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Local Fourier Analysis of Vanka Smoother Based Multigrid for Staggered Discretization of Biot's Consolidation Problem

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ABSTRACT

Numerical simulation is increasingly prominent in the field of continuum mechanics, and a very important aspect is the efficient resolution of the resulting systems after the discretization of the models. Biot's models are widely used nowadays in a great variety of fields. They lead to computationally complex problems for which traditional simulations become too expensive. Therefore, fast numerical algorithms have to be designed for their solution.Here, we deal with the design of monolithic multigrid solvers for the highly efficient solution of the resulting algebraic systems, by using the local Fourier analysis. Special smoothers suitable for saddle point problems, as the Vanka type relaxations, are analyzed by using this technique.A special local Fourier analysis strategy is proposed for the study of the Vanka smoothers, giving rise to a very accurate tool to predict the asymptotic convergence factors of the resulting multigrid method. By using this approach, an efficient multigrid solver is obtained for the solution of the quasi-static Biot's model for soil consolidation.

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1. *Introduction To Biot's Consolidation Model:*

Poroelasticity theory mathematically describes the interaction between the deformation of an elastic porous material and the fluid flow inside of it. Maurice Biot, known as the founder of the theory of poroelasticity, established the three-dimensional mathematical formulation in the forties, see for example Biot (1941), and nowadays, the analysis and numerical simulation of Biot's models has become a trend topic due to their large variety of applications, ranging from geomechanics and petrol engineering to biomechanics or even food processing more recently.

Here, we consider the classical quasi-static Biot's model for soil consolidation, which can be formulated as a coupled system of partial differential equations for the unknowns displacement and pressure. Since we restrict ourselves to the two-dimensional case, for simplicity in the presentation, we will denote by u the horizontal displacement of the solid matrix, by v the vertical displacement, and by p the pore pressure of the fluid. Following this notation, the governing equations for a homogeneous, isotropic and incompressible medium read

$$\begin{aligned} -(\lambda+2\mu)u_{xx} - \mu\, u_{yy} - (\lambda+\mu)v_{xy} + p_x &= 0, \\ -(\lambda+\mu)u_{xy} - \mu\, v_{xx} - (\lambda+2\mu)\, v_{yy} + p_y &= 0, \\ (u_x + v_y)_t - k\,\Delta p &= f, \end{aligned} \tag{1}$$

where λ and μ are the so-called Lamé coefficients, and k = κ/η is the hydraulic conductivity, which is given as the quotient between the permeability of the porous medium, κ, and the viscosity of the fluid, η. The divergence of the displacements, appearing in the third equation, represents the volume increase rate of the system, that is, a measure of the change in porosity of the soil, and at the initial time t=0, it is assumed to be equal to zero.

It is well-known that for materials with low permeability and/or when small time steps are considered, strong non-physical oscillations can appear in the pressure approximation field of the numerical solution if the discretization of the system is not carefully chosen, see for example Gaspar *et al* (2003), Aguilar *et al* (2008), Ferronato *et al* (2010), Haga *et al* (2012), Favino *et al* (2013) and Phillips *et al* (2009). Different strategies have been proposed by several authors to overcome these difficulties and to obtain oscillation-free solutions independently of the space and time discretization parameters. Among these, here we consider a staggered grid discretization that leads to a finite difference scheme which mimic the properties of the continuous problem, see Gaspar *et al* (2006).

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The considered staggered grid is displayed in Figure 1. The pressure unknowns are located at the vertices of the grid, whereas the horizontal and vertical displacement unknowns are placed at the midpoints of the horizontal and vertical edges, respectively.

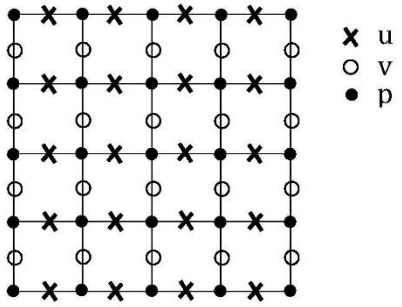


Fig. 1: Staggered grid for displacement and pressure unknowns.

