

## Non-fourier Heat Conduction in a Long Cylindrical Media with Insulated Boundaries and Arbitrary Initial Conditions

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**Abstract:** The non-Fourier transient heat conduction in a long cylindrical media with insulated boundaries under the influence of arbitrary initial conditions is investigated analytically. The solution is expressed in the form of Bessel series. The obtained analytical closed-form solution can be used for validation of numerical solutions. The resulting solution is valid for any choice of integrable initial conditions. The significant feature of proposed solution is its generality. In order to show the applicability of the presented solution, two numerical examples are solved.

**Key words:** Non-Fourier conduction, Hyperbolic conduction, Insulated boundaries, Cylindrical media, Arbitrary initial conditions

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### INTRODUCTION

The Fourier law, a parabolic heat-conduction model, which states the relation between heat flux and temperature gradient, is appropriate for many practical heat conduction conditions (Carslaw, H.S., and J.C. Jaeger, 1959; Arpaci, V.S., 1966).

While the heat-transfer situations include extremely high temperature gradients, temperatures near absolute zero, extremely large heat fluxes, and extremely short transient duration, the heat propagation speed is finite, and the Fourier law should be modified. The modified models include the thermal wave model, the phase-lag concept, and the dual-phase-lag (DPL) (Bertman, B., and D.J. Sandiford, May 1970; Qiu, T.Q., T. Juhasz, 1994; Kim, W.S., L.G. Hector Jr, 1990; Al-Nimr, M.A., and M.A., Hader, 2001; Chen, J.K., and J.E. Beraun, 2001; Tzou, D.Y., and K.S. Chiu, 2001; Tzou, D.Y., 2002).

The propagation speed of thermal wave in many homogenous and nonhomogenous materials is finite. The classical thermal wave theory indicates that there is a time delay, the relaxation time, between the heat flux vector and the temperature gradient across a material. The values of relaxation time for homogenous and nonhomogeneous substances are in the order of  $10^{-8}$  to  $10^{-12}$  and  $10^{-3}$  to  $10^3$  s, respectively (Kaminski, W., 1990; Mitra *et al.* 1995). Kaminski (1990) considered the physical meaning of relaxation time in the hyperbolic-heat conduction equation for materials with a nonhomogenous inner structure, and discussed the differences of penetration time, heat flux, and temperature profiles between the hyperbolic heat-conduction equation (HHCE) and parabolic heat-conduction equation (PHCE) from experimental and calculated results. Mitra *et al.* (1990) presented the experimental evidence of the wave nature of heat propagation and demonstrated that the hyperbolic heat-conduction model is accurate.

The classical theory of heat conduction is based on the un-physical property that heat propagates at an infinite speed. On such basis, the constitutive equation governing heat flow is given by Fourier's law.

$$q = -\lambda \nabla \theta \quad (1)$$

where the material constant  $\lambda$  is the thermal conductivity. When this relation is incorporated in the local energy balance equation

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$$\nabla \cdot q = -\rho c \frac{\partial \theta}{\partial t} \tag{2}$$

the classical parabolic heat conduction equation

$$\frac{\partial \theta}{\partial t} = a \Delta \theta \tag{3}$$

where  $a = \lambda / \rho c$  and  $\Delta$  are thermal diffusivity, mass density, specific heat capacity and Laplace's differential operator, respectively. Eq. (2) yields temperature solutions which imply an infinite speed of heat propagation.

A modified non-Fourier heat flux equation has been developed by several different approaches (Weymann, H.D., 1967; Gurtin, M.E. and A.C. Pipkin, 1968; Maurer, M.J., 1969; Taitel, Y., 1972; Luikov, A.V., V.A. Bubnov, 1976; Frankel, J.I., B. Vick, 1985; Barletta, A. and E. Zanchini, 1997; Kronberg, A.E., A.H. Benneker, 1998). When heat waves are important Fourier's conduction law, which connects the heat flux  $q$  to the temperature, must be modified by adding an extra thermal inertia term. Vernotte (1958) and Cattaneo (1958) independently proposed a different constitutive equation for conduction heat transfer in the form

$$q + \tau \frac{\partial q}{\partial t} = -\lambda \nabla \theta \tag{4}$$

where  $\tau$  is the so-called relaxation time (a non-negative constant). When Eq. (4) is used in conjunction with the local energy balance Eq. (2), a general hyperbolic equation governing the non-Fourier heat conduction results and can be written in the form

$$\frac{\partial \theta}{\partial t} + \tau \frac{\partial^2 \theta}{\partial t^2} + Q(x,t) = a \Delta \theta \tag{5}$$

where  $Q$  is the source term and  $\sqrt{a/\tau}$  denotes the propagation speed of temperature wave.

