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The Canonical Product of the Differential Equation with One Turning Point and Singular Point

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Abstract: We consider the differential equation

$$-y'' + q(x)y = \rho^2 \phi^2(x) y \qquad for \qquad x \in I := [0,1], \quad (*)$$

where the weight function f φ^2 is real and has one zero in the open interval (0,1), so called turning point, and the assumption that is odd order. This turning point is admitted to be pole of the function q. Using the asymptotic solution as well as the distribution of positive and negative eigenvalue, we derive the canonical product of a particular solution of the Sturm-Liouville in one turning point case.

Key words: Turning point; Singular point; Eigenvalues; Sturm-Liouville; Infinite products.

INTRODUCTION

We consider the differential equation

$$-y'' + q(x)y = \rho^2 \phi^2(x) y, \qquad 0 \le x \le 1. \tag{1}$$

Here $\lambda=\rho^2$ is the eigenvalue parameter. We assume that the weight function f^2 is real and has one zero in the open interval (0,1), so called turning point, and the assumption that is odd order. This turning point is admitted to be pole of the function q. Differential equations with turning points have various applications in mathematics, elasticity, optics, geophysics and other branches of natural sciences (see (Eberhard *et al.,* 1994; Goldenveizer *et al.,* 1997) and the references therein). The importance of asymptotic analysis in obtaining information on the solution of a Sturm-Liouville equation with multiple turning points was realized by Leung (Leung, 1977), Olver (1977a; 1977b), Heading (Headig, 1977), and Eberhard, Freiling and Schneider in (Eberhard *et al.,* 1994). The results of Dorodnicyn (Dorodnicyn, 1960), Kazarinoff (Kazarinoff, 1958), Langer, (1931), Dyachenko, (2000), Marasi and Jodayree in (2006) and Neamaty (1999), Neamaty and Dabbaghian (2007) bring important innovations to the asymptotic approximation of solutions of Sturm-Liouville equations. It is necessary to point out that applying asymptotic solutions for studying inverse problem in turning points cases, is more complicated and practically is not convenient to use. Especially in deriving the asymptotic formulas, one should apply Bessel function type. In addition a more difficult and challenging task is to shape the asymptotic behavior of the solutions and corresponding eigenvalues. So the inverse problem of reconstructing the potential function from the given spectral information and corresponding dual equation cannot be studied by using the asymptotic forms. In fact, in asymptotic methods one cannot generally express the exact solution in closed form. Indeed in methods connected with dual equations, the closed form of the solution is needed. The representing solution of the infinite product form plays an important role for investigating the corresponding dual equations. In the previous article (Kheiri and Jodayree, 2003), authors considered the following Sturm-Liouville equation (1). It is assumed that q(x) is a real function that is Lebesgue integrable on the interval [0,1], $\lambda=\rho^2$ is the spectral parameter and

$$\phi^2(x) = (x - x_1)^{4m+1} \phi_0(x),$$

where $0 < x_1 < 1$, $m \in N$, $\varphi_0 > 0$ for $x \in [0,1]$, φ_0 is a twice continuously differentiable on [0,1], and $\varphi^2(x)$

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has one zero in [0,1], so called turning point. The solution y(x,λ) of such and equation with initial conditions y(0, λ) = 0, y′(0, λ) = 1 was found to have the infinite product form

$$y(x,\lambda)=\frac{1}{2}\left|\phi(x)\phi(0)\right|^{\frac{-1}{2}}p(x)\prod_{k\geq 1}\frac{\lambda-\lambda_k(x)}{z_k^2(x)},\qquad 0\leq x<x_1 \tag{2}$$

$$y(x,\lambda)=\frac{1}{2}\left|\phi(x)\phi(0)\right|^{\frac{-1}{2}}\pi p^{\frac{1}{2}}(x_1)f^{\frac{1}{2}}(x)\cos\frac{\pi\mu}{2}$$
$$\times\prod_{k\geq 1}\frac{(\lambda-r_k(x))p^2(x_1)}{\tilde{j}_k^2}\prod_{k\geq 1}\frac{(u_k(x)-\lambda)f^2(x)}{\tilde{j}_k^2},\qquad x_1<x<1, \tag{3}$$