For the time discretization we use an implicit Euler scheme and then, on each time step the following stationary discrete operator is considered

$$\mathbf{L}_h = \begin{bmatrix} L_h^{uu} & L_h^{uv} & L_h^{up} \\ L_h^{vu} & L_h^{vv} & L_h^{vp} \\ L_h^{pu} & L_h^{pv} & L_h^{pp} \end{bmatrix} = \begin{bmatrix} -(\lambda+2\mu)\partial_{xx} - \mu\,\partial_{yy} & -(\lambda+\mu)\partial_{xy} & \partial_x \\ -(\lambda+\mu)\partial_{xy} & -\mu\,\partial_{xx} - (\lambda+2\mu)\partial_{yy} & \partial_y \\ \partial_x & \partial_y & -k\,(\partial_{xx}+\partial_{yy}) \end{bmatrix}, \tag{2}$$

where the involved scalar operators are given as follows

$$-\partial_{xx} = \frac{1}{h^2}[-1 \quad 2 \quad -1], \quad -\partial_{yy} = \frac{1}{h^2}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad -\partial_{xy} = \frac{1}{h^2}\begin{bmatrix} 1 & * & -1 \\ * & * & * \\ -1 & * & 1 \end{bmatrix},$$

$$\partial_x = \frac{1}{h}[-1 \quad * \quad 1], \qquad \partial_y = \frac{1}{h}\begin{bmatrix} 1 \\ * \\ -1 \end{bmatrix}.$$

This discretization, apart from avoiding the non-physical oscillations, can easily be combined with fast solvers like geometric multigrid methods, as we will see in Section 2, where a monolithic multigrid solver based on Vanka type relaxation is proposed for the solution of the resulting algebraic system of equations. The proposed multigrid method will be analyzed by using the local Fourier analysis technique in Section 3. Also in this section, a special analysis will be developed for the study of the Vankatype smoothers considered here. Finally, in Section 4, the results from the LFA will be compared with the asymptotic convergence factors of the real multigrid code, showing a very accurate match between the theoretical and the practical results.

2. Vanka Smoother Based Multigrid For Poroelasticity:

Since their development in the 70's, multigrid methods (see Hackbusch (1985), Stüben *et al* (1982) and Trottenberg *et al* (2001), for example) have been proved to be among the most efficient numerical algorithms for solving the large sparse systems of equations arising from the discretization of partial differential equations, achieving asymptotically optimal complexity at least for elliptic problems. These methods are iterative solvers mainly based on the acceleration of the convergence of classical iterative methods by using solutions obtained on coarser meshes as corrections. Two ideas are involved in the development of multigrid methods: the first one is the fact that some classical iterative methods have a strong smoothing effect on the components of the error corresponding to the high frequencies (high oscillating error components), and the second one is that a smooth error can be well represented on a coarser grid. These observations suggest the following structure of a two-grid cycle:

1. Perform ν_1 iterations of an iterative relaxation method S_h on the fine grid (pre-smoothing)
2. Compute the defect of the current fine grid approximation
3. Restrict the defect to the coarse grid by using a restriction operator $R_{h,2h}$
4. Solve the coarse grid defect equation
5. Interpolate the correction to the fine grid using a prolongation operator $P_{2h,h}$
6. Add the interpolated correction to the current fine grid approximation
7. Perform ν_2 iterations of an iterative relaxation method S_h on the fine grid (post-smoothing)

Following this algorithm, the two-grid error transformation operator is given by

$$M_{h,2h} = S_h^{\nu_2}\left(I_h - P_{2h,h}(L_{2h})^{-1}R_{h,2h}L_h\right)S_h^{\nu_1}, \tag{3}$$

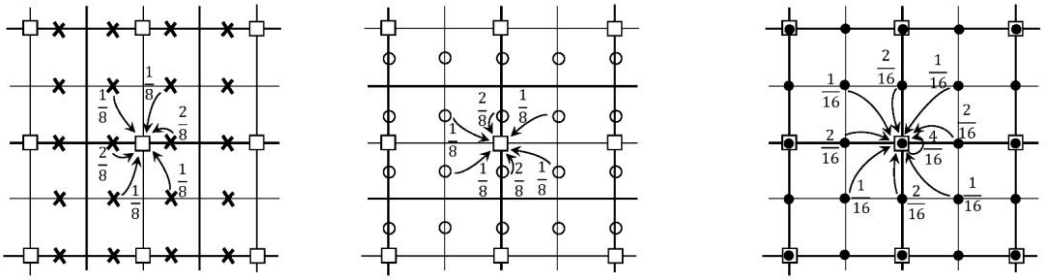
where I_h denotes the identity and the subscript "2h" indicates that the coarse grid is obtained by doubling the mesh size in each space direction, which is called "standard coarsening". Instead of inverting L_{2h}, the coarse-grid equation can be solved by recursive application of this procedure, yielding a multigrid method.

