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Best Multiplier Approximation of Unbounded Periodic Functions Using Fejer Operators

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Abstract

In this paper, we study the best multiplier approximation of unbounded functions in L_(P,∅_n) (B) -space, B=[-π,π] by using the trigonometric Fourier series by means of the averaged modulus of smoothness..

Key words: Multiplier convergence, Multiplier integral

INTRODUCTION

The approximation by trigonometric polynomials is well investigated. The rate of convergence in uniform and integral norm was described (Lorentz, G.G., 1986) by the classical modulus of smoothness due to D..Jackson, S.N. Bernstein. Many authors studied the approximation theory for continuous and bounded functions on [a,b] like Bhaya .Also later many researchers using the concept of Multiplier Approximation in Approximation theory (Jassim, S.K. and Abeer Mahadi, 2017) which is study the uniform convergence of multiplier approximation of unbounded functions by Bernstien Durrmeyer operators in tow and d-dimensions. In this work we will discuss the Best multiplier approximation of periodic unbounded functions using Fejer operators.

2. Definitions and notations:

In this section we introduce some definitions and results that are used throughout this work.

Definition 2.1 (Hardy, G.H., 1949):

A series $\sum_{n=0}^{\infty} a_n$ is called a multiplier convergence if there is a sequence$\{\emptyset_n\}_{n=0}^{\infty}$such that $\sum_{n=0}^{\infty} a_n \emptyset_n < \infty$ and $\{\emptyset_n\}$is called a multiplier for the convergence.

Example 1:

The series $\sum_{n=1}^{\infty} \frac{1}{n}$divergent series and the sequence$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ convergent sequence. since $\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent series then the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a multiplier convergent.

Note:

If $\sum a_n$ is convergent series then it is multiplier convergent, this by taken $\{\emptyset_n\}_{n=0}^{\infty} = \{1\}_{n=0}^{\infty}$.But the reverse is incorrect, see above example

Definition 2.2:

For any real valued function f if there is a sequence $\{\emptyset_n\}_{n=0}^{\infty}$ such that $\int_B f\emptyset_n(x) < \infty$, then we say that $\emptyset_n$ is a multiplier for the integral.

Definition 2.3:

Let $L_p(B)$ be the space of all bounded measurable functions with the norm $\|f\|_{Lp} = \|f\|_p = (\int_B |f(x)|^p)^{\frac{1}{p}} < \infty$,1 $\leq p < \infty$.**Now** for any real valued function $f$we can define the multiplier integral norm as follows $\|f(.)\|_{p,\emptyset_n} = Sup\left\{(\int_B |f\emptyset_n(x)|^p\, dx)^{\frac{1}{p}} : x \in B\right\}$

Where $\emptyset_n$ is called the multiplier for the integral.

Definition 2.4:

Let $L_{P,\emptyset_n}(B)$,be the space of all real valued unbounded functions$f$ such that $\int_B f\emptyset_n(x)\, dx < \infty$ with the norm $\|f\|_{P,\emptyset_n} = \sup\left\{(\int_B |f\emptyset_n(x)|^p\, dx)^{\frac{1}{p}} : x \in B\right\}$, where $\emptyset_n$is the multiplier for the integral ,and $B = [-\pi, \pi]$

Definition2.5:

For$f \in L_{P,\emptyset_n}(B)$,we will define the following concepts.

1. $\omega^k(f,\delta)_{P,\emptyset_n} = sup_{|h|<\delta}\left\|\Delta_h^k f(\cdot)\right\|_{P,\emptyset_n}$, the multiplier modulus of smoothness of order k of function f where$\Delta_h^k(f,x) = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} f\left(x - \frac{kh}{2} + ih, x \mp kh2 \in B\right.$ the kth symmetric difference of the function f.

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2. The averaged modulus of smoothness of order k, (τ-modulus) of function f is the following function of $\delta \in \left[0, \frac{2\pi}{k}\right]$

$$\tau^k(f,\delta)_{P,\emptyset_n} = \|\omega^k(f,.;\delta)\|_{P,\emptyset_n} = \sup\left\{\int_{-\pi}^{\pi}(\omega^k(f,.,\delta)\emptyset_n)^p dx)^{\frac{1}{p}}\right\}$$

Where $\omega^k(f,.;\delta) = sup_{|h|<\delta}\left\|\Delta_h^k f(\cdot)\right\|_{\infty,\emptyset_n}$ for

$t \text{ and } t+kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right] \cap [-\pi,\pi]$

3. $\omega^{r,\theta}(f,\delta)_{p,\emptyset} = sup\left\|\Delta_{h,\theta}^r\ (f,.)\ \right\|_{p,\emptyset_n}$ where $\Delta_{h,\theta}^r(f,x) = \sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k} f\left(x - \frac{rh\theta}{2} + kh\theta\right), x \pm \frac{rh\theta}{2} \in B$, $\theta(x) = \sqrt{x-x^2}$, $x \in [0,1]$

Is the multiplier Ditzian –Totik modulus of smoothness of f .

