Australian Journal of Basic and Applied Sciences, 11(13) October 2017, Pages: 20-24

AUSTRALIAN JOURNAL OF BASIC AND APPLIED SCIENCES

ISSN:1991-8178 EISSN: 2309-8414
Journal home page: www.ajbasweb.com

New Subclass of Multivalent Functions with Negative Coefficients inAnalytic Topology

Lieth Majed

Lieth A. Majed, lecture, department of mathematics College of Science,Diyala,Iraq.

Address For Correspondence:
Lieth Majed, Lieth A. Majed, lecture, department of mathematics College of Science, Diyala,Iraq,
E-mail: majed@sciences.uodiyala.edu.iq

ARTICLE INFO

Article history:
Received 10 August 2017
Accepted 1 October 2017
Available online 18 October 2017

Keywords:
Negative coefficients, closure theorem,weighted mean and integral operator.

ABSTRACT

In this present paper, we establish new subclass of multivalent functions with negative coefficients in unit disk $\nabla = \{z \in \mathbb{C}: |z| < 1\}$. We obtain some properties, like, theorem of coefficient inequality, weighted mean, subordination theorems.AMS subject classification: 30C45.

INTRODUCTION

Let $\mathcal{D}_p$be denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}\, z^{n+p}, (p \in \mathbb{N}) \tag{1}$$

which are analytic and n+p- valent in the open unit disk
U={z∈ $\mathbb{C}$: $|z| < 1$}.
Let$\mathcal{N}\mathcal{D}_p$ denote the subclass of $\mathcal{D}_p$of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p}\, z^{n+p}, \qquad (a_{n+p} \geq 0, z \in \nabla). \tag{2}$$

Note that the authors defined and studied some classes of analytic functions like the form (1) in (Ibrahim, W., D. Maslina, 2010) and (Anas, A. and D. Maslina, 2015).

For a function $f \in \mathcal{N}\mathcal{D}_p$, let the Komatuoperator [6] defined by

$$k_{c,p}^{\delta} f(z) = \frac{(c+p)^{\delta}}{\Gamma(\delta)} \frac{1}{z^c} \int_0^z t^{c-1} \left(log \frac{z}{t}\right)^{\delta-1} f(t)dt$$
$$= z^p - \sum_{n=1}^{\infty} \left(\frac{c+p}{c+p+n}\right) a_{n+p} z^{n+p} \; (c > -p, \delta > 0) \tag{3}$$

We suppose$\mathcal{N}(p,n,c,\mu,\beta)$denote the subclass of $\mathcal{N}\mathcal{D}_p$consisting of functions f which satisfy:

$$\left|\frac{(p)(p-1)z^{p-2} - k_{c,p}^{\delta} f(z))''}{\mu\left(k_{c,p}^{\delta} f(z)\right)'' + 2\mu(p)(p-1)z^{p-2}}\right| < \beta, \tag{4}$$

ToCite ThisArticle: Lieth Majed., New Subclass of Multivalent Functions with Negative Coefficients inAnalytic Topology. ***Aust. J. Basic & Appl. Sci.,*** *11(13): 20-24, 2017*

where $\delta > 0, 0 < \mu < 1, 0 < \beta \leq p$. and $k_{c,p}^{\delta} f$ is given by (3).

Theorem (1):

Let the function $f \in ₦\mathcal{D}_p$ be defined by (2).Then

$f \in ₦(p, n, c, \mu, \beta)$ if and only if

$$\sum_{n=1}^{\infty}\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)a_{n+p} \leq 3\mu\beta(p)(p-1), \qquad (5)$$

where $\delta > 0, 0 < \mu < 1, 0 < \beta \leq p$.