Various cases of hyperbolic heat conduction for finite and semi-infinite media under different initial and boundary conditions were studied analytically and numerically (Ozisik, M.N., and B. Vick, 1984; Gembarovic, J. and V. Majernik, 1988; Kar, A., C.L. Chan, 1992; Hector Jr, L.G., W.S. Kim, 1992; Tang, D.W. and N. Araki, 1996; Glass, D.E., M.N. Ozisik, 1985; Chen, H.T. and J.Y. Lin, 1993). Recently, Moosaie (2008) solved analytically the case of a finite medium with insulated boundaries under the influence of arbitrary initial conditions. Also, Moosaie solved the problems of hyperbolic heat conduction in a finite medium subjected to arbitrary periodic surface disturbance and arbitrary non-periodic surface disturbance (Moosaie, A., 2007; Moosaie, A., 2008).

The present work considers the (1+1)-dimensional non-Fourier conduction equation in a long cylindrical media with insulated boundaries and arbitrary initial conditions. The analytical solution of the heat conduction equation is derived by using separation of variables method. Finally, the solutions for the two special choices of initial conditions are presented as two examples that demonstrate the applicability of the proposed solution procedure.

**Analysis:**

Let us consider a long circular cylinder of radius  $R$  composed of an isotropic heat conducting material with insulated boundaries in which one-dimensional heat conduction and constant thermal properties prevail. The governing Eq. (5) in this case and with the assumption of vanishing source term reduces to

$$\frac{\partial \theta}{\partial t} + \tau \frac{\partial^2 \theta}{\partial t^2} = a \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right) \tag{6}$$

In this study, we are interested in the non-Fourier heat conduction in a long cylindrical media with insulated boundary conditions and arbitrary initial conditions. The boundary and initial conditions for such a problem are as follows

$$\frac{\partial}{\partial r} \theta(0,t) = 0, \quad \frac{\partial}{\partial r} \theta(R,t) = 0 \tag{7a}$$

$$\theta(r,0) = F(r), \quad \frac{\partial}{\partial t} \theta(r,0) = G(r) \tag{7b}$$

where  $F$  and  $G$  are arbitrary functions of the spatial variable  $r$ . Separation of variables in Eqs. (6) and (7) leads to the following eigenvalue problem

$$\frac{d^2 \phi}{d r^2} + \frac{1}{r} \frac{d \phi}{d r} + \lambda_n^2 \phi = 0 \tag{8a}$$

$$\frac{d}{d r} \phi(0) = 0, \quad \frac{d}{d r} \phi(R) = 0 \tag{8b}$$

for which the eigenfunctions and eigenvalues are given by

$$\phi_n(r) = J_0(\lambda_n r), \quad J_0'(\lambda_n R) = 0, \quad n = 0, 1, 2, \dots \tag{9a}$$

$J_0$  is the Bessel function of the first kind of order 0. The eigenfunctions form an orthogonal set in terms of which we can express the solution of the original problem as (Arpaci, V.S., 1966).

$$\theta(r,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) J_0(\lambda_n r) \tag{10}$$

By substituting this solution into Eq. (6) we come up to the following ordinary differential equation for  $a_0(t)$

$$\tau \ddot{a}_0(t) + \dot{a}_0(t) = 0 \tag{11a}$$

which has the following general solution

$$a_0(t) = A_0 + B_0 e^{-t/\tau} \tag{11b}$$

Moreover, the unknown functions  $a_0(t)$  must fulfill the following equation

$$\tau \ddot{a}_n(t) + \dot{a}_n(t) + k_n^2 a_n(t) = 0 \tag{12a}$$

where

$$k_n^2 = a \lambda_n^2 \tag{12b}$$

The general solution to this homogeneous ordinary differential equation is given by

$$a_n = e^{\frac{-t}{2\tau}} (A_n \cos \alpha_n t + B_n \sin \alpha_n t) \tag{12c}$$

where

$$\alpha_n = \frac{\sqrt{4\tau k_n^2 - 1}}{2\tau} \tag{12d}$$

Therefore, the temperature distribution becomes

$$\theta(r, t) = A_0 + B_0 e^{-t/\tau} + \sum_{n=1}^{\infty} e^{\frac{-t}{2\tau}} (A_n \cos \alpha_n t + B_n \sin \alpha_n t) J_0(\lambda_n r) \tag{13}$$

The integration constants  $A_n$  and  $B_n$  ( $n = 0, 1, \dots$ ) can be calculated by enforcing initial conditions (7b).

$$\theta(r, 0) = A_0 + B_0 + \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = F(r) \tag{14}$$

Utilizing the orthogonality of Bessel functions, one deduces that

$$A_0 + B_0 = \frac{2}{R^2} \int_0^R r F(r) dr \tag{15a}$$

$$A_n = \frac{\int_0^R r F(r) J_0(\lambda_n r) dr}{\int_0^R r J_0^2(\lambda_n r) dr} \tag{15b}$$