Definition 2.6:

Let $f \in L_{P,\emptyset_n}(B)$ then the degree of best multiplier approximation of a function f with respect to trigonometric polynomial $g_n \in \Pi_n$ is given by $E_n(f)_{P,\emptyset_n} = in\,f\{\|f - g_n\|_{P,\emptyset_n}, g_n \in \Pi_n\}$, where Π_n be the set of all trigonometric polynomials

Definition 2.7:

For $f \in L_{P,\emptyset_n}(B)$, $P \geq 1$, the multiplier Fourier series of the function f at any point x is given by

$f(x) \cong \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$,where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x)\cos kx\ dx\ , b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x)\sin kx\ \ dx$$

Definition 2.8:

For $2\pi-$ periodic function $f \in L_{P,\emptyset_n}(B)$ let us define the nth partial sum S_n of multiplier Fourier series as

$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$, where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x)\cos kx\, dx\, , b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(x)\sin kx\, dx$$

Proposition 2.9:

For $f \in L_{P,\emptyset_n}(B), B = [-\pi,\pi]$ we have

$$S_n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\frac{\sin(2n+1)\frac{t-x}{2}}{2\sin\frac{t-x}{2}}dt$$

Proof:

$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$, where $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\cos kt\ dt$, $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\sin kt\ dt$. Then $S_n(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)dt$

$$+\sum_{k=1}^{n}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\cos kt\ \cos kx\ dt\ + \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\sin kt\ \sin kx\ dt\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\left[1 + 2\sum_{k=1}^{n}(\cos kx \cdot \cos kt + \sin kx \cdot \sin kt)\right] dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\right] dt$$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\, D_n(t-x)\ dt$,where

$D_n(t-x) = \frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)$ Called Dirichlete kernel for $t-x$.

Now to prove $D_n(t) = \frac{\sin(2n+1)\frac{t}{2}}{2\sin\frac{t}{2}}$

$$\sin(2n+1)\frac{t}{2} = \sin\frac{t}{2} + \sin\frac{3t}{2} - \sin\frac{t}{2} + \sin\frac{5t}{2} - \sin\frac{3t}{2} + \cdots + \sin\frac{2n+1}{2}t\ - \sin(2n-1)\frac{t}{2}$$

$$= \sin\frac{t}{2} + \sin\frac{t}{2}\cdot 2\cos t + 2\sin\frac{t}{2}\cdot\cos t + 2\sin\frac{t}{2}\cdot\cos 3t + \cdots$$

$+2\sin\frac{t}{2}\cos nt = \sin\frac{t}{2}[1 + 2\sum_{k=1}^{n}\cos kt]$, then

$\sin\ (2n+1)\frac{t}{2} = \sin\ \frac{t}{2}[1 + 2\sum_{k=1}^{n}\cos kt]$.

So that $\frac{\sin(2n+1)\frac{t}{2}}{\sin\frac{t}{2}} = [[1 + 2\sum_{k=1}^{n}\cos kt]]$.

Then $\frac{\sin(2n+1)\frac{t}{2}}{2\sin\frac{t}{2}} = \frac{1}{2} + \sum_{k=1}^{n}\cos kt$

Therefore $S_n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\frac{\sin(2n+1)\frac{t-x}{2}}{2\sin\frac{t-x}{2}}\ dt$

Definition2.10:

For $f \in L_{P,\emptyset_n}(x)$, let $S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$, be the multiplier partial sums of Fourier series then we will be define the multiplier Fejer operator as follows :

$$\delta_n(f,x) = \frac{S_0(f,x) + S_1(f,x) + \cdots S_n(f,x)}{n+1}$$

Proposition2.11:

For $f \in L_{p,\emptyset_n}(B), B = [-\pi,\pi]$, we get that $\delta_n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t-x)\, k_n(t)dt$,where $k_n(t) = \left[\frac{\sin(n\frac{t}{2})}{\sin\frac{t}{2}}\right]^2$