The result is sharp for the function

$$f(z) = z^p - \frac{3\mu\beta(p)(p-1)}{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)} z^{n+p}, n \geq 1.$$

Proof:

Assume that the inequality (5) holds true and let $|z| = 1$,then

from (4), we obtain

$$|(p)(p-1)z^{p-2} - (f(z))''| - \gamma\left|\mu(f(z))'' + 2\mu(p)(p-1)z^{p-2}\right|$$

$$= \left|\sum_{n=1}^{\infty}(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p}z^{n+p-2}\right|$$

$$-\gamma\left|3\mu(p)(p-1)z^{p-2} - \sum_{n=1}^{\infty}\mu(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p}z^{n+p-2}\right| \qquad (6)$$

$$\leq \sum_{n=1}^{\infty}(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p} - 3\,\beta\mu(p)(p-1) + \sum_{n=1}^{\infty}\beta\mu(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p}$$

$$= \sum_{n=1}^{\infty}\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\gamma\mu+1)a_{n+p} - 3\beta\mu(p)(p-1) \leq 0,$$

by hypothesis.

Hence by maximum modulus principle, $f \in ₦(p, n, c, \mu, \beta)$.

Conversely, Let$f \in ₦(p, n, c, \mu, \beta)$. Then

$$\left|\frac{(p)(p-1)z^{p-2} - (k_{c,p}^{\delta}f(z))''}{\mu\left(k_{c,p}^{\delta}f(z)\right)'' + 2\mu(p)(p-1)z^{p-2}}\right| < \beta, (z \in \nabla).$$

That is

$$\left|\frac{\sum_{n=1}^{\infty}(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p}z^{n+p-2}}{3\mu(p)(p-1)z^{p-2} - \sum_{n=1}^{\infty}\mu(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p}z^{n+p-2}}\right| < \beta, \qquad (7)$$

Since$Re(z) \leq |z|$for all z$(z \in \nabla)$, we get

$$Re\left\{\frac{\sum_{n=1}^{\infty}(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p}z^{n+p-2}}{3\mu(p)(p-1)z^{p-2} - \sum_{n=1}^{\infty}\mu(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p}z^{n+p-2}}\right\} \leq \beta, \qquad (8)$$

we choose the value of z on the real axis so that $(k_{c,p}^{\delta}f(z))''$ is real.

$$\sum_{n=1}^{\infty}(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p}z^{n+p-2}$$

$$\leq 3\mu\gamma(p)(p-1)z^{p-2} - \sum_{n=1}^{\infty}\beta\mu(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p}z^{n+p-2}.$$

Letting z $\to 1^-$, through real values,

$$\sum_{k=2}^{\infty}(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p}$$

$$\leq 3\mu\beta(p)(p-1) - \sum_{n=1}^{\infty}\beta\mu(n+p)(n+p-1)\left(\frac{c+p}{c+p+n}\right)a_{n+p},$$

we obtain inequality (5).

Finally, sharpness follows, if we take

$$f(z) = z^p - \frac{3\mu\beta\,(p)(p-1)}{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)} z^{n+p}, n \geq 1. \tag{9}$$

Corollary (1):

Let $f \in ₦(p,n,c,\mu,\beta)$. Then

$$a_{n+p} \leq \frac{3\mu\beta\,(p)(p-1)}{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)}, n \geq 1. \tag{10}$$

In the following theorem, we obtain weighted mean is in the class $₦(p,n,c,\mu,\beta)$

Definition (1)[5]:

Let f_1 and f_2 be in the class $₦(p,n,c,\mu,\beta)$Then the weighted mean ₩of $f_1$and f_2 is given by:

$$₩(z) = \frac{1}{2}[(1-j)f_1(z) + (1+j)f_2(z)], \quad 0 < j < 1.$$

Theorem (2):

Let f_1 and f_2 be in the class$₦(p,n,c,\mu,\beta)$Then the weighted mean$w_j$off_1 and$f_2$is also in the class $₦(p,n,c,\mu,\beta)$

Proof:

By Definition (1), we have $₩(z) = \frac{1}{2}[(1-j)f_1(z) + (1+j)f_2(z)]$,

$$= \frac{1}{2}\left[(1-j)\left(z^p - \sum_{n=1}^{\infty} a_{n+p,1}\, z^{n+p}\right) + (1+j)\left(z^p - \sum_{n=1}^{\infty} a_{n+p,2}\, z^{n+p}\right)\right]$$

$$= z^p - \sum_{n=1}^{\infty} \frac{1}{2}\left[(1-j)a_{n+p,1} + (1+j)a_{n+p,2}\right] z^{n+p}.$$