The initial rate of change of temperature implies that

$$\frac{\partial}{\partial t} \theta(r, 0) = -\frac{1}{\tau} B_0 + \sum_{n=1}^{\infty} \left( \frac{-1}{2\tau} A_n + B_n \alpha_n \right) J_0(\lambda_n r) = G(r) \tag{16}$$

which yields

$$B_0 = \frac{-2\tau}{R^2} \int_0^R r G(r) dr \tag{17a}$$

$$B_n = \frac{1}{\alpha_n} \left\{ \frac{\int_0^R r G(r) J_0(\lambda_n r) dr}{\int_0^R r J_0^2(\lambda_n r) dr} + \frac{1}{2\tau} A_n \right\} \tag{17b}$$

Eqs. (15a) and (17a) give the following explicit relation for  $A_0$ .

$$A_0 = \frac{2}{R^2} \int_0^R r (F(r) + \tau G(r)) dr \tag{18}$$

**Numerical Examples:**

In order to show how the presented solution works, the obtained general solution is reduced to two special cases. For the first example, the initial rate of temperature change is defined as

$$G(r) = \frac{\theta_0}{\tau} \left( 1 - \left( \frac{r}{R} \right)^2 \right) \tag{19}$$

The initial temperature distribution is assumed to be zero, i.e.

$$F(r) = 0 \tag{20}$$

The temperature distribution can be readily obtained by computing coefficients  $A_n$  and  $B_n$  via Eqs. (15b), (17a), (17b) and (18). Furthermore, we introduce the following dimensionless quantities.

$$X = \frac{r}{R}, \quad \bar{\theta} = \frac{\theta}{\theta_0}, \quad Fo = \frac{at}{R^2}, \quad Ve = \frac{a\tau}{R^2} \tag{21}$$

The dimensionless temperature at  $X=0$ ,  $X=0.5$  and  $X=1$  as a function of Fourier number  $Fo$  for different Vernotte numbers  $Ve^2$  is plotted in Figures (1-12). Also, the dimensionless temperature at  $Ve^2 = 0.1$  as a function of  $X$  for different Fourier numbers  $Fo$  is plotted in Figures (13-18).

As another numerical example, we consider the initial temperature in the form of a function given by

$$F(r) = \theta_0 \left( 1 - \left( \frac{r}{R} \right)^2 \right) \tag{22}$$

The initial rate of temperature change is assumed to be zero.

$$G(r) = 0 \tag{23}$$

The temperature distribution due to this choice of initial conditions can be calculated by introducing these initial conditions to Eqs. (15b), (17a), (17b) and (18). The obtained results are presented in Figures (19-32).

