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A New Method for Solving Linear Fractional Programming Problems

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Abstract: In this paper a new method is suggested for solving the problem in which the objective function is a linear fractional function, and where the constraint functions are in the form of linear inequalities. The proposed method is based mainly upon solving this problem algebraically using the concept of duality. Since the earlier methods based on the vertex information may have difficulties as the problem size increases, our method appears to be less sensitive to the problem size. An example is given to clarify the developed theory and the proposed method.

Key words: linear fractional programming; Duality concept

INTRODUCTION

Linea r fraction maximum problems (i.e. ratio objective that have numerator and denominator) have attracted considerable research and interest , since they are useful in production planning, financial and corporate planning, health care and hospital planning.

Several methods to solve this problem are proposed in (1962), Charnes and Kooper have proposed their method depends on transforming this (LFP) to an equivalent linear program . Another method is called up dated objective function method derived from Bit ran and Novaes (1973) is used to solve this linear fractional program by solving a sequence of linear programs only re-computing the local gradient of the objective function .Also some aspects concerning duality and sensitivity analysis in linear fraction program was discussed by Bit ran and Magnant I (1976), and Singh .C. (1981) in his paper made a useful study about the optimality condition in fractional programming.

The suggested method in this paper depends mainly on the solving linear fractional functions, and where the constraint functions are in the form of linear inequalities, the proposed method is based mainly upon solving this problem algebraically using the concept of duality. Since the earlier methods based on the vertex information may have difficulties as the problem size increases, our method appears to be less sensitive to the problem size. An example is given to clarify the developed theory and the proposed method.

In section 2 some notations and definitions of the (LFP) problem is given while in section 3 we give the main result of this method together with a simple example and finally section 4 gives a conclusion remarks about this proposed method .

Notations and Definitions:

The problem of concern arises when a linear fractional function is to be maximized on a convex polyhedral set *X*. This problem can be formulated mathematically as follows, and will be denoted by *(LFP)*

$$\begin{aligned} &\text{Maximize} \quad F(x) = \frac{c^T x + \alpha}{d^T x + \beta} \\ &\text{Subject to;} \quad x \in X = \{x, Ax \le b\} \end{aligned} \tag{2-1}$$

where $x \in R^n$, A is an $(m + n) \times n$ matrix, we point out that the nonnegative conditions are included in the set of constraints. Also c and d are n-vectors, , $b \in R^{m+n}$, and α,β are scalers. It is assumed that the feasible solution set *X* is a compact set i. e. bounded and closed. Moreover, $d^T x + \beta > 0$

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everywhere in X.

This problem can also be formulated as

$$\begin{aligned} &Maximize \quad F(x)=\frac{c^{T}x+\alpha}{d^{T}x+\beta} \\ &Subject\ to; \quad a^{i}x\le b_{i} \quad i=1,2...,m+n. \end{aligned} \tag{2-2}$$

Here a^{i} represents the i[th] row of the given matrix A. It should be noted that in the non-degenerate case, an extreme point of the feasible region X lies on some n-linearly independent subset of X.

In (2.2) we shall assume that $\beta \neq 0$, then an equivalent form of (2.2) can be formulated as

$$\begin{aligned} &Maximize \quad F(x)=(c^{T}-\frac{\alpha}{\beta}d^{T})\frac{x}{d^{T}x+\beta}+\frac{\alpha}{\beta} \\ &Subject\ to \quad (A+\frac{\alpha}{\beta}d^{T})\frac{x}{d^{T}x+\beta}\le\frac{b}{\beta} \end{aligned} \tag{2.3}$$

If we define $\frac{x}{d^{T}x+\beta}\ge 0$ then (2.3) can be written in the form

$$\begin{aligned} &Maximize\ F(y)=(c^{T}-\frac{\alpha}{\beta}d^{T})y+\frac{\alpha}{\beta} \\ &Subject\ to\ (A+\frac{\alpha}{\beta}d^{T})y\le\frac{b}{\beta} \end{aligned} \tag{2.4}$$

Simply (2.4) can be written in the form

$$\begin{aligned} &Maximize \quad F(y)=P^{T}y+\frac{\alpha}{\beta} \\ &Subject\ to \quad Gy\le g \end{aligned} \tag{2.5}$$

$$Where \quad p^{T}=(c^{T}-\frac{\alpha}{\beta}d^{T}),\ G=(A+\frac{b}{\beta}d^{T}),\ and\ g=\frac{b}{\beta}$$

In the above definition of y , we can get $x=\beta\frac{y}{1-d^{T}y}$ (2.6)