From the previous algorithm, it is clear that many details are open for discussion and decision, since all the components have to be properly chosen. It is well-known that the performance of multigrid methods strongly depends on the choice of their components. There are no rules to facilitate this demanding task, but we will see that the local Fourier analysis is very useful to the design of the algorithm.

Regarding the coarse-grid correction part of the algorithm, geometric inter-grid transfer operators, dictated by the staggered location of the unknowns, are considered here. This results in 6-point restriction operators for the displacements and the full-weighting restriction operator for the pressure unknowns, as seen in Figure 2. As the prolongation operators we use the common bilinear interpolation of neighboring coarse-grid unknowns in the staggered grid. Finally, on the coarse grids, we apply direct coarse-grid discretization.

On the other hand, the efficiency and robustness of a multigrid algorithm is strongly influenced by the smoother. Moreover, for the problem we are dealing

with, an additional difficulty appears, since it results in a system of saddle point type aspect. An overview of multigrid methods for discretizations on rectangular grids of this type of problems is presented in Oosterlee *et al* (2008), where coupled or box-relaxation and decoupled distributive relaxation methods appear as the most suitable for this kind of problems. Due to the fact that for some systems of equations it is a challenge to design an efficient distributive relaxation scheme, box-relaxation seems to be the best option.It consists of decomposing the mesh into small subdomains and treating them separately, that is, all the equations corresponding to the points in each subdomain are solved simultaneously as a system. Therefore, one relaxation step consists of a loop over all subdomains, solving for each one the system arising from the corresponding equations. Many variants of box-type smoothers can be considered, they can differ in the choice of the subdomains which are solved simultaneously, and in the way in which the local systems to be solved are built.



Restriction for u Restriction for v Restriction for p

Fig. 2: Six-point restriction operators for horizontal and vertical displacement unknowns, $u$and v, and full-weighting restriction for the pressure unknowns.

This class of smoothers was introduced by Vanka(1986) to solve the finite difference discretization on rectangular grids of the Navier-Stokes equations. Since then, much literature can be found about the application of this type of smoothers, mainly in the field of Computational Fluid Dynamics (John (1999), John *et al* (2000) and Turek (1999)), but also in the context of Computational Solid Mechanics, for example in Wobker *et al* (2009). For the staggered-grid discretization of the poroelasticity problem considered here, a point-wise Vanka smoother was proposed in Gaspar (2004). It is based on simultaneously updating all unknowns appearing in the discrete divergence operator in the pressure equation. This way of building the blocks is very common in box-relaxations used for Stokes and Navier-Stokes problems. This approach implies that four unknowns corresponding to displacements and one pressure unknown, see Figure 3, are relaxed simultaneously, making necessary to solve a 5 × 5 system for each grid point. Then, we iterate over all grid points in lexicographic order, and for each of them the corresponding box is solved.

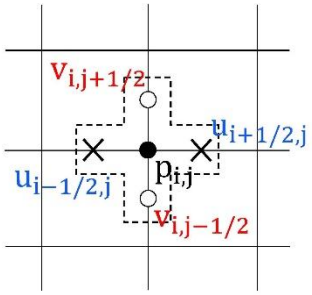


Fig. 3: Unknowns coupled in the Vanka smoother.

In this full variant, all the unknowns in the system are considered coupled. Therefore, the need of solving such systems makes these smoothers expensive. A simplified variant can be considered by coupling each displacement unknown in the system only with itself and with the corresponding pressure unknown. In this way, the solution of the resulting systems becomes much cheaper, making the use of this diagonal version very appealing in practice.However, it is observed that the diagonal version of the point-box smoother can be less robust with respect to some applications, so this is something that we will need to study in the following sections.

3. Local Fourier Analysis:

The main tool to quantitatively analyze the convergence properties of a multigrid algorithm is the local mode analysis or local Fourier analysis, introduced by Achi Brandt in Brandt (1977) and

developedfurther in several papers (see Brandt (1984) and Brandt (1994)). A good introduction to this analysis can be foundin the booksTrottenberg *et al* (2001), Stüben *et al* (1982) and Wesseling (1992), and the LFA monograph Wienands*et al* (2005).This analysis is based on a Fourier decomposition of the error function in eigenvectors of the discretization of the differentialoperator, and the study of the behavior of each operator involved in the multigrid methodon these components. Local Fourier analysis assumes that all the operations involved in the multigrid algorithm are local processes, that is, the operations performed on each unknown only dependon the information on nearby neighbors, what allows to neglect the effect of boundaryconditions. By imposing some assumptions on the discrete operator: linear differentialoperators with constant coefficients, a basis of complex exponential eigenfunctions of the operator, called Fourier components,can be obtained. Summarizing, this analysis mainly simplifies the computation of the spectral radius of the k-grid iteration matrix, by considering the matrix representation of this operator with respect to such basis of eigenfunctions, which results in a block-diagonal matrix.