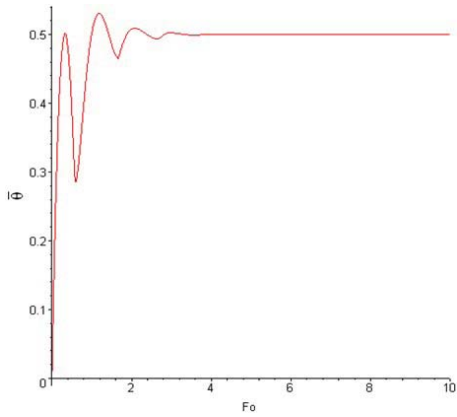


Fig. 1: Dimensionless Temperature at  $X=0, Ve^2 = 0.1$

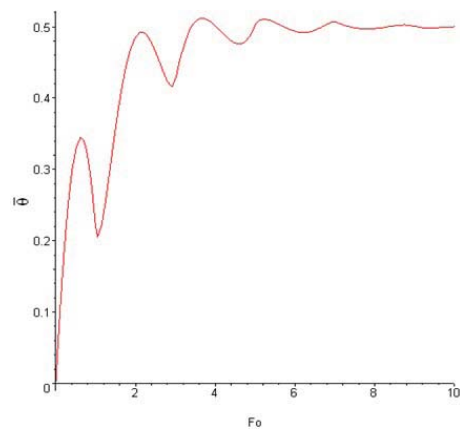


Fig. 2: Dimensionless Temperature at  $X=0, Ve^2 = 1$

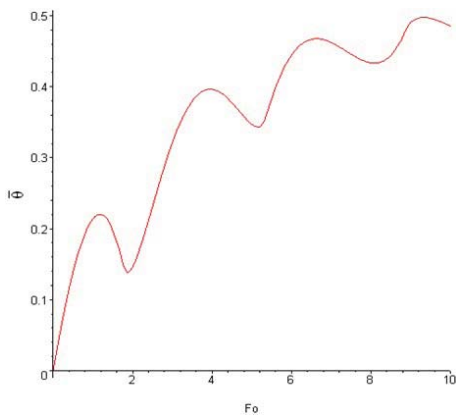


Fig. 3: Dimensionless Temperature at  $X=0, Ve^2 = 10$

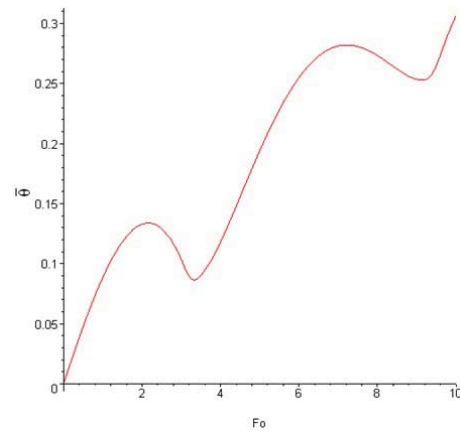


Fig. 4: Dimensionless Temperature at  $X=0, Ve^2 = 100$

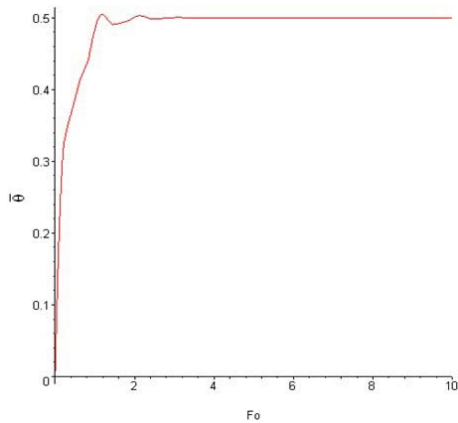


Fig. 5: Dimensionless Temperature at  $X=0.5, Ve^2 = 0.1$

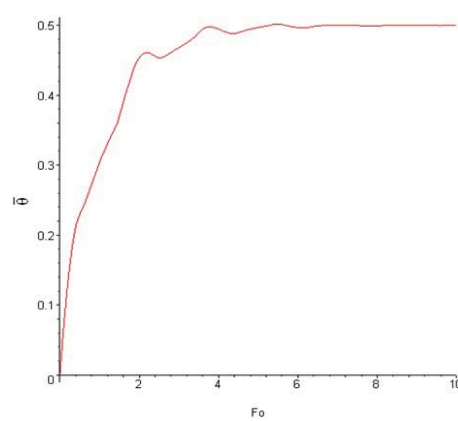


Fig. 6: Dimensionless Temperature at  $X=0.5, Ve^2 = 1$

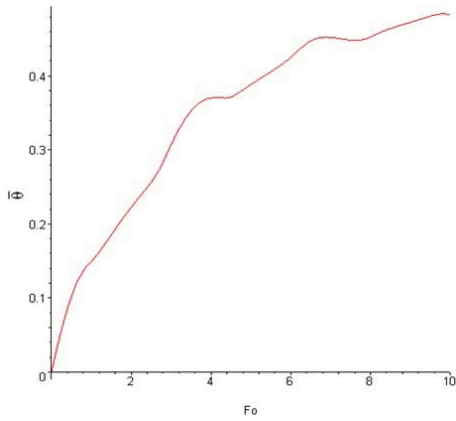


Fig. 7: Dimensionless Temperature at  $X=0.5, Ve^2=10$