Consider the dual of the linear program of (2.5) in the form

Minimize $w = u^{T} g$

$$\begin{aligned} &\text{Subject to } u^{T} G = p^{T} \\ &\qquad u \ge 0 \end{aligned} \tag{2.7}$$

On multiply the set of constraints of this dual problem by $T = (T_1 \mid T_2)$, $T_1 = p\,(p^{T}p)^{-1}$, and the column of the matrix T_2 constitute the bases of $N(p^{T}) = \{v;\ p^{T} v = 0\}$. (2.8)

We have $u^{T}G\ T_1 = 1$, $u^{T}GT_2 = 0$ and $u \ge 0$. In the case when $GT_2 \neq 0$, an s x (m+n) matrix Q of non-negative entries is defined such that $QGT_2 = 0$, this matrix will play an important role to find the optimal value of the above problem as the maximum value of w on the interval on the real line defined by $W=\{w \in R \mid QGT_1 w \le Q\ g\}$

Simply the above representation can be written as $W = \{w \in R \mid Zw \leq z\}$

Where $Z = QGT_1$, and $z = Q\,g$.

Also a sub matrix $\overline{Q}$ of the given matrix Q satisfying $\overline{Q}GT_1 = 1$ will be important for specifying the dual values needed for solving the linear fraction programming (2.1) . This dual values satisfied the well known Kuhn-Takucer condition [2] , [3] for a point y^k to be an optimal solution of the above program (2.5) we must exist

$$u \geq 0 \text{ such that } G_r^t u = p, \text{ or simply}$$
$$u = (G_r G_r^t)^{-1} G_r p \qquad (2\text{-}9)$$

Here Gr is a submatrix of the given matrix G containing only the coefficients of the set of active constraints at the current point y^k . Also from the complementary slackness theorem we have for the above set of active constraints the corresponding dual variables must be positive. Hence a sub matrix $\overline{Q}$ of the given matrix Q satisfying $\overline{Q}GT_1 = 1$ will be important for detecting the dual values needed for specifying the set of active constraints corresponding to the above positive dual values due to the complementary slackness theorem for the above set of active constraints .

New Method for Solving (LFP) Problems:

Our method for solving (LFP) problems summarize as follows:

1- compute $T_1 = (p^T p)^{-1} p$, and the matrix T_2 as in (2.8)
2- Find the matrix Q of non-negative entries such that $QGT_2 = 0$,
3- Find a sub matrix $\overline{Q}$ of the given matrix Q satisfying $\overline{Q}\ GT_1 = 1$
4- In the rows of $\overline{Q}$ for every positive entry determine the corresponding active constraint in the given matrix $G\ T_1$
5- Solve an n x n system of linear equations for these set of active constraints to get the optimal solution y*. Then using (2.6) to get the optimal solution of the (LFP) problem defined by (2.1).

Remark 3-1:

The matrix Q of non-negative entries such that $QGT_2 = 0$, is considered as the a polar matrix of the given matrix GT_2

Remark 3-2:

With d =0 in (LFP) ,the above problem reduces to linear programming problem (LP) , and hence our method can be used to solve the (LP) as a special case of this (LFP) using the same argument

Example: Consider the following (LFP)

Maximize $z = \dfrac{x_1 + x_2 + 3}{x_2 + 1}$

Subject to $x_1 + 2x_2 \leq 6$
$-x_1 \leq 0, -x_2 \leq 0$

For this (LFP) we have

$c^t = (1\ 1)$, $d^t = (0\ 1)$, $\alpha = 3$, $\beta = 1.$, then we have

$$T_1 = \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \text{ and } G\ T_2 \begin{pmatrix} 10 \\ -2 \\ -1 \end{pmatrix}$$

$$\text{Which gives } Q = \begin{pmatrix} 1 & 5 & 0 \\ 1 & 0 & 10 \end{pmatrix}$$

The second row in Q satisfies $\overline{Q}\, GT_1 = 1$. This indicates that the first and the third constraints in G are the only active set of constraints ,on solving $y_1 + 8\ y_2 = 6$, $y_2 = 0$, we get $y^{t*} = (6\ 0)$ as the optimal solution for the equivalent problem which finally on using (2.6) gives $x^{t*} = (6\ 0\ )$ as the optimal solution of our linear fraction program with optimal value $z^* = 9$.

Conclusion:

Anew method for solving linear fractional functions with constraint functions are in the form of linear inequalities is given. The proposed method is based mainly upon solving this problem algebraically using the concept of duality. Since the earlier methods based on the vertex information may have difficulties as the problem size increases, our method appears to be less sensitive to the problem size.

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