where the sequence $\{u_k(x)\}$ represents the sequence of positive eigenvalues and $\{r_k(x)\}$ the sequence of negative eigenvalues of the Dirichlet problem associated with (1) on [0,x], for each x in $(0,x_1]$. The sequence $\{\lambda_k(x)\}$, for each fixed x, $0 < x \leq x_1$, represents the sequence of negative eigenvalues of the Dirichlet problem for Eq.(1) on the closed interval [0,x], where

$$p(x)=\int_0^x|\phi(t)|\,dt \qquad for \qquad 0\leq x<x_1,$$

$$f(x)=\int_{x_1}^x|\phi(t)|\,dt \qquad for \qquad x_1<x\leq 1,$$

$$z_k(x)=\frac{k\pi}{p(x)} \qquad for \qquad k=1,2,...,$$

And $\tilde{j}_k, j_k, k=1,2,....,$ and the positive zeros of $J_1'(x), J_{\frac{1}{3}(x)}$, respectively. In this paper we obtain the infinite product representation of solution of (1) in a case where the weight function has one zero that it is admitted to be pole of the function q.

2notations and Preliminary Results:

Let C(x,λ) be the solution of (1) corresponding to the initial conditions C(0,λ)=1, C′(0,λ)=0. In order to represent the solution C(x,λ) as an infinite product we use a suitable fundamental system of solutions (FSS) for Eq.(1) as constructed in [4]. Introducing some terminology at this point we write

1) $I_{1,\varepsilon}=[0,x_1-\varepsilon]\cup[x_1-\varepsilon,x_1+\varepsilon]\cup[x_1+\varepsilon,1],$

$$\mu_1=\frac{1}{2+l_1}\in(0,1],\eta_1:=\mu_1(1+4A_1)^{\frac{1}{2}} \quad \text{with} \quad \arg\eta_1\in(-\frac{\pi}{2},\frac{\pi}{2}],$$

and

[1]: = 1 + $O(\rho^{-1})$.

2) $\varphi^2(x)$ is real and has in (0,1) one zero x_1, of order l_1 where l_1 is odd. In the terminology of [4], x_1 is of type IV. The functions

$$\phi_0:I_{1,\varepsilon}\to R,\qquad \phi_0(x):=(x-x_1)^{-l_1}\phi^2(x),$$

are non-vanishing and real-analytic; denote

$$k_0:=\phi_0(x_1).$$

3) q has the form

$$q(x)=A_1(x-x_1)^{-2}+B_1(x-x_1)^{-1}+C_1(x),\qquad (x\in I_{1,\varepsilon},x\neq x_1),$$

With constants A_1, B_1 and a bounded real-analytic function C_1.
4) For $x \in [0,1]$ let

$$R_{\pm}(x) := \int_0^x |\phi_{\pm}(t)|\,dt \qquad with \quad \phi_{\pm}^2(t) := \max\{0, \pm\phi^2(t)\}$$

According to the type of x_1 we know from [4, Theorem 2,6] that in the sector

$$S_{-1} = \{\rho \mid \arg\rho \in [-\frac{\pi}{4}, 0]\},$$

there exist a fundamental system of solutions of (1) $\{Z_1^{IV}(x,\rho), Z_2^{IV}(x,\rho)\}$, and such that

$$Z_{\upsilon_1}^{IV_N}(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} e^{\rho\int_{x_1}^x |\phi(t)|dt}[1], & 0 \le x < x_1, \\ |\phi(x)|^{-\frac{1}{2}} e^{\frac{-i\pi}{4}} \left\{ \dfrac{\sin\frac{1}{2}(l_1+3)\eta_1\pi}{\sin\eta_1\pi} e^{i\rho\int_{x_1}^x |\phi(t)|dt}[1] + i\dfrac{\sin\frac{1}{2}(l_1+1)\eta_1\pi}{\sin\eta_1\pi} e^{-i\rho\int_{x_1}^x |\phi(t)|dt}[1] \right\}, \end{cases}$$