Proof:

Since $\delta_n(f,x) = \frac{1}{n+1}[s_0(f,x) + s_1(f,x) + \cdots s_n(f,x)] = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\frac{[D_0(t-x)+D_1(t-x)+\cdots D_n(t-x)]}{n+1} = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)k_n(t-x)\,dt$, where $k_n(t) = \frac{1}{n+1}[D_0(t) + D_1(t) + \cdots D_n(t)]$ and since $D_n(t) = \frac{\sin\left(\frac{2n+1}{2}\right)t}{2\sin\frac{t}{2}}$,we get

$$k_n(t) = \frac{\sin\frac{t}{2} + \sin\frac{3t}{2} + \cdots + \sin(\frac{2n-1}{2})\,t}{2n\sin\frac{t}{2}} \cdot \frac{2n\sin\frac{t}{2}}{2n\sin\frac{t}{2}}$$

$$k_n(t) = \frac{(1-\cos t) + (\cos t - \cos 2t) + \cdots + (\cos(n-1)t - \cos t)}{4n\sin^2\frac{t}{2}}$$

$k_n(t) = \frac{1-\cos t}{4n\sin^2\frac{t}{2}} = \frac{2\sin^2\frac{nt}{2}}{4n\sin^2\frac{t}{2}} = \frac{\sin^2\frac{nt}{2}}{2n\sin^2\frac{t}{2}}$ There for $\delta_n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)k_n(t-x)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)\frac{\sin^2(\frac{n(t-x)}{2})}{2n\sin^2(\frac{t-x}{2})}\,dt$

Definition2.12 ([5] Jassim, S.K. and N.J. Mohamed, 2010):

Let f *and* g be tow functions then we say that $f(x) = O\{g(x)\}$ If $|f(x)| < A.g(x)$, x goes to some given limit. A is a constant and $g(x) \neq 0$. In particular, O (1) means bounded function. By $f(x) = 0\{g(x)\}$ we mean that $(f(x)/g(x) > 0$ as x tends to a given limit .In particular, 0(1) means a function which tends to zero.

Definition 2.13:

For the space $L_{p,\emptyset_n}(B)$, and for $s = 1,2,\cdots$Let define the classes $W_{p,\emptyset_n}^{s}$,$W_{P\emptyset_n}^{s,\alpha}$ as follows $W_{p,\emptyset_n}^{s} = \{f \in L_{P,\emptyset_n}(B): f^{(s-1)}$, is absolutely continuous and $f^{(s)} \in L_{p,\emptyset_n}(B)\}$
$W_{P\emptyset_n}^{s,\alpha} = \{f \in W_{p,\emptyset_n}^{s}: f^{(s)} \in \text{Lip}(\alpha, L_{P,\emptyset_n}), \alpha$ is Lipschitz constant$\}$

Definition 2.14([6] Yunus, E.Y. and H.A. Ahnet, 2017):

Let $(\rho_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. We consider two means of the Fejer operators defined by $\delta_n^*(f,x) = \frac{1}{K_n}\sum_{m=0}^{n}\rho_{n-m}.S_m(f,x)$
$\delta_n^{**}(f,x) = \frac{1}{K_n}\sum_{m=0}^{n}\rho_m.S_m(f,x)$,where $K_n = \sum_{m=0}^{n}\rho_m$

Notes:

1- If $\rho_n = 1$ for all $n \in N$ both of $\delta_n^*(f,x)$ and $\delta_n^{**}(f,x)$ are equal to the Fejer operators $\delta_n(f,x) = \frac{1}{n+1}\sum_{m=0}^{n} S_m(f,x)$

2- $\delta_n^*(f,x) = \frac{1}{\rho_0+\rho_1+\rho_2\cdots\rho_n}[\rho_n S_0 + \rho_{n-1}S_1 \cdots \rho_0 S_n]$

3- $\delta_n^{**}(f,x) = \frac{1}{\rho_0+\rho_1+\rho_2\cdots\rho_n}[\rho_0 S_0 + \rho_1 S_1 \cdots \rho_n S_n]$

Lemma2.1 (Jassim, S.K. and N.J. Mohamed, 2010):

For the Dirichlete kernel $D_n(t)$ we have $\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = 1$

Lemma 2.2(Jassim, S.K. and N.J. Mohamed, 2010):