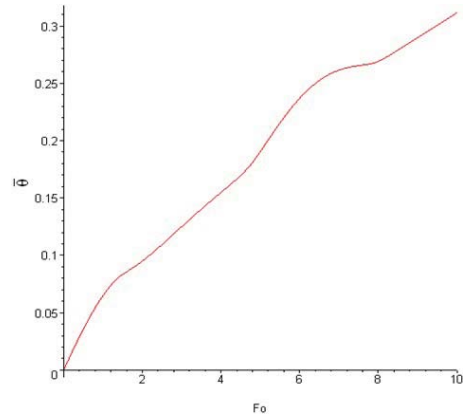


Fig. 8: Dimensionless Temperature at  $X=0.5, Ve^2=100$

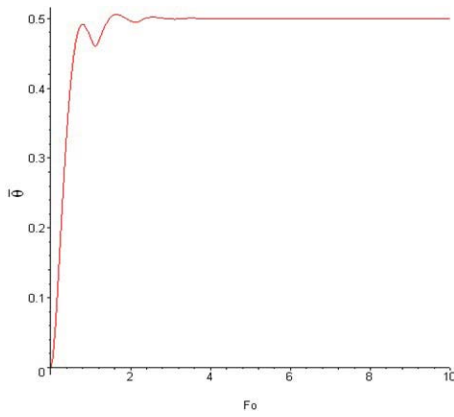


Fig. 9: Dimensionless Temperature at  $X=1, Ve^2=0.1$

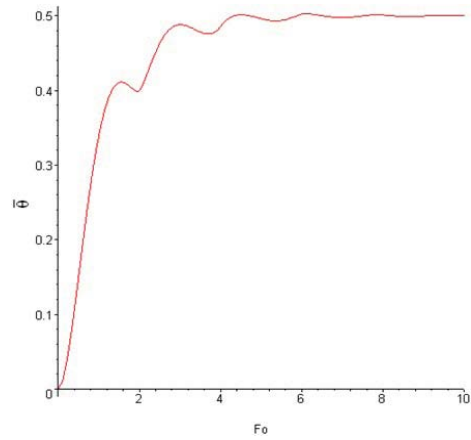


Fig. 10: Dimensionless Temperature at  $X=1, Ve^2=1$

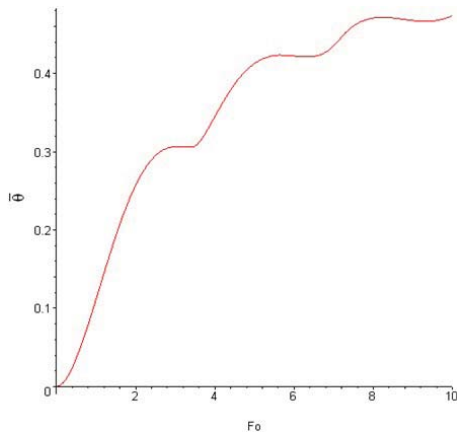


Fig. 11: Dimensionless Temperature at  $X=1, Ve^2=10$

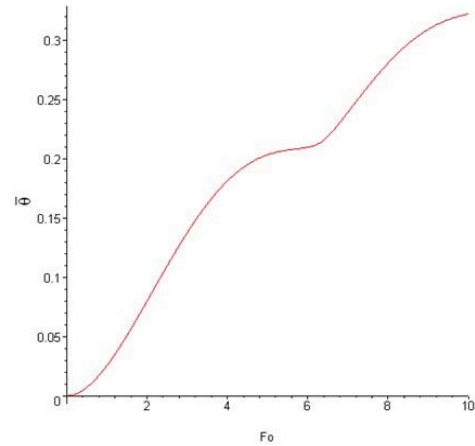
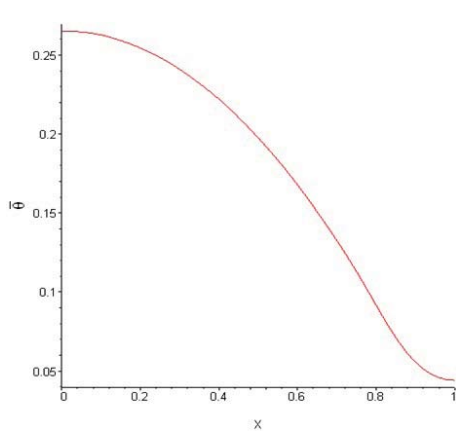
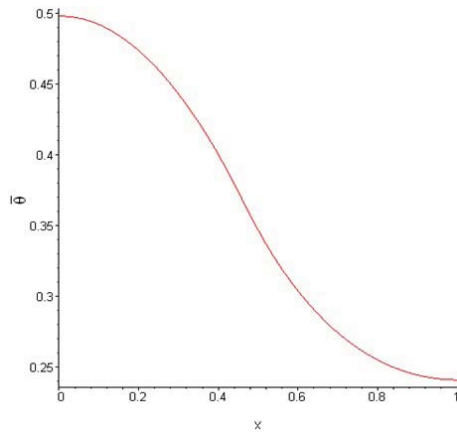


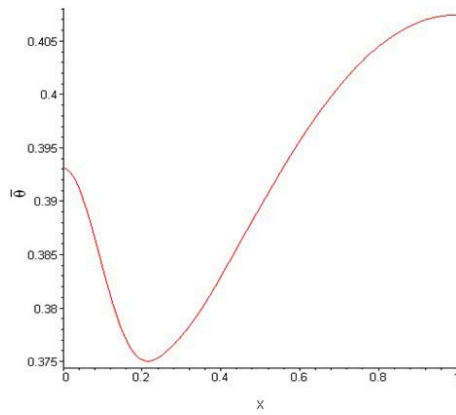
Fig. 12: Dimensionless Temperature at  $X=1, Ve^2=100$



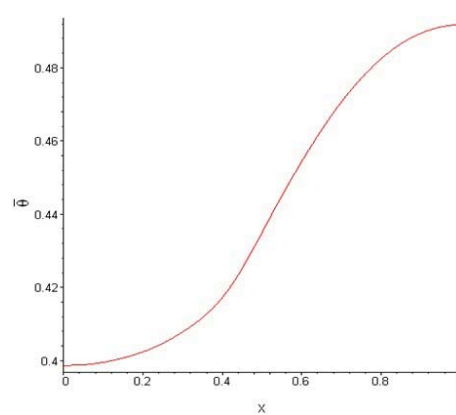
**Fig. 13:** Dimensionless Temperature at  $Fo=0.1, Ve^2=0.1$