$$x_1 < x < 1,$$

$$Z_2^{IV_N}(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} e^{-\rho\int_{x_1}^x |\phi(t)|dt}[1], & 0 \le x < x_1, \\ |\phi(x)|^{-\frac{1}{2}} \dfrac{\sin\eta_1\pi}{\sin\frac{1}{2}(l_1+3)\eta_1\pi} e^{-i\rho\int_{x_1}^x |\phi(t)|dt - i\frac{\pi}{4}}[1], & x_1 < x < 1, \end{cases} \tag{5}$$

$$W(\rho) = -2\rho[1].$$

In the sequel we also need $\{Z_1^{IV_N}(x_1,\rho), Z_2^{IV_N}(x_1,\rho)\}$.
From [4] we have

$$Z_1^{IV_N}(x_1,\rho) = e^{-\frac{i\pi}{4}} 2^{-\frac{\mu_1}{2}} \rho^{\frac{1-\mu_1}{2}} k_0^{-\frac{\mu_1}{4}} \left[\frac{\sin\frac{1}{2}(l_1+3)\eta_1\pi}{\sin\eta_1\pi} V_1 + \frac{\sin\frac{1}{2}(l_1+1)\eta_1\pi}{\sin\eta_1\pi} V_2 \right],$$

where

$$V_1 = -\frac{\sqrt{2\pi}}{2} \kappa_0^{\frac{\mu_1}{4}} e^{(\eta_1+\frac{1}{2})\frac{\pi i}{2}} 2^{\frac{\mu_1}{2}} \psi_1(x_1) \xi_1^{\frac{\mu_1}{2}-\eta_1} i \csc\eta_1\pi \frac{2^{\eta}}{\Gamma(1-\eta_1)} + \psi_1(x_1)\xi_1^{\frac{\mu_1}{2}-\eta_1} \frac{O(1)}{\rho^{2\mu_1}},$$

and

$$V_2 = \frac{\sqrt{2\pi}}{2} \kappa_0^{\frac{\mu_1}{4}} e^{-(\eta_1+\frac{1}{2})\frac{\pi i}{2}} 2^{\frac{\mu_1}{2}} \psi_1(x_1) \xi_1^{\frac{\mu_1}{2}-\eta_1} i \csc\eta_1\pi \frac{2^{\eta_1}}{\Gamma(1-\eta_1)} + \psi_1(x_1)\xi_1^{\frac{\mu_1}{2}-\eta_1} \frac{O(1)}{\rho^{2\mu_1}}.$$

Consequently

$$Z_1^{IV_N}(x_1,\rho) = -\frac{\sqrt{2\pi}}{2} \rho^{\frac{1-\mu_1}{2}} \psi_1(x_1) \xi_1^{\frac{\mu_1}{2}-\eta_1} \csc\eta_1\pi \frac{2^{\eta_1}}{\Gamma(1-\eta_1)}$$

$$\times \left\{ ie^{\frac{i\pi\eta_1}{2}} \frac{\sin\frac{1}{2}(l_1+3)\eta_1\pi}{\sin\eta_1\pi} - i e^{\frac{-i\pi\eta_1}{2}} \frac{\sin\frac{1}{2}(l_1+1)\eta_1\pi}{\sin\eta_1\pi} \right\} [1]. \tag{6}$$

Similarly

$$Z_2^{IV_N}(x_1,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1-\mu_1}{2}}\psi_1(x_1)\xi_1^{\frac{\mu_1}{2}-\eta_1}e^{\frac{-i\eta_1\pi}{2}}\frac{1}{\sin\frac{1}{2}(l_1+3)\eta_1\pi}\times\frac{2^{\eta_1}}{\Gamma(1-\eta_1)}[1]. \tag{7}$$

3asymptotic Form of the Solution:

We consider the differential equation (1) with the following conditions $C(0,\lambda)=1$, $C'(0,\lambda)=0$.

