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Homotpy Analysis Method for Solving Fractional Differential Equations

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ABSTRACT

We present some approximate analytical solution to the extraordinary fractional differential equations by applying the homotopy analysis method (HAM). While compared with the Adomain decomposition method (ADM) and the homotopy perturbation method (HPM), the HAM contains the auxiliary convergence-control parameter $\hbar$ and the control function H(x,t), which provides a useful way to adjust and control the convergence region of solution series. The numerical results reveal that HAM is accurate and effective when it is applied to the perturbed PDEs.

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INTRODUCTION

The HAM distinguishes itself from various other analytical methods in four important aspects. First, it is a series expansion method that is not directly dependent on small or large physical parameters. Thus, it is applicable for not only weakly but also strongly nonlinear problems, going beyond some of the inherent limitations of the standard perturbation methods. Second, the HAM is an unified method for the Lyapunov artificial small parameter method, the delta expansion method, the Adomian decomposition method,(Adomian, G., 1994) and the homotopy perturbation method. (Liang, Songxin; Jeffrey, J. David, 2009 ; Sajid, M., T. Hayat, 2008) The greater generality of the method often allows for strong convergence of the solution over larger spacial and parameter domains. Third, the HAM gives excellent flexibility in the expression of the solution and how the solution is explicitly obtained. It provides great freedom to choose the basis functions of the desired solution and the corresponding auxiliary linear operator of the homotopy. Finally, unlike the other analytic approximation techniques, the HAM provides a simple way to ensure the convergence of the solution series.

The homotopy analysis method is also able to combine with other techniques employed in nonlinear differential equations such as spectral methods (Motsa, S.S., *et al*., 2010) and Padé approximants. It may further be combined with computational methods, such as the boundary element method to allow the linear method to solve nonlinear systems. Different from the numerical technique of homotopy continuation, the homotopy analysis method is an analytic approximation method as apposed to discrete computational method. Further, the HAM uses the homotopy parameter only on a theoretical level to demonstrate that a nonlinear system may be split into an infinite set of linear systems which are solved analytically, while the continuation methods require solving a discrete linear system as the homotopy parameter is varied to solve the nonlinear system.

In the last twenty years, the HAM has been applied to solve a growing number of nonlinear ordinary/partial differential equations in science, finance, and engineering. (Liao, S.J., 2012 ; Vajravelu, K., Van Gorder, 2013) For example, multiple steady-state resonant waves in deep and finite water depth (Xu, D.L., *et al*., 2012) were found with the wave resonance criterion of arbitrary number of traveling gravity waves; this agreed with Phillips' criterion for four waves with small amplitude. Further, a unified wave model applied with the HAM, (Liao, S.J., 2013) admits not only the traditional smooth progressive periodic/solitary waves, but also the progressive solitary waves with peaked crest in finite water depth. This model shows peaked solitary waves are consistent solutions along with the known smooth ones. Additionally, the HAM has been applied to many other nonlinear problems such as nonlinear heat transfer, (Abbasbandy, S., 2006) the limit cycle of nonlinear dynamic systems, (Chen, Y.M., J.K. Liu, 2009) the American put option, (Zhu, S.P., 2006) the exact Navier-Stokes

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equation, (Turkyilmazoglu, M., 2009) the option pricing under stochastic volatility, (Park, Sang-Hyeon; Kim, Jeong-Hoon, 2011) the electrohydrodynamic flows, (Mastroberardino, A., 2011) the Poisson–Boltzmann equation for semiconductor devices, (Nassar, Christopher, J. *et al*., 2011) and others.

Fractional Calculs:

Definition 1:

A real function $f(t),\ t>0$, is said to be in the space C_μ, $\mu \in R$, if there exists a real number $n>\mu$, such that $f(t)=t^n f_1(t)$, where $f_1(t)\in C[0,\infty)$ and it is said to be in the $C_\mu{}^k$ if and only if $f^{(k)} \in C_\mu$, $k \in N$.

Definition 2:

The Riemann-Liouville fractional integral operator of order or $\nu>0$ of a function $f \in C_\mu$, $\mu \ge -1$, is defined as (Oldham, Keith B., Spanier, Jerome, 1974; Miller, Kenneth S., Ross, Bertram, eds. 1993; Samko, S., *et al*., 1993 ; Carpinteri, A., F. Mainardi, eds., 1998; Podlubny, Igor, 1998)

$$I_a^\nu f(t)=\frac{1}{\Gamma(\nu)}\int_a^x (t-r)^{\nu-1} f(r)dr.$$

$$I_0^\alpha f(t)=I^\alpha f(t)\quad I^0 f(t)=f(t).$$

Remark:

The Riemann-Liouville integral operator has the following properties:

i) $I_a^\nu I_a^\gamma = I_a^{\nu+\gamma}$

ii) $I_a^\nu (t-b)^\beta = \dfrac{\Gamma(\beta+1)}{\Gamma(\nu+\beta+1)}(t-b)^{\nu+\beta}$

where $f \in C_\mu$, $\mu \ge -1$, $\nu,\gamma \ge 0$ and $\beta \ge -1$.