Since f_1 and f_2 are in the class$₦(p,n,c,\mu,\beta)$so by Theorem (1), we get

$$\sum_{n=1}^{\infty}\left(\frac{c+p}{c+p+n}\right)\left((n+p)(n+p-1)\right)(\beta\mu+1)a_{k,1} \leq \mu\beta(p)(p-1),$$

and

$$\sum_{n=1}^{\infty}\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)a_{k,2} \leq \mu\beta(p)(p-1).$$

Hence

$$\sum_{n=1}^{\infty}\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)]\frac{1}{2}\left[(1-j)a_{n+p,1} + (1+j)a_{n+p,2}\right]$$

$$= \frac{1}{2}\ (1-j)\sum_{n=1}^{\infty}\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)a_{n+p,1}$$

$$+ \frac{1}{2}(1+j)\sum_{n=1}^{\infty}\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)a_{n+p,1}$$

$$\leq \frac{1}{2}(1-j)\mu\beta(p)(p-1) + \frac{1}{2}(1+j)\mu\gamma(p)(p-1) = \mu\beta(p)(p-1).$$

Therefore,$₩ \in ₦(p,n,c,\mu,\beta)$.

In, Littlewood (1925) proved the following subordination theorem.

(see also Duren 1983).

Theorem (2) [4]:

if $f$and g are analytic in U with $f \prec g$

then for $\alpha > 0$ and z = r $e^{i\vartheta}$ and (0< $r < 1$)

$\int_0^{2\pi}|f(\mathrm{z})|^{\alpha}\,d\vartheta \le \int_0^{2\pi}|g(z)|^{\alpha}\,d\vartheta$ (11)

We will make use the above theorem to prove.

Theorem (3):

Let $f \in \mathbb{N}(p,n,c,\mu,\beta)$and suppose that $f$is defined by

$$f(z) = z^p - \frac{3\mu\beta\,(p)(p-1)}{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\gamma\mu+1)} z^{n+p}, n \ge 1. \quad (12)$$

If there exists an analytic function w given by

$$[w(z)]^n = \frac{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\gamma\mu+1)}{\mu\beta\,(p)(p-1)} \sum_{n=1}^{\infty} a_{n+p}\, z^{n+p}, \quad (13)$$

then for z = r $e^{i\vartheta}$ and (0< $r<1$)

$$\int_0^{2\pi}\left|f\left(\mathrm{r}\, e^{i\vartheta}\right)\right|^{\alpha} d\vartheta \le \int_0^{2\pi}\left|f\left(\mathrm{r}\, e^{i\vartheta}\right)\right|^{\alpha} d\vartheta, \qquad (\alpha>0).$$

Proof:

Let $f(z)$of the form (2) and $f_n(z)$ defined by (12), then we must show that

$$\int_0^{2\pi}\left|1-\sum_{n=1}^{\infty} a_k\, z^k\right|^{\alpha} d\vartheta \le \int_0^{2\pi}\left|1-\frac{3\mu\beta(p)(p-1)}{\left(\frac{c+p}{c+p+n}\right)\left((n+p)(n+p-1)\right)(\beta\mu+1)} z^k\right|^{\alpha} d\vartheta.$$

By applying Littlewood ̓s subordination theorem, it would suffice to show that

$$1-\sum_{n=1}^{\infty} a_{n+p}\, z^{n+p} \prec 1-\frac{3\mu\beta(p)(p-1)}{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)} z^{n+p}.$$

By setting

$$1-\sum_{n=1}^{\infty} a_{n+p}\, z^{n+p} = 1-\frac{3\mu\beta(p)(p-1)}{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)} [w(z)]^{n+p}.$$

We find that

$$[w(z)]^{n+p} = \frac{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)}{3\mu\beta(p)(p-1)} \sum_{n=1}^{\infty} a_{n+p}\, z^{n+p},$$

Which readily yields$w(0) = 0$.

