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Legendre Multi-wavelets Direct Method for Solving Fredholm Integral Equations of the Second Kind

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Abstract: In this paper,We use the continuous Legendre multi-wavelets on the interval [0, 1) to solve Fredholm integral equations of the second kind. To do so, we reduced the solution of Fredholm integral equation to the solution of algebraic equations. Illustrative examples are included to show the high accuracy of the estimation, and to demonstrate validity and applicability of the technique.

Key words: Fredholm integral equation; Legendre multi-wavelets; Multiresolution of analysis(MRA)

INTRODUCTION

In the recent years, there has been an increase usage among scientists and engineers to apply wavelet technique as well as a numerical solution to solve both linear and nonlinear problems. The main advantage of the wavelet technique is its ability to transform complex problems into a system of algebraic equations. Several numerical methods for approximating the solution of this kind of integral equations are known, and many different basic functions have been used, such as orthogonal bases and wavelets. In (Maleknejad, K., *et al*, 2002), the problem is solved by using the rationalized wavelets. Spline functions (Maleknejad, K., D. Rahbar, 2000), Haar wavelets (Maleknejad, K., F. Mirzaee, 2005), Triangular orthogonal functions (Babolian, E., H.R. Marzban, 2008), Monte Carlo method (Farnoosh, R., M. Ebrahimi, 2008) and Coifman wavelet (Maleknejad, K., T. Lotfi, 2007) are also applied for solving these problems. The hybrid functions are also used in several literatures, such as (Maleknejad, K., M.T. Kajani, 2003; Kajani, M.T., A.H. Vencheh, 2005). In this paper, we present the application of the linear Legendre multi-wavelets as basis functions in Galerkin's method for numerical solution of the second kind Fredholm integral equations. Numerical experiments show that the Legendre multi-wavelets has a good degree of accuracy.

2 Legendre Multi-wavelets and its Properties:

2.1 Wavelets and Legendre Multi-wavelets:

A wave is usually defined as an oscillating function of time or space, such as a sinusoid. Fourier analysis is wave analysis, it expands a signal or function in term of sinusoid. A wavelet is a "small wave", which has its energy concentrated in time to give a tool for the analysis of transient, nonstationary, or time-varying phenomena. Wavelet constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as (Boggess, A., F.J. Narcowich, 2001)

$$\psi a,b(t) = |a|^{-1}\psi\left(\frac{t-b}{a}\right), \quad a,b \in \mathrm{R}, \quad a \neq 0.$$

If we restrict the parameters a and b to the discrete values as $a = a_0^{-k}, b = nb_0a_0^{-k}$, where $a_0 > 1$, $b_0 > 0$, n and k are positive integers, we have the following of discrete wavelets:

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$$\psi n,k(t) = |a|^{\frac{k}{2}} \psi(a_0^k t - nb_0),$$

which from a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{n,k}(t)$ forms an orthonormal basis (Boggess, A., F.J. Narcowich, 2001).

Definition 2.1:
The increasing sequence $\{V_j\}_{j\in z}$ of subset of $L^2(\mathbb{R})$ with scaling function φ is called MRA if it satisfies the conditions

1. $\cup_j V_j$ is dense in $L^2(\mathbb{R})$,
2. $\cap_j V_j = \{0\}$,
3. $f(t) \in V_j \Leftrightarrow f(2^{-j}t) \in V_0$
4. $\{\varphi(t-n)\}_{n\in Z}$ is an orthogonal basis for V_0.

By definition of MRA the sequence $\{2^j \varphi(2^j t - n)\}_{n\varepsilon Z}$ forms an orthonormal basis for V_j. For constructing the linear Legendre multi-wavelets, we replace the single scaling function $\varphi(t)$ of one real variable t with a vector of scaling functions $\Phi(t) = [\varphi_1(t), \ldots, \varphi_m(t)]^T$, where the operator $\Phi(t)$ satisfies a matrix version of the usual scaling equation and the corresponding wavelet functions are defined in a similar matrix manner. Hence, $\Phi(t)$ satisfies an equation of the form