Applying the $\{Z_1^{IV_N}(x_1,\rho), Z_2^{IV_N}(x_1,\rho)\}\quad for\ \ x\in I_{1,\varepsilon}$, where x_1 is of type IV. We have

$C(x,\rho)=C_1Z_1(x,\rho)+C_2Z_2(x,\rho)$.

That using of Cramer's rule leads to the equation

$$C(x,\rho)=\frac{1}{-2\rho}(Z_2'(0,\rho)Z_1(x,\rho)-Z_1'(0,\rho)Z_2(x,\rho)). \tag{8}$$

Taking (4)-(5) into account we derive

$$C(x,\rho)=\begin{cases}\frac{1}{2}|\phi(x)|^{\frac{-1}{2}}|\phi(0)|^{\frac{1}{2}}\left(e^{\rho\int_0^x|\phi(t)|dt}[1]-e^{-\rho\int_0^x|\phi(t)|dt}[1]\right), & 0\le x<x_1.\\ \\ \frac{1}{2}|\phi(x)|^{\frac{-1}{2}}|\phi(0)|^{\frac{1}{2}}\left(M_1(\rho)e^{i\rho\int_{x_1}^x|\phi(t)|dt}[1]+M_2(\rho)e^{-i\rho\int_{x_1}^x|\phi(t)|dt}[1]\right), & x_1<x\le 1,\end{cases} \tag{9}$$

where

$$M_1(\rho)=\frac{\sin\frac{1}{2}(l_1+3)\eta_1\pi}{\sin\eta_1\pi}e^{\rho\int_0^{x_1}|\phi(t)|dt-i\frac{\pi}{4}}$$

$$M_2(\rho)=i\frac{\sin\frac{1}{2}(l_1+1)\eta_1\pi}{\sin\eta_1\pi}e^{\rho\int_0^{x_1}|\phi(t)|dt-i\frac{\pi}{4}}+\frac{\sin\eta_1\pi}{\sin\frac{1}{2}(l_1+3)\eta_1\pi}e^{-\rho\int_0^{x_1}|\phi(t)|dt-i\frac{\pi}{4}}. \tag{10}$$

By virtue of (9) and (10), the following estimates are also valid:

$$C(x,\rho)=\begin{cases}\frac{1}{2}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}e^{\rho\int_0^x|\phi(t)|dt}[[1]], & 0\le x<x_1,\\ \\ \frac{1}{2}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}\frac{\sin\frac{1}{2}(l_1+3)\eta_1\pi}{\sin\eta_1\pi}e^{\rho\int_0^{x_1}|\phi(t)|dt+i\rho\int_{x_1}^x|\phi(t)|dt-i\frac{\pi}{4}}[[1]], & x_1<x<1,\end{cases} \tag{11}$$

Similarly, using of (6), (7) and (8) for $x=x_1$ we find that

$$C(x_1,\rho)=-\frac{1}{2\rho}\Bigg(i\rho|\phi(0)|^{\frac{1}{2}}e^{\rho\int_0^{x_1}|\phi(t)|dt}\frac{\sqrt{2\pi}}{2}\rho^{\frac{1-\mu_1}{2}}\psi_1(x_1)\xi_1^{\frac{\mu_1}{2}-\eta_1}\csc\eta_1\pi\, e^{i(\frac{l_1+2}{2})\eta_1\pi}\frac{2^{\eta_1}}{\Gamma(1-\eta_1)}$$

$$-\rho|\phi(0)|^{\frac{1}{2}}e^{-\rho\int_0^x|\phi(t)|dt}\frac{\sqrt{2\pi}}{2}\rho^{\frac{1-\mu_1}{2}}\psi_1(x_1)\xi_1^{\frac{\mu_1}{2}-\eta_1}\csc\eta_1\pi\frac{2^{\eta_1}}{\Gamma(1-\eta_1)}\times\frac{\sin\eta_1\pi}{\sin\frac{1}{2}(l_1+3)\eta_1\pi}\Bigg)$$

$$= i\frac{|\phi(0)|^{\frac{1}{2}}\sqrt{2\pi}\,\rho^{\frac{1-\mu_1}{2}}\psi_1(x_1)\xi_1^{\frac{\mu_1}{2}-\eta_1}\csc\eta_1\pi e^{i(\frac{l_1+1}{2})\eta_1\pi}2^{\eta_1}}{4\Gamma(1-\eta_1)}e^{\rho\int_0^{x_1}|\phi(t)|dt}[[1]]. \quad (12)$$