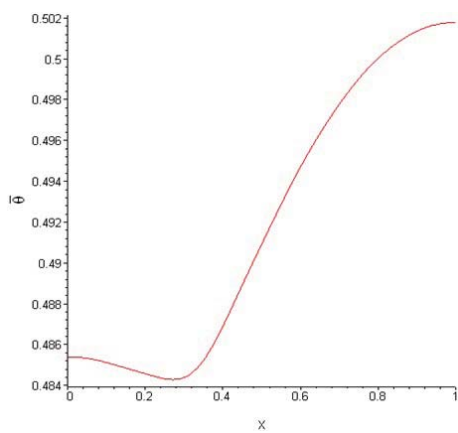
**Fig. 14:** Dimensionless Temperature at  $Fo=0.3, Ve^2=0.1$



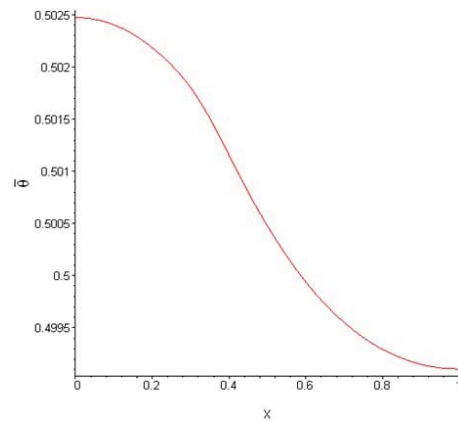
**Fig. 15:** Dimensionless Temperature at  $Fo=0.5, Ve^2=0.1$



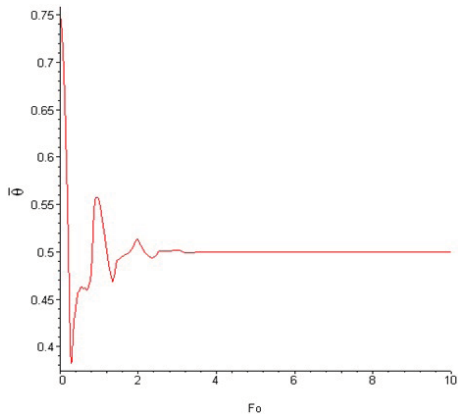
**Fig. 16:** Dimensionless Temperature at  $Fo=0.8, Ve^2=0.1$



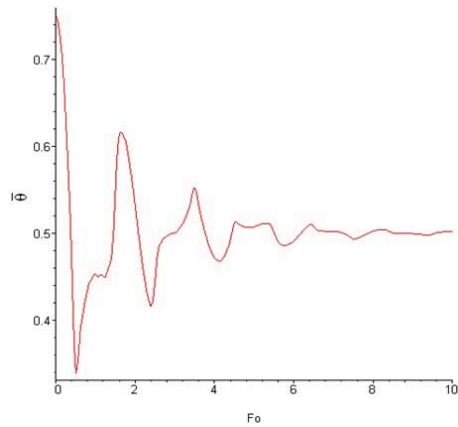
**Fig. 17:** Dimensionless Temperature at  $Fo=1.5, Ve^2=0.1$



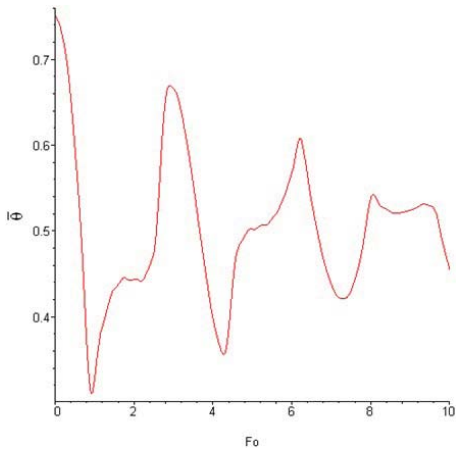
**Fig. 18:** Dimensionless Temperature at  $Fo=3, Ve^2=0.1$



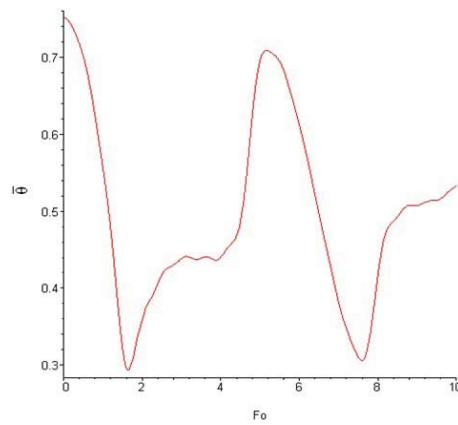
**Fig. 19:** Dimensionless Temperature at  $X=0.5, Ve^2 = 0.1$