$$\Phi(t) = \sum_k A_k \Phi(2t-k) \tag{2.1}$$

where A_k is a finite set of matrices, and the corresponding wavelet functions are defined by

$$\Psi(t) := \sum_k B_k \Phi(2t-k) \tag{2.2}$$

where $\Psi(t) = [\psi^0(t), \ldots, \psi^m(t)]^T$ is the vector of corresponding wavelet functions. By taking $m = 2$, we describe scaling functions $\varphi_0(x)$ and $\varphi_1(x)$ as follows:

$$\varphi_0(t) = 1, \quad \varphi_1(t) = \sqrt{3}(2t-1), \quad 0 \le t < 1. \tag{2.3}$$

In definition of MRA, V_j is linear span of $\{,2^j\varphi_0(2^jt-n), \quad 2^j\varphi_1(2^jt-n)\}_{n\in Z}$

Now, let $\psi^0(t)$ and $\psi^1(t)$ be the corresponding mother wavelets, then by MRA we have

$$\begin{aligned} \psi^0(t) &= a_0\varphi_0(2t) + a_1\varphi_1(2t) + a_2\varphi_0(2t-1) + a_3\varphi_1(2t-1), \\ \psi^1(t) &= b_0\varphi_0(2t) + b_1\varphi_1(2t) + b_2\varphi_0(2t-1) + b_3\varphi_1(2t-1) \end{aligned} \tag{2.4}$$

and by applying suitable conditions (Khellat, F., S.A. Yousefi, 2006) on $\psi^0(t)$ and $\psi^1(t)$ the explicit formula for Legendre mother wavelets will obtain as

$$\psi^0(t)=\begin{cases}-\sqrt{3}(4t-1), & 0\le \mathrm{t}<\dfrac{1}{2},\\ \sqrt{3}(4t-3), & \dfrac{1}{2}\le \mathrm{t}<1,\end{cases} \tag{2.5}$$

$$\psi^1(t)=\begin{cases}6t-1, & 0\le \mathrm{t}<\dfrac{1}{2},\\ 6t-5), & \dfrac{1}{2}\le \mathrm{t}<1,\end{cases} \tag{2.6}$$

and the family $\{\psi_{kn}^j\}=\left\{2^{\frac{k}{2}}\psi^j(2^k t-n)\right\}$, k is any nonnegative integer, n = 0, 1,· · · , 2^k-1 and j=0,1,

forms an orthonormal basis for L^2 ($\mathbb{R}$). The next four functions are given by

$$\psi_{10}^0(t)=\begin{cases}-\sqrt{6}(8t-1), & 0\le t<\dfrac{1}{4},\\ \sqrt{6}(8t-3), & \dfrac{1}{4}\le t<\dfrac{1}{2},\\ 0 & \dfrac{1}{2}\le t<1,\end{cases}$$

$$\psi_{11}^0(t)=\begin{cases}0 & 0\le t<\dfrac{1}{2},\\ -\sqrt{6}(8t-5), & \dfrac{1}{2}\le t<\dfrac{3}{4},\\ \sqrt{6}(8t-7) & \dfrac{3}{4}\le t<1,\end{cases}$$

$$\psi_{10}^0(t)=\begin{cases}\sqrt{2}(12t-1), & 0\le t<\dfrac{1}{4},\\ \sqrt{6}(8t-5), & \dfrac{1}{4}\le t<\dfrac{1}{2},\\ 0 & \dfrac{1}{2}\le t<1,\end{cases}$$

$$\psi_{11}^{0}(t)=\begin{cases}0 & 0\le t<\dfrac{1}{2},\\ -\sqrt{2}(12t-7), & \dfrac{1}{2}\le t<\dfrac{3}{4},\\ \sqrt{2}(12t-11) & \dfrac{3}{4}\le t<1,\end{cases}$$

3 Legendre Multi-wavelets Direct Method:

In this section we have used the Legendre multi-wavelets to approximate the functions with one variable or two variables then by substituting of these approximations in the Fredholm integral equation, the equation has transformed into a system of algebraic equations.

