31×10^{0}	0.46×10^{-1}	0.10×10^{-1}	0.22×10^{-2}	0.30×10^{-3}	0.31×10^{-4}
N	9	10	11	12	13	14	15
$\|e_\infty\|$	0.53×10^{-5}	0.52×10^{-6}	0.48×10^{-7}	0.52×10^{-8}	0.29×10^{-9}	0.14×10^{-9}	0.27×10^{-10}

4 Conclusion:

An efficient and accurate numerical scheme based on the Legendre-spectral method proposed for solving the nonlinear system of Fredholm-Volterra integral equations. The Gaussian integration method with the Lagrange interpolation were employed to reduce the problem to the solution of non-linear algebraic equations. Illustrative examples were given to demonstrate the validity and applicability of the method. The results show that the method is very simple and accurate. In fact by selecting a few collocation points excellent numerical results are obtained.

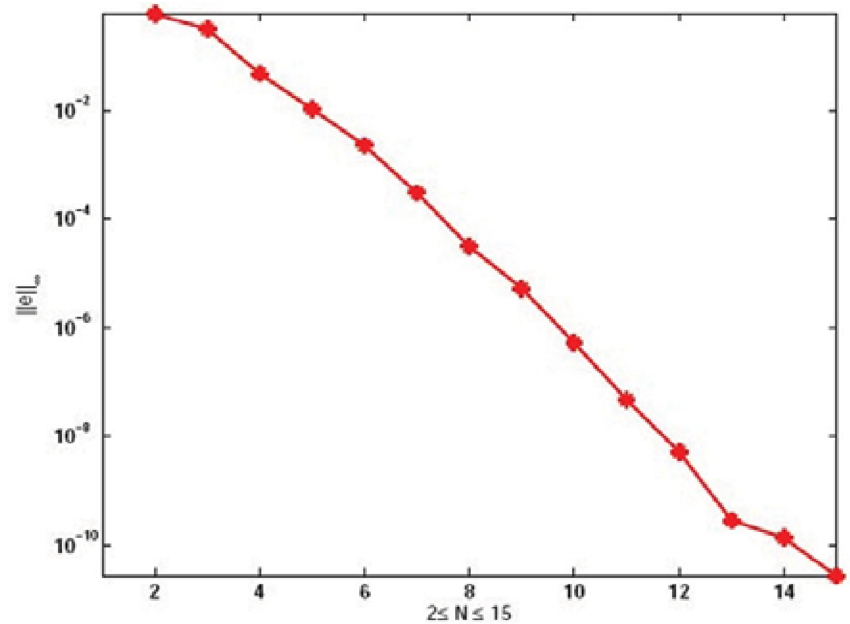

Fig. 2: Error estimate for Example 3.

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