For the Fejer kernel$k_n(t)$ we have $\frac{1}{\pi}\int_{-\pi}^{\pi} k_n(t)\,dt = 1$

Lemma2.3:

For $f \in L_{p,\emptyset_n}(B)$, we have
1.$\omega^{k+1,\theta}(f,\delta)_{p,\emptyset_n} \le C\omega^{k,\theta}(f,\delta)_{p,\emptyset_n}$
2. $\omega^{k,\theta}(f,\delta)_{p,\emptyset_n} \le \omega^{k}(f,\delta)_{p,\emptyset_n} \le \tau^{k}(f,\delta)_{p,\emptyset_n}$
3.$\omega^{k,\theta}(f,\delta)_{p,\emptyset_n} \le C(p,k)\|f\|_{p,\emptyset_n}$

Proof:

Let $f \in L_{p,\emptyset_n}(B)$then $f\emptyset_n(x)$ is a multiplier integral function on B this means $\|f\emptyset_n(.)\|_p = \{\int_B |f\emptyset_n(x)|^p\,dx\}^{\frac{1}{p}}$,**now**
1. By (Rebecca, M., Brannon, 2003) we have
$\omega_{r+1}^{\theta}(f\emptyset_n(x),\delta)_p \le C\,\omega_r^{\theta}(f\emptyset_n(x),\delta)_p$. Then we have
$\omega^{k+1,\theta}(f,\delta)_{p,\emptyset_n} \le C\,\omega^{k,\theta}(f,\delta)_{p,\emptyset_n}$.
2. By (Rebecca, M., Brannon, 2003) we have
$\omega_r^{\theta}(f\emptyset_n(x),\delta)_p \le \omega_p(f\emptyset_n(x),\delta)_p \le \tau^{r}(f\emptyset_n(x),\delta)_p$,then we get
$\omega^{k,\theta}(f,\delta)_{p,\emptyset_n} \le \omega^{k}(f,\delta)_{p,\emptyset_n} \le \tau^{k}(f,\delta)_{p,\emptyset_n}$.
3. By (Rebecca, M., Brannon, 2003) we have
$\omega_r^{\theta}(f\emptyset_n(x),\frac{1}{n})_p \le C(p,k)\|f\emptyset_n(.)\|_p$,thus$\omega^{k,\theta}(f,n^{-1})_{p,\emptyset_n} \le C(p,k)\|f\|_{p,\emptyset_n}$

Lemma2.4(Carothers, N.L., 1998) [Minkoswskis Inequality]:

For $f, g \in L_p(B), p \ge 1$ Then $f+g \in L_p(B)$, and$\{\int_B |f(x)+g(x)|^p dx\}^{\frac{1}{p}} \le \{\int_B |f(x)|^p dx\}^{\frac{1}{p}} + \{\int_B |g(x)|^p dx\}^{\frac{1}{p}}$

Lemma2.5:
For $f \in L_{P,\emptyset_n}(B)$, we have $\|\delta_n(f,.)\|_{p,\emptyset_n} \le C.\|f\|_{p,\emptyset_n}$, where C is a constant

Proof:

$$\|\delta_n(f,.)\|_{p,\emptyset_n} = sup\{\int_{-\pi}^{\pi}[\frac{1}{2n\pi}\int_{-\pi}^{\pi} f\emptyset_n(t-x)K_n(t)dt]^p\, dx\}^{\frac{1}{p}} \le \sup\{\int_{-\pi}^{\pi}[\frac{1}{2n\pi}\int_{-\pi}^{\pi}|f\emptyset_n(t-x)|^p dx]^{\frac{1}{p}}.K_n(t)\, dt\}$$

$$\le \frac{1}{2n\pi}\sup\{[\int_{-\pi}^{\pi}|f\emptyset_n(t-x)|^p\, dx]^{\frac{1}{p}}.\int_{-\pi}^{\pi}K_n(t)dt\}$$

$$\le sup\{\int_{-\pi}^{\pi}|f\emptyset_n(t-x)|^p dx\}^{\frac{1}{p}}.\frac{1}{2n\pi}\int_{-\pi}^{\pi}K_n(t)dt \le \|f\|_{p,\emptyset_n}.C$$

Then $\|\delta_n(f,.)\|_{p,\emptyset_n} \le C\|f\|_{p,\emptyset_n}$

Lemma2.6:
For $f \in W^{s,1}_{P\emptyset_n}$ we have $\tau^k(f,\frac{1}{n})_{p,\emptyset_n} = O(\frac{1}{n})$