3.1 Function Approximation:

A function *f(t)* defined over [0, 1) may be expanded as

$$f(t)=f_0\phi_0(t)+f_1\phi_1(t)+\sum_{k=0}^{\infty}\sum_{j=0}^{1}\sum_{n=0}^{\infty}f_{kn}^{j}\psi_{kn}^{j}(t) \tag{3.1}$$

where

$$f_0=\left\langle f(t),\phi_0(t)\right\rangle,\quad f_1=\left\langle f(t),\phi_1(t)\right\rangle,\quad f_{kn}^{j}=\left\langle f(t),\psi_{kn}^{j}(t)\right\rangle \tag{3.2}$$

In equation (3.2), (.,.) denoting the inner product. If the infinite series of equation (3.1) is truncated, then it can be written as

$$f(t)\approx f_0\phi_0(t)+f_1\phi_1(t)+\sum_{k=0}^{M}\sum_{j=0}^{1}\sum_{n=0}^{2^k-1}f_{kn}^{j}\psi_{kn}^{j}(t)=F^T\Psi(t) \tag{3.3}$$

Where Ψ (t) and F are given by

$$F=\left[f_0,f_1,f_{00}^{0},f_{00}^{1},\ldots,f_{M0}^{0},f_{M1}^{0},\ldots,f_{M(2^M-1)}^{0},\ldots,f_{M0}^{1},f_{M1}^{1},\ldots f_{M(2^M-1)}^{1}\right]^T,$$

$$\Psi(t)=\left[\phi_0(t),\phi_1(t),\psi_{00}^{0}(t),\psi_{00}^{1}(t),\ldots,\psi_{M0}^{0}(t),\psi_{M1}^{0}(t),\ldots,\psi_{M(2^M-1)}^{0}(t),\ldots,\psi_{M0}^{1}(t),\psi_{M1}^{1}(t),\ldots,\psi_{M(2^M-1)}^{1}(t)\right]^T$$

We relabel these functions as follows

$$\zeta_1(t)=\phi_0(t),\zeta_2(t)=\phi_0(t),\zeta_3(t)=\psi_{00}^{0}(t),\zeta_4(t)=\psi_{00}^{1}(t),\ldots,\zeta_{2M+2}(t)=\psi_{M(2^M-1)}^{1}(t)$$

3.2 Solving Fredholm Integral Equation of the Second Kind:

Consider Fredholm integral equation of the second kind

$$\mathrm{y(t)}=\mathrm{x(t)}+\int_0 k(t,,s)y(s)ds, \tag{3.4}$$

where $x\in L^2$ [0, 1), $k\in L^2$ ([0, 1) × [0, 1)) and y is an unknown function. If we approximate *x, y* and K by (3.1)-(3.1) as follows

$$x(t) = X^T \Psi(t), y(t) = X^T \Psi(t), k(t,s) = \Psi^T(t) K \Psi(s)$$

where

$$K = [k_{i,j}]_{2^{(M+2)} \times 2^{(M+2)}} \quad k_{i,j} = \int_0^1 \int_0^1 k(t,s) \zeta_i(s) \zeta_j(t) ds dt.$$

With substituting in (3.4) we have

$$\Psi^T(t) Y = \int_0^1 \Psi^T(t) K \Psi(s) \Psi^T(s) Y ds + \Psi^T(t) X$$

so,

$$\Psi^T(t) Y = \Psi^T(t) K Y = \Psi^T(t) X$$

then,

$$(I - K) Y = X.$$

By solving this linear system we can find the vector Y.

4 Numerical Examples:

In three examples below $|y - \tilde{y}|$ shows the absolute value of difference between the exact solution with the numerical solution. Table 1, 2 and 3 show the values of $|y - \tilde{y}|$ corresponding to Legendre multi-wavelets method for example 1, 2 and 3 respectively. The computations associated with the examples were performed using Maple 12.

Example 4.1:

Consider Fredholm integral equation of the second kind given in (Han Danfu, Shang Xufeng, 2007)

$$y(t) = e^{(2t = \frac{1}{3})} - \frac{1}{3} \int_0^1 e^{(2t = \frac{1}{3})} y(s) ds, \tag{4.1}$$

The exact solution of this problem is $y(t) = e^{2t}$.