Definition 3:

The Caputo's fractional derivative of f is given by

$$D^\nu f(t)=\frac{1}{\Gamma(k-\nu)}\int_0^t (t-r)^{k-\nu-1} f^{(k)}(r)dr,$$

or

$$D^\nu f(t)=I^{k-\nu}D^k f(t).$$

where

$f \in C_{-1}{}^k$, $k-1<\nu\le k, k\in N, t\ge 0$.

Noting to above definitions, for $f \in C_\mu{}^k$, $\mu \ge -1$, $k-1<\nu\le k$, $k \in N$, and $t>0$, we have

i) $D_a^\nu I_a^\nu f(t)=f(t)$

ii)

$$I_a^\nu D_a^\nu f(t)=f(t)-\sum_{j=0}^{k-1} f^{(j)}(0^+)\frac{(t-a)^j}{j!}.$$

Definition 4:

The Mittage-Leffler function $E_\nu(z)$ for $\nu>0$ and $z\in C$ is defined by the series representation [19-23]

$$E_\nu(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\nu+1)},\qquad \nu>0,\qquad z\in C.$$

Homotopy Analysis Method:

Consider a general nonlinear differential equation

$$\mathcal{N}[u(x)]=0$$

where $\mathcal{N}$ is a nonlinear operator. Let $\mathcal{L}$ denote an auxiliary linear operator, $u_0(x)$ an initial guess of $u(x)$, and c_0 a constant (called the convergence-control parameter), respectively. Using the embedding parameter $q \in [0,1]$ from homotopy theory, one may construct a family of equations,

$$(1-q)\mathcal{L}[U(x;q)-u_0(x;q)]=c_0\,q\,\mathcal{N}[U(x;q)],$$

called the zeroth-order deformation equation, whose solution varies continuously with respect to the embedding parameter $q \in [0,1]$. This is the linear equation

$$\mathcal{L}[U(x;q)-u_0(x;q)]=0,$$

with known initial guess $U(x;0)=u_0(x)$ when $q=0$, but is equivalent to the original nonlinear equation $\mathcal{N}[u(x)]=0$, when $q=1$, i.e. $U(x;1)=u(x)$). Therefore, as q increases from 0 to 1, the solution $U(x;q)$ of the zeroth-order deformation equation varies (or deforms) from the chosen initial guess $u_0(x)$ to the solution $u(x)$ of the considered nonlinear equation.

Expanding $U(x;q)$ in a Taylor series about $q=0$, we have the homotopy-Maclaurin series

$$U(x;q)=u_0(x)+\sum_{m=1}^{\infty} u_m(x)\,q^m.$$

Assuming that the so-called convergence-control parameter c_0 of the zeroth-order deformation equation is properly chosen that the above series is convergent at $q = 1$, we have the homotopy-series solution

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x).$$

From the zeroth-order deformation equation, one can directly derive the governing equation of $u_m(x)$

$$\mathcal{L}[u_m(x) - \chi_m u_{m-1}(x)] = c_0\, R_m[u_0, u_1, \cdots, u_{m-1}],$$

called the m^{th}-order deformation equation, where $\chi_1 = 0$ and $\chi_k = 1$ for $k > 1$, and the right-hand side R_m is dependent only upon the known results u_0, u_1, ..., u_{m-1} and can be obtained easily using computer algebra software. In this way, the original nonlinear equation is transferred into an infinite number of linear ones, but without the assumption of any small/large physical parameters.

Since the HAM is based on a homotopy, one has great freedom to choose the initial guess $u_0(x)$, the auxiliary linear operator $\mathcal{L}$, and the convergence-control parameter c_0 in the zeroth-order deformation equation. Thus, the HAM provides the mathematician freedom to choose the equation-type of the high-order deformation equation and the base functions of its solution. The optimal value of the convergence-control parameter c_0 is determined by the minimum of the squared residual error of governing equations and/or boundary conditions after the general form has been solved for the chosen initial guess and linear operator. Thus, the convergence-control parameter c_0 is a simple way to guarantee the convergence of the homotopy series solution and differentiates the HAM from other analytic approximation methods. The method overall gives a useful generalization of the concept of homotopy.

4) Solution of Extraordinary Fractional Differential Equations Via HAM:

$$D^p y(t) + \lambda D^\alpha y(t) = ty \quad \text{where} \quad p = 1,2 \quad and\ 0 < \alpha < 1 \tag{1}$$
$$y(0) = k_0$$
$$y'(0) = k_1$$

$$y_k(t) = h[D^p y_{k-1} + \lambda D^\alpha y_{k-1} - t y_{k-1}] \tag{2}$$

Applying the operator J^α, the inverse operator of D^α, on both sides of (2), we obtain

$$y_k = \chi_k y_k(t) + h[y_{k-1} + \lambda D^{\alpha-p} y_{k-1} - D^{-p} t y_{k-1}], \quad k \geq 1 \tag{3}$$