Proof:
Since $f \in W^{s,1}_{P\emptyset_n}$ then $f \in W^{s}_{P\emptyset_n}$ and $f \in L_{P,\emptyset_n}(B)$, thus $f\emptyset_n \in L_p(B)$,then $\tau_k(f\emptyset_n,n^{-1})_p = O(n^{-1})$(see[9]). There for $\tau^k(f,\frac{1}{n})_{p,\emptyset_n} = O(\frac{1}{n})$ ■

3. Main results:
In this section we will used all above lemmas to state and prove the theorems of best multiplier approximation of $f \in L_{P,\emptyset_n}(B), B = [-\pi,\pi]$ using Fejer operators by means of the averaged modulus of smoothness of order k.

Theorem 3.1:
For $f \in L_{p,\emptyset_n}(B)$, we get that $E_n(f)_{p,\emptyset_n} \le C(k)\omega^{k,\theta}(f,n^{-1})_{p,\emptyset_n}$

Proof:

Let Π_n be the set of all trigonometric polynomials, $f \in L_{p,\emptyset_n}(B)$, then $\|f\emptyset_n(.)\|_p = \left[\int_B |f\emptyset_n(x)|^p dx\right]^{\frac{1}{p}}$,**exists** Then by (Sendov Theorem [8]) there is a polynomial $T_n^* \in \Pi_n$ such that $\|f\emptyset_n(\cdot) - T_n^*\|_p \le C_1(k)\omega_k^\theta(f\emptyset_n(\cdot),\frac{1}{n})_p$,(see [10]). Then we get $\|f - T_n^*\|_{p,\emptyset_n} \le C_1(k)\omega^{k,\theta}(f,n^{-1})_{p,\emptyset_n}$, **but** $E_n(f)_{p,\emptyset_n} = \inf_{T_n\in\pi_n}\{\|f - T_n\|_{p,\emptyset_n}\} = \|f - T_n^*\|_{p,\emptyset_n} \le C(k)\omega^{k,\theta}(f,n^{-1})_{P,\emptyset_n}$ Thus $E_n(f)_{P,\emptyset_n} \le C(k)\omega^{k,\theta}(f,n^{-1})_{P,\emptyset_n}$, where T_n^* is the best multiplier approximation of f.

Theorem3.2:
Let $f \in L_{p,\emptyset_n}(B)$, $B = [-\pi,\pi]$, $1 \le p < \infty$ and $\{\emptyset_n\}_{n=0}^\infty$ be a monotonic sequence of positive real numbers then we have $\|f - \delta_n^*\|_{p,\emptyset_n} \le C(k)\tau^k(f,\frac{1}{n})_{p,\emptyset_n}$

Proof:

Let T_n be the best multiplier approximation of f and since $\delta_n^*(T_n,x) = T_n$ we get $\|f - \delta_n^*\|_{p,\emptyset_n} = \left[\int_B |(f(x) - \delta_n^*(f,x))\emptyset_n|^p dx\right]^{\frac{1}{p}} = \left[\int_B |(f(x) - T_n + T_n -\right.$ *δn∗f,x)∅nPdx1P*

$$\le \left[\int_B |(f(x) - T_n)\emptyset_n|^p dx\right]^{\frac{1}{p}} + \left[\int_B |(T_n - \delta_n^*(f,x))\emptyset_n|^p dx\right]^{\frac{1}{p}}$$

$$= \|f - T_n\|_{p,\emptyset_n} + \left[\int_B |(\delta_n^*(T_n,x) - \delta_n^*(f,x))\emptyset_n|^p dx\right]^{\frac{1}{p}}$$

$$= \|f - T_n\|_{p,\emptyset_n} + \|\delta_n^*(T_n,\cdot) - \delta_n^*(f,\cdot)\|_{p,\emptyset_n}$$

$$= \|f - T_n\|_{p,\emptyset_n} + \|\delta_n^*(T_n - f)\|_{p,\emptyset_n} = \|f - T_n\|_{p,\emptyset_n} + C_1\|T_n - f\|_{p,\emptyset_n}$$

Using theorem (3.1) and lemma (2.3(2)) we have

$$\|f - \delta_n^*(f,\cdot)\|_{p,\emptyset_n} \le CE_n(f)_{p,\emptyset_n} \le C(k)\omega^{k,\emptyset_n}\left(f,\frac{1}{n}\right)_{p,\emptyset_n} \le C(k)\tau^k\left(f,\frac{1}{n}\right)_{p,\emptyset_n}$$