We solve (4.1) by using the method with $M = 1$, $M = 2$ and $M = 4$ in section 4.

Table 1: Numerical results of Example 1

x	$\|y - \tilde{y}\|$		
	M = 1	M = 2	M = 4
0	2.5558e-02	5.7622e-03	3.3240e-04
0.1	1.1457e-02	1.8086e-04	1.7555e-05
0.2	5.4625e-04	3.3920e-03	2.1378e-04
0.3	2.5056e-03	4.2190e-03	2.6075e-04
0.4	2.0132e-02	6.0825e-04	2.7300e-05
0.5	6.9473e-02	1.5663e-02	9.0463e-04
0.6	3.1145e-02	4.9150e-04	4.5361e-05
0.7	1.4849e-03	9.2203e-03	5.8203e-04
0.8	6.8107e-03	1.1468e-02	7.0778e-04
0.9	5.4724e-02	1.6535e-03	6.9118e-05

Example 4.2:

Next, consider Fredholm integral equation of the second kind

$$y(t) = \int_0^1 (st+t)y(s)ds + x(t) \tag{4.2}$$

where

$$x(t) = \begin{cases} \frac{8}{3}t - \frac{1}{3}, & 0 \le t < \frac{1}{3}, \\ -\frac{1}{3}t + \frac{2}{3}, & \frac{1}{3} \le t < \frac{2}{3}, \\ -\frac{10}{3}t + \frac{8}{3}, & \frac{2}{3} \le t < 1, \end{cases}$$

and exact solution

$$y(t) = \begin{cases} 3t, & 0 \le t < \frac{1}{3}, \\ 1, & \frac{1}{3} \le t < \frac{2}{3}, \\ 3-3t, & \frac{2}{3} \le t < 1. \end{cases}$$

We solve (4.2) by using the method with $M = 1$, $M = 2$ and $M = 4$ in section 4.

Table 2: Numerical results of Example 2

x	$\lvert y-\tilde{y}\rvert$		
	M = 1	M = 2	M = 4
0	3.30e-10	3.30e-10	3.30e-10
0.1	2.00e-10	2.00e-10	2.00e-10
0.2	1.00e-10	1.00e-10	1.00e-10
0.3	3.00e-10	1.11e-02	4.00e-10
0.4	2.22e-02	4.00e-10	4.00e-10
0.5	5.56e-02	0.00e-00	0.00e-00
0.6	2.22e-02	4.00e-10	4.00e-10
0.7	3.00e-10	1.11e-02	4.00e-10
0.8	1.00e-10	1.00e-10	1.00e-10
0.9	2.00e-10	2.00e-10	2.00e-10

Example 4.3:

Finaly, consider the following Fredholm integral equation of the second kind

$$y(t) = 0.9t^2 + \int_0^1 0.5t^2 s^2 y(s)ds,$$

for which the exact solution is y (t) = t^2.
The results obtained for y (t) with $M = 1$, $M = 2$ and $M = 3$ are presented in Table 3.

Table 3: Numerical results of Example 3

x	$\lvert y-\tilde{y}\rvert$		
	M = 1	M = 2	M = 4
0	1.0416e-02	2.6042e-03	1.6276e-04
0.1	4.5832e-03	1.0417e-04	6.5104e-06
0.2	4.1714e-04	1.1458e-03	7.1614e-05
0.3	4.1775e-04	1.1458e-03	7.1614e-05
0.4	4.5813e-03	1.0429e-04	6.5109e-06
0.5	1.0419e-02	2.6043e-03	1.6276e-04
0.6	4.5789e-03	1.0444e-04	6.5113e-06
0.7	4.2257e-04	1.0444e-03	7.1613e-05
0.8	4.2438e-04	1.1453e-03	7.1613e-05
0.9	4.5735e-03	1.0478e-04	6.5126-06

5 Conclusion:

The Legendre multi-wavelets has been applied for solving Fredholm integral equation of the second kind by reducing an integral equation into a system of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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