Consequently ,the first few terms of the HAM series solution are as followes:

$$y_1 = \chi_1 y_0(t) + h[y_0 + \lambda D^{\alpha-p} y_0 - D^{-p} t y_0]$$

$$y_1 = h(1 + \lambda D^{\alpha-p} - D^{-p} t) y_0$$

$$y_2 = \chi_2 y_1(t) + h(1 + \lambda D^{\alpha-p} - D^{-p} t) y_1$$

$$y_2 = h(1 + \lambda D^{\alpha-p} - D^{-p} t) y_0 + h^2(1 + \lambda D^{\alpha-p} - D^{-p} t)(1 + \lambda D^{\alpha-p} - D^{-p} t) y_0$$

$$y_2 = h\big(1 + h(1 + \lambda D^{\alpha-p} - D^{-p} t)\big)(1 + \lambda D^{\alpha-p} - D^{-p} t) y_0$$

$$y_3 = \chi_3 y_2(t) + h(1 + \lambda D^{\alpha-p} - D^{-p} t) y_2$$

$$y_3 = h\big(1 + h(1 + \lambda D^{\alpha-p} - D^{-p} t)\big)(1 + \lambda D^{\alpha-p} - D^{-p} t) y_0$$
$$+ h^2\big(1 + h(1 + \lambda D^{\alpha-p} - D^{-p} t)\big)(1 + \lambda D^{\alpha-p} - D^{-p} t)^2 y_0$$

$$y_3 = h(1 + \lambda D^{\alpha-p} - D^{-p} t) y_0 + h^2(1 + \lambda D^{\alpha-p} - D^{-p} t)^2 y_0$$
$$+ h^2(1 + \lambda D^{\alpha-p} - D^{-p} t)^2 y_0 + h^3(1 + \lambda D^{\alpha-p} - D^{-p} t)^3 y_0$$

$$y_3 = h\left(1 + h(1 + \lambda D^{\alpha-p} - D^{-p}t)\right)^2 (1 + \lambda D^{\alpha-p} - D^{-p}t) y_0$$

$$y_3 = h\left(1 + h(1 + \lambda D^{\alpha-p} - D^{-p}t)\right)^3 (1 + \lambda D^{\alpha-p} - D^{-p}t) y_0$$

$$y_4 = h\left(1 + h(1 + \lambda D^{\alpha-p} - D^{-p}t)\right)^4 (1 + \lambda D^{\alpha-p} - D^{-p}t) y_0$$

$$y_5 = h\left(1 + h(1 + \lambda D^{\alpha-p} - D^{-p}t)\right)^4 (1 + \lambda D^{\alpha-p} - D^{-p}t) y_0$$

And so on. Hence, the HAM series solution is

$$y(t) = y_0 + y_1 + y_2 + y_3 + \cdots$$

$$y(t) = y_0 + h(1 + \lambda D^{\alpha-p} - D^{-p}t) y_0 + h\left(1 + h(1 + \lambda D^{\alpha-p} - D^{-p}t)\right)$$

$$(1 + \lambda D^{\alpha-p} - D^{-p}t) y_0 + h\left(1 + h(1 + \lambda D^{\alpha-p} - D^{-p}t)\right)^2 (1 + \lambda D^{\alpha-p} - D^{-p}t) y_0 + \cdots$$

$$y(t) = \sum_{i=0}^{1} k_i t^i + \sum_{n=1}^{\infty} h\left(1 + h(1 + \lambda D^{\alpha-p} - D^{-p}t)\right)^{n-1} (1 + \lambda D^{\alpha-p} - D^{-p}t) \sum_{i=0}^{1} k_i t^i$$

If we take h= -1, we get

$$y(t) = \sum_{i=0}^{1} k_i t^i + \sum_{n=1}^{\infty} (-1)^n (\lambda D^{\alpha-p} + D^{-p}t)^{n-1} (1 + \lambda D^{\alpha-p} - D^{-p}t) \sum_{i=0}^{1} k_i t^i$$

5)Numerical Results:

Example1:

Consider the following non-homogeneous fractional differential equation

$$Dy(t) + D^{\frac{1}{2}} y(t) = ty$$

$$y_0 = y(0) = 0$$

$$y_1 = -1 - \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} + \frac{1}{2} t^2$$

$$y_2 = \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} + t - \frac{8}{15\sqrt{\pi}} t^{\frac{5}{2}} - \frac{1}{2} t^2 - \frac{1}{\sqrt{\pi}} t^{\frac{7}{2}} + \frac{1}{4} t^4$$

And so on. Hence, the HAM series solution is

$$y(t) = y_0 + y_1 + y_2 + y_3 + \cdots$$

If we take h= -1, we get

$$y(t) = \exp\left(\frac{8}{3\sqrt{\pi}} t^2 - \sqrt{t}\right)^2 \operatorname{erf}\left(\sqrt{t} - \frac{8}{3\sqrt{\pi}} t^2\right)$$

Conclusion:

Homotopy analysis method has been known as a powerful scheme for solving many functional equations such as algebraic equations, ordinary and partial differential equations, integral equations and so on.The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approxi-mations which a rerapidly convergent to the exact solution. In this work, the homotopy analysis method has been successfully applied to fractional differential equations. In the example 1, the solution are the same as those results given by Adomian decomposition method (Samir Hadid and Ahcene Merad).

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