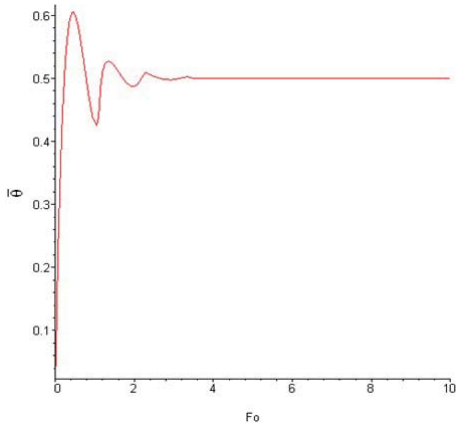
**Fig. 20:** Dimensionless Temperature at  $X=0.5, Ve^2 = 1$



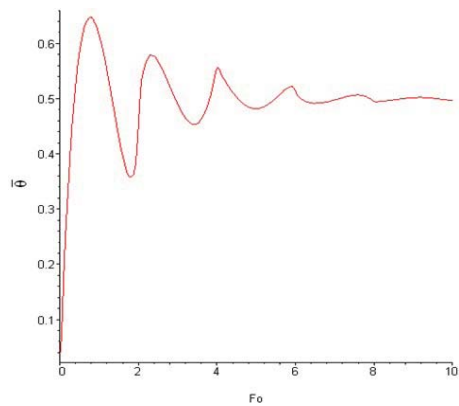
**Fig. 21:** Dimensionless Temperature at  $X=0.5, Ve^2 = 10$



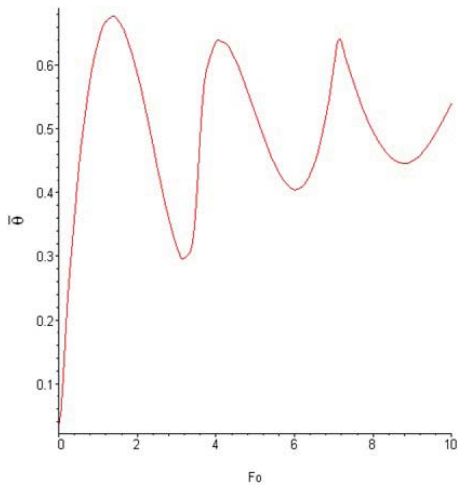
**Fig. 22:** Dimensionless Temperature at  $X=0.5, Ve^2 = 100$



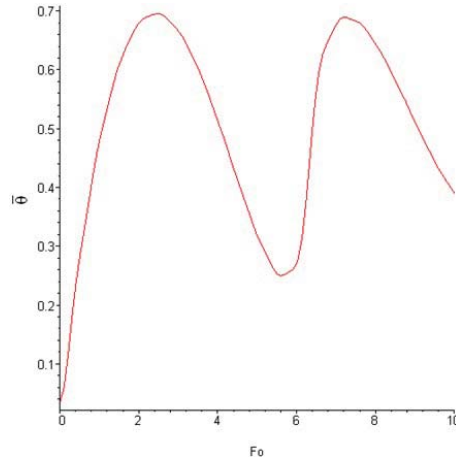
**Fig. 23:** Dimensionless Temperature at  $X=1, Ve^2 = 0.1$



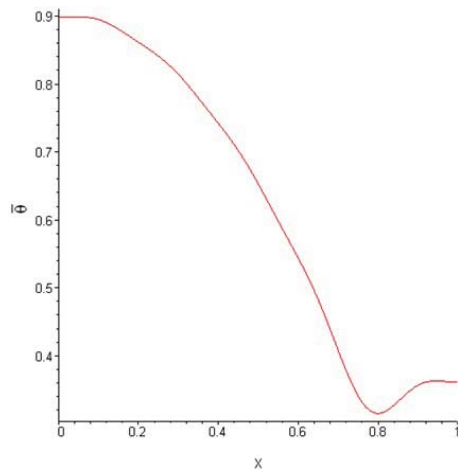
**Fig. 24:** Dimensionless Temperature at  $X=1, Ve^2 = 1$