4distribution of the Eigenvalue:

We consider the boundary value problem $L_1 = L_1(\phi^2(x), q(x), b)$ for Eq.(1) with boundary conditions $y(0,\lambda)=1$, $y'(0,\lambda)=0$, $y(b,\lambda)=0$.

The boundary value problem L_1 for $b \in (0, x_1)$ has a countable set of negative eigenvalues $\{\lambda_n^-(b)\}_{n\geq 0}$. From (Tamarkin, 1927) we have

$$\sqrt{-\lambda_n^-(b)} = \frac{n\pi}{\int_0^b |\phi(t)|\,dt} + O(\frac{1}{n}), \quad (13)$$

and for x = x_1 similarly from (12) we have

$$\sqrt{-\lambda_n^-(x_1)} = \frac{n\pi + (\frac{\pi\eta_1}{2} - \frac{\pi}{4})}{\int_0^{x_1} |\phi(t)|\,dt} + O(\frac{1}{n}). \quad (14)$$

The spectrum $\{l_n\}$ of boundary value problem L_1 for $x_1 < b$, consist of two sequences of negative and positive eigenvalues: $\{\lambda_n(b)\} = \{\lambda_n^-(b)\} \cup \{\lambda_n^+(b)\}$, $n \in N$, such that

$$\sqrt{\lambda_n^+(b)} = \frac{n\pi - \frac{\pi}{4}}{\int_{x_1}^b |\phi(t)dt} + O(\frac{1}{n}), \quad (15)$$

$$\sqrt{-\lambda_n^-(b)} = \frac{n\pi - \frac{\pi}{4}}{\int_0^{x_1} |\phi(t)dt} + O(\frac{1}{n}). \quad (16)$$

5Main results:

Since the solution C(x,r) of the Sturm-Liouville equation defined by a fixed set of initial conditions is an entire function of r for each fixed $x \in [0,1]$, thus it follows from the classical Hadamard's factorization theorem that such solution is expressible as an infinite product.

For fixed $b \in (0, x_1)$by Halvorsen's result (Halvorsen, 1987), C(b, ρ) is an entire function of order $\frac{1}{2}$. Therefore we can use Hadamard's theorem to represent the solution in the form where h(b) is a function independent of l but may depend on b and the infinite number of negative eigenvalues, $\{\lambda_n(b)\}_{n=1}^{\infty}$ form the zero set of C(b, l) for each b. Since $C(b, \lambda_n(b)) = 0$, these l_n(b) correspond to eigenvalues of the boundary value problem L_1 on the closed interval [0,b], 0 < b < x_1. We rewrite the infinite product as

$$C(b,\lambda) = h(b)\prod(1 - \frac{\lambda}{\lambda_n(b)}) = h_1(b)\prod(\frac{\lambda - \lambda_n(b)}{z_n^2}) \quad (18)$$

with

$$h_1(b) := h(b)\prod \frac{-z_n^2}{\lambda_n(b)} ,$$

where

$$z_n = \frac{n\pi}{R-(b)} \quad .$$

Now (13) implies that $\frac{-z_m^2}{\lambda_m(b)} = 1 + O(\frac{1}{m^2})$. It follows from the results of [6] that the infinite product $\prod \frac{-z_m^2}{\lambda_m(b)}$ is absolutely convergent on any compact subinterval of $(0, x_1)$. The function $\frac{-z_m^2}{\lambda_m(b)}$ is continuous and so the O-term is uniformly bounded in b.