Furthermore, by using (5), we obtain

$$|[w(z)]|^{n+p} = \left|\sum_{n=1}^{\infty} \frac{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)}{3\mu\beta(p+\eta)(p+\eta-1)} a_{n+p} z^{n+p}\right|$$

$$\le \sum_{n=1}^{\infty} \frac{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)}{3\mu\beta(p)(p-1)} a_{n+p}|z|^{n+p}$$

$$\le |z|^p \sum_{n=0}^{\infty} \frac{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)}{3\mu\beta(p)(p-1)} a_{n+p}$$

$$\le |z| < 1.$$

Theorem (4):

Let $\alpha > 0$. If$f \in \mathbb{N}(p,n,c,\mu,\beta)$and

$$f(z) = z^p - \frac{3\mu\beta(p)(p-1)}{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)} z^{n+p}, n \geq 1$$

then for z = r $e^{i\vartheta}$ and (0< $r < 1$),

$$\int_0^{2\pi} \left|f'(\mathrm{r}\, e^{i\vartheta})\right|^\alpha d\vartheta \leq \int_0^{2\pi} \left|f'_k(\mathrm{r}\, e^{i\vartheta})\right|^\alpha d\vartheta, \tag{14}$$

Proof:

$$f'(z) = (p)z^{p-1} - \sum_{k=2}^{\infty} (n+p)a_{n+p}\, z^{n+p-1},$$

$$f'\ \ (z) = (p)z^{p-1} - \frac{3\mu\beta(p)(p-1)(n+p)}{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\gamma\mu+1)} z^{n+p-1}, n \geq 1.$$

It is sufficient to show that

$$1 - \sum_{n=1}^{\infty} \left(\frac{p+n}{p}\right) a_{n+p}\, z^{n+p} \prec 1 - \frac{3\mu\gamma(p)(p-1)}{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\gamma\mu+1)} \left(\frac{n+p}{p}\right) z^{n+p}.$$

By setting

$$1 - \sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right) a_{n+p}\, z^{n+p} = 1 - \frac{3\mu\beta(p+\eta)(p+\eta-1)}{\left(\left(\frac{c+p}{c+p+n}\right)(k+p+\eta)(k+p+\eta-1)\right)(\beta\mu+1)} \left(\frac{n+p}{p}\right) [w(z)]^{n+p},$$

hence

$$[w(z)]^{n+p} = \sum_{n=1}^{\infty} \frac{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)}{\mu\beta(p)(p-1)} a_{n+p} z^{n+p}$$

Which readily yields$w(0) = 0$.

By using Theorem (1), we obtain

$$|[w(z)]|^{n+p} = \left|\sum_{n=1}^{\infty} \frac{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)}{\mu\beta(p)(p-1)} a_{n+p} z^{n+p}\right|$$

$$\leq |z|^p \sum_{n=0}^{\infty} \frac{\left(\left(\frac{c+p}{c+p+n}\right)(n+p)(n+p-1)\right)(\beta\mu+1)}{\mu\beta(p)(p-1)} a_{n+p}$$

$$\leq |z| < 1.$$

Conclusion:

We obtain the properties theorem of coefficient inequality, closure theorem ,weighted mean and integral operator.

REFERENCES

Anas,A.and D. Maslina, 2015.New subclass of p-valent functions with negativecoefficients.,AIP .C.P.

Duren,P.L., 1983. Univalent Functions, Grundlehren der MathematischenWisswnschaften (Vol.259),Springer-Verlag, New York.

Ibrahim, W., D. Maslina,2010. On certain classes of multivalent analytic functions., J. Mathematics and Statistics., 6(3): 271-275.

Littlewood,L.E., 1925 On inequalities in the theory of functions, Prec. London Math. Soc.,23: 481-519.

Miller,S.S.and P.T. Mocanu,2000. Differential subordinations :Theory and Applications, Series on Monographs and Text Books in Pure and Applied Mathematics (Vol. 225), Marcel Dekker, New York and Basel.

Salim,T.O., 2010Aclass of multivalent function involving a generalized linear operator and subordination, Int. J. Open Prob. Comp. Anal., 2(2): 82-94.