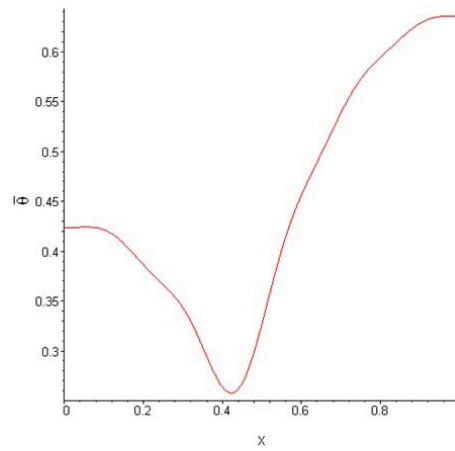
**Fig. 25:** Dimensionless Temperature at  $X=1, Ve^2 = 10$



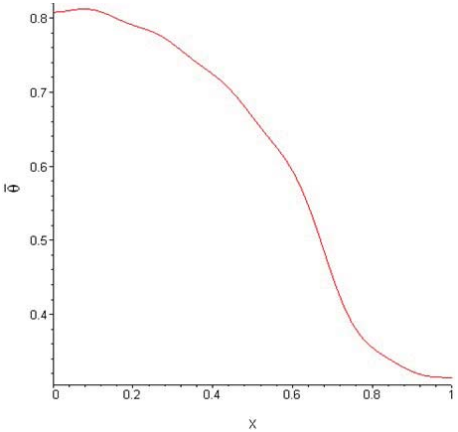
**Fig. 26:** Dimensionless Temperature at  $X=1, Ve^2 = 100$



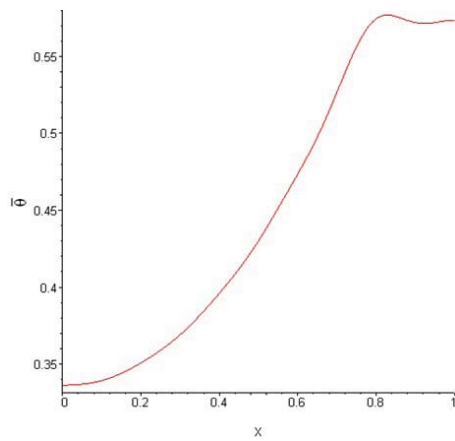
**Fig. 27:** Dimensionless Temperature at  $Fo=0.4, Ve^2 = 10$



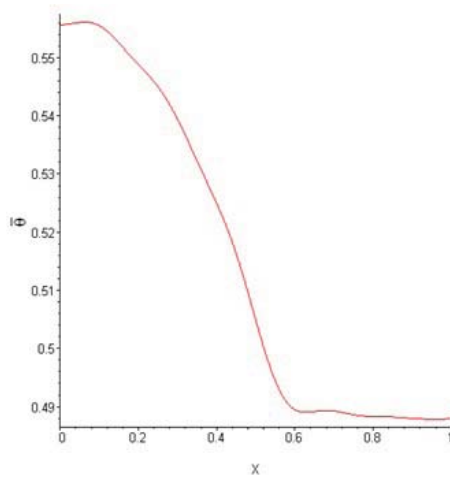
**Fig. 28:** Dimensionless Temperature at  $Fo=1, Ve^2 = 10$



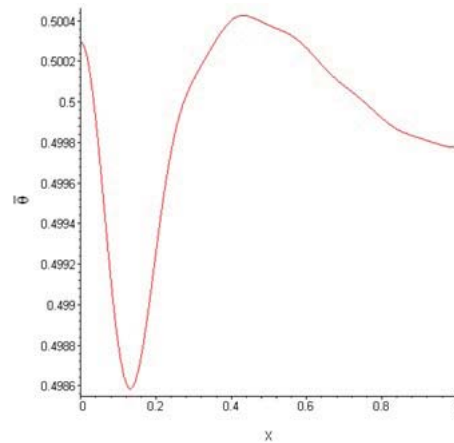
**Fig. 29:** Dimensionless Temperature at  $Fo=3, Ve^2 = 10$



**Fig. 30:** Dimensionless Temperature at  $Fo=7.5, Ve^2 = 10$



**Fig. 31:** Dimensionless Temperature at  $Fo=1.5, Ve^2 = 10$



**Fig. 32:** Dimensionless Temperature at  $Fo=30, Ve^2 = 10$

**Conclusions:**

In this paper, the (1+1)-dimensional non-Fourier heat conduction equation was solved analytically for a long cylindrical media under the influence of arbitrary initial conditions and insulated boundaries. The solution is expressed in terms of Bessel series. The important feature of this solution is its generality, namely, it is valid for any choice of initial conditions and it can be used for validation of numerical recipes. In order to show how the proposed method of solution works, two numerical examples have been solved. The graphs demonstrate that the time needed for reaching steady-state situation is increased with increase of the relaxation time  $\tau$ . This result could be expected, because the speed of propagation of heat waves is inversely proportional to relaxation time via the relation  $\sqrt{a/\tau}$ . Another conclusion from the graphs is that the shape of the heat wave is inherited from the shape of initial temperature distribution within the medium.

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