Theorem 1. Let C(x, λ) be the solution of (1) satisfying the initial conditions C(0, λ) = 1, C″(0, λ) = 0. Then for 0 < x < x_1,

$$C(x,\lambda) = |\phi(x)|^{-\frac{1}{2}} |\phi(0)|^{\frac{1}{2}} R_{\_}(x) \prod_{m\geq 1} \frac{\lambda - \lambda_m(x)}{z_m^2} \tag{19}$$

where the sequence $\lambda_m(x)$, $m \geq 1$, represents the sequence of negative eigenvalues of the boundary value problem L_1 on [0,x].**proof.** Let $\{\lambda_n(b)\}$ be the eigenvalues of the boundary value problem L_1 on [0,b], for fixed b = x, 0 < x < x_1 then according to [10] we have

$$\prod (\frac{\lambda - \lambda_m(x)}{z_m^2}) = \frac{\sinh R_{\_}(x)\sqrt{\lambda}}{R_{\_}(x)\sqrt{\lambda}} \left(1 + O(\frac{\log n}{n})\right). \tag{20}$$

Thus from (11), (18), we obtain $h_1(x) = \dfrac{C(x,\lambda)}{\prod (\frac{\lambda - \lambda_m(x)}{z_m^2})} = |\phi(x)|^{-\frac{1}{2}} |\phi(0)|^{\frac{1}{2}} R_{\_}(x).$

Similarly for b = x_1 again by Hadamard's theorem we that

$$C(x_1,\lambda) = A\prod (1 - \frac{\lambda}{\lambda_n(x_1)}) \tag{21}$$

where A is constant. Let j_n, n = 1,2,… be the sequence of positive zeros of the Bessel function of order h_1, then (see [1])

$$\frac{-j_n^2}{R_{\_}^2(x_1)\lambda_n(x_1)} = 1 + O(\frac{1}{n^2}),$$

So the infinite product

$$\prod \frac{-j_n^2}{R_{\_}^2(x_1)\lambda_n(x_1)}$$

are absolutely convergent. Consequently we may write as before,

$$C(x_1,\lambda) = A_1 \prod \frac{(\lambda - \lambda_n(x_1))R_{\_}^2(x_1)}{j_n^2}, \tag{22}$$

where

$$A_1 = A\prod \frac{-j_n^2}{R_-^2(x_1)\lambda_n(x_1)}.$$

Theorem 2. For b = x_1,

$$c(x_1,\lambda) = \frac{|\phi(0)|^{\frac{1}{2}} V(x_1) R_-(x_1)^{\frac{1}{2}+\eta_1} e^{i(\frac{l_1+1}{2})\eta_1\pi}}{2\mu_1} \times \prod_{n\geq 1} \frac{(\lambda-\lambda_n(x_1))R_-^2(x_1)}{j_n^2}$$

where $V(x_1) = \lim_{x\to x_1} \phi^{-\frac{1}{2}}(x)\left\{\int_{x_1}^{x} |\phi(t)|\,dt\right\}^{\frac{1-\mu_1}{2}-\eta_1}$.

proof. According to (Jodayree and Mingarelli, 2000) the infinite product

$$\prod_{m\geq 1} \frac{(\lambda-\lambda_m(x_1))R_-^2(x_1)}{j_m^2}$$

is an entire function of l, whose roots are precisely $\lambda_m(x_1)$, $m \geq 1$. Moreover

$$\prod_{m\geq 1} \frac{(\lambda-\lambda_m(x_1))R_-^2(x_1)}{j_m^2} = 2^{\mu_1}\Gamma(1-\mu_1)[i\sqrt{\lambda}R_-(x_1)]^{-\mu_1} J_{\eta_1}(i\sqrt{\lambda}R_-(x_1))(1+O(\frac{\log n}{n}))$$

Uniformly on the Circles $|\lambda| = \frac{n^2\pi^2}{R_-^2(x_1)}$.

Thus it follows from (12),

$$A_1 = \frac{C(x_1,\lambda)}{\prod \frac{(\lambda-\lambda_m(x_1))R_-^2(x_1)}{j_m^2}} = \frac{|\phi(0)|^{\frac{1}{2}} V(x_1) R_-(x_1)^{\frac{1}{2}+\eta_1} e^{i(\frac{l_1+1}{2})\eta_1\pi}}{2\mu_1}.$$

Similarly for b = x, x_1 < x < 1, the boundary value problem L_1 on [0,x] has an infinite number of positive and negative eigenvalues which are denoted by $\{\lambda_n(x)\} = \{\lambda_n^+(x)\} \cup \{\lambda_n^-(x)\}$, respectively.