Thus $\|f - \delta_n^*(f,\cdot)\|_{P,\emptyset_n} \le C\tau^k\left(f,\frac{1}{n}\right)_{P,\emptyset_n}$ ■

Theorem 3.3:
Let $\in L_{p,\emptyset_n}(B)$, $B = [-\pi,\pi]$, $1 \le p < \infty$,and $\{\emptyset_n\}_{n=0}^\infty$ be a sequence of positive real numbers then we have $\|f - \delta_n^{**}(f,\cdot)\|_{p,\emptyset_n} \le C\tau^k\left(f,\frac{1}{n}\right)_{p,\emptyset_n}$

Proof:
Let T_n be the best multiplier approximation of f ,then $\|f - \delta_n^{**}(f,\cdot)\|_{p,\emptyset_n} = \|f - T_n + T_n - \delta_n^{**}(f,\cdot)\|_{p,\emptyset_n} \le \|f - T_n\|_{p,\emptyset_n} + \|T_n - \delta_n^{**}(f,\cdot)\|_{p,\emptyset_n} = \|f - T_n\|_{p,\emptyset_n} + \|\delta_n^{**}(T_n,.) - \delta_n^{**}(f,\cdot)\|_{p,\emptyset_n} = \|f - T_n\|_{p,\emptyset_n} + \|\delta_n^{**}(T_n - f)\|_{p,\emptyset_n} \le \|f - T_n\|_{p,\emptyset_n} + C_1\|T_n - f\|_{p,\emptyset_n} = E_n(f)_{p,\emptyset_n} + C_1E_n(f)_{p,\emptyset_n} = CE_n(f)_{p,\emptyset_n} \le C(k)\omega^{k,\theta}(f,\frac{1}{n})_{p,\emptyset_n}$,this by theorem (3.1).Then by using lemma(2.3(2)) we get $\|f - \delta_n^{**}(f,\cdot)\|_{p,\emptyset_n} \le C(k)\omega^{k,\theta}(f,\frac{1}{n})_{p,\emptyset_n} \le C\tau^k\left(f,\frac{1}{n}\right)_{p,\emptyset_n}$

Thus $\|f - \delta_n^{**}(f,\cdot)\|_{p,\emptyset_n} \le C\tau^k\left(f,\frac{1}{n}\right)_{p,\emptyset_n}$ ■

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Notes:

From the above theorems we can get the results of Yunus E.Y. and Ahmet H.A. [6]. In the $L_{p,\emptyset_n}$ −space for the derivatives of functions**.**

Corollary3.5:

For $1 \le p < \infty$, $s \in N$,and for all $f \in W^{s,1}_{P\emptyset_n}$ we have $\left\|f^{(s)} - \delta_n^*(f^{(s)},.)\right\|_{p,\emptyset_n} = O(\frac{1}{n})$

Proof:

Since $f \in W^{s,1}_{P\emptyset_n}$,then $f^{(s)} \in L_{p,\emptyset_n}(B)$, using theorem (3.2) and lemma(2.6) we get $\left\|f^{(s)} - \delta_n^*(f^{(s)},.)\right\|_{p,\emptyset_n} \le C(k)\tau^k\left(f^{(s)},\frac{1}{n}\right)_{p,\emptyset_n} = C(k)O(\frac{1}{n})$ Then $\| f^{(s)} -$ *δn∗(f(s),.)p,∅n=CkO1n=O1n* ■

Corollary 3.6:

Let $1 \le p < \infty$, $s \in N$, then for all $f \in W^{s,1}_{P\emptyset_n}$ we have $\left\|f^{(s)} - \delta_n^{**}(f^{(s)},.)\right\|_{p,\emptyset_n} = O(\frac{1}{n})$

Proof:

Since $f \in W^{s,1}_{P\emptyset_n}$ then $f^{(s)} \in L_{p,\emptyset_n}(B)$ using theorem (3.3) and lemma (2.6) we get $\left\|f^{(s)} - \delta_n^{**}(f^{(s)},.)\right\|_{p,\emptyset_n} \le C(k)\tau^k\left(f^{(s)},\frac{1}{n}\right)_{p,\emptyset_n} = C(k)O(\frac{1}{n})$
Then $\left\|f^{(s)} - \delta_n^{**}(f^{(s)},.)\right\|_{p,\emptyset_n} = C(k)O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right)$ ■

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