By Hadamard's theorem, the solution on [0,x], x_1 < x < 1 is of the form

$$C(x,\lambda) = g(x)\prod_{n\geq 1}(1-\frac{\lambda}{\lambda_n^-(x)})(1-\frac{\lambda}{\lambda_n^+(x)}). \tag{24}$$

Let $\tilde{j}_n, n = 1,2,...,$ be the positive zeros of $J_1'(z)$, derivative of the Bessel function of order one. The distribution of $\tilde{j}_n$ is of the form

$$\tilde{j}_n = m^2\pi^2 - \frac{m\pi^2}{2} + O(1), \tag{25}$$

(see [1]). Consequently, we have

$$\frac{\tilde{j}_n}{R_+^2(x)\lambda_n^+(x)} = 1 + O(\frac{1}{n^2}),$$

$$\frac{-\tilde{j}_n}{R_-^2(x)\lambda_n^-(x)} = 1 + O(\frac{1}{n^2}). \tag{26}$$

Consequently, the infinite products

$$\prod \frac{\tilde{j}_n}{R_+^2(x)\lambda_n^+(x)}$$

and

$$\prod \frac{-\tilde{j}_n}{R_-^2(x)\lambda_n^-(x)} \tag{27}$$

are absolutely convergent for each $x \in (x_1, 1)$. Therefore we may write

$$C(x,\lambda) = g_1(x)\prod \frac{(\lambda - \lambda_n^-(x))R_-^2(x_1)}{\tilde{j}_n^2}\prod \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2} \tag{28}$$

with

$$g_1(x) = g(x)\prod \frac{\tilde{j}_n}{R_+^2(x)\lambda_n^+(x)}\prod \frac{-\tilde{j}_n}{R_-^2(x)\lambda_n^-(x)}.$$

Theorem 3. For $x_1 < x < 1$,

$$C(x,\lambda) = \frac{\pi}{8}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}(R_-(x)R_+(x))^{\frac{1}{2}}$$

$$\frac{\sin\frac{1}{2}(l_1+3)\eta_1\pi}{\sin\eta_1\pi} \times \prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x))R_-^2(x_1)}{\tilde{j}_n^2}\prod_{n\geq 1} \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2}. \tag{29}$$

proof:

From Lemmas 2 and 3 of [10] the infinite products

$$\prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x_1))R_-^2(x_1)}{\tilde{j}_n^2}\prod_{n\geq 1} \frac{(\lambda_n^+(x_1) - \lambda)R_+^2(x)}{\tilde{j}_n^2}.$$

are entire functions of l for fixed x, those roots are precisely $\lambda_n^-(x)$and $\lambda_n^+(x)$, n ³ 1, respectively. Moreover

$$\prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x))R_-^2(x_1)}{\tilde{j}_n^2}\prod_{n\geq 1} \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2}$$

$$= \frac{4e^{R_-(x)\sqrt{\lambda}}}{\pi R_-^{\frac{1}{2}}(x)R_+^{\frac{1}{2}}(x)\sqrt{\lambda}}\left\{\cos(R_+(x)\sqrt{\lambda} - \frac{\pi}{4}) + O(\frac{1}{\sqrt{\lambda}})\right\},$$

as $\lambda \to \infty$. Thus by using of the asymptotic expansion of C(x,l) in [11] we get

$$g_1(x) = \frac{C(x,\lambda)}{\prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x))R_-^2(x_1)}{\tilde{j}_n^2}\prod_{n\geq 1} \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2}}$$

$$= \frac{\pi}{8}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}(R_-(x)R_+(x))^{\frac{1}{2}}\frac{\sin\frac{1}{2}(l_1+3)\eta_1\pi}{\sin\eta_1\pi}.$$

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