To perform this analysis, we first introduce the infinite two-dimensional staggered grid employed in this work. In order to do this, we notice that it is composed of three types of grid-points on which we can decompose the global grid as $G_h = G_h^1 \cup G_h^2 \cup G_h^3$, where
$G_h^j = \{(x_h^j, y_h^j) = (k_1, k_2)h + (\delta_1, \delta_2); k_1, k_2 \in \mathbb{Z}\}$,

with (δ_1, δ_2) differently defined on the three types of nodes composing the staggered grid, that is,

- $(\delta_1, \delta_2) = \left(\frac{h}{2}, 0\right)$ for G_h^1 (grid-points for the horizontal displacement unknowns),
- $(\delta_1, \delta_2) = \left(0, \frac{h}{2}\right)$ for G_h^2 (grid-points for the vertical displacement unknowns),
- $(\delta_1, \delta_2) = (0, 0)forG_h^3$ (grid-points for the pressure unknowns).

This definition of the infinite grid makes necessary to define suitable vector-grid functions as Fourier components, yielding a more involved local Fourier analysis, different to the standard case for collocated grids on which all the unknowns are at the same locations. In this way, it is fulfilled that the following grid-functions

$$\boldsymbol{\varphi}_h(\boldsymbol{\theta}, \boldsymbol{x}_h) := \begin{pmatrix} e^{i\boldsymbol{\theta} x_h^1/h} \\ e^{i\boldsymbol{\theta} x_h^2/h} \\ e^{i\boldsymbol{\theta} x_h^3/h} \end{pmatrix}$$

with $\boldsymbol{\theta} = (\theta_1, \theta_2) \in (-\pi, \pi]^2$ and $\boldsymbol{x}_h = (x_h^1, x_h^2, x_h^3), x_h^j \in G_h^j$, form a unitary basis of vector-valued Fourier modes, yielding the so-called Fourier space
$\mathcal{F}(G_h) := \text{span}\{\boldsymbol{\varphi}_h(\boldsymbol{\theta}, \boldsymbol{x}_h); \boldsymbol{\theta} \in (-\pi, \pi]^2\}$.

By considering vector operators $\mathbf{L}_h$defined on the infinite grid and satisfying the assumptions of the LFA, it is fulfilled that the Fourier modes $\boldsymbol{\varphi}_h(\boldsymbol{\theta}, \boldsymbol{x}_h)$ are formal eigenfunctionsof $\mathbf{L}_h$. More precisely, the following relation reads

$$\mathbf{L}_h \boldsymbol{\varphi}_h(\boldsymbol{\theta}, \boldsymbol{x}_h) = \widetilde{\mathbf{L}_h}(\boldsymbol{\theta}) \boldsymbol{\varphi}_h(\boldsymbol{\theta}, \boldsymbol{x}_h) \quad (4)$$

where$\widetilde{\mathbf{L}_h}(\boldsymbol{\theta})$is the so-called symbol of operator $\mathbf{L}_h$, which denotes the representation of the discrete operator on the Fourier space. In particular, the Fourier symbol of discrete operator given in (2) can be easily computed by obtaining the Fourier symbols of the scalar operators involved, yielding

$$\widetilde{\mathbf{L}_h}(\boldsymbol{\theta}) = \begin{bmatrix} \widetilde{L_h^{uu}}(\boldsymbol{\theta}) & \widetilde{L_h^{uv}}(\boldsymbol{\theta}) & \widetilde{L_h^{up}}(\boldsymbol{\theta}) \\ \widetilde{L_h^{vu}}(\boldsymbol{\theta}) & \widetilde{L_h^{vv}}(\boldsymbol{\theta}) & \widetilde{L_h^{vp}}(\boldsymbol{\theta}) \\ \widetilde{L_h^{pu}}(\boldsymbol{\theta}) & \widetilde{L_h^{pv}}(\boldsymbol{\theta}) & \widetilde{L_h^{pp}}(\boldsymbol{\theta}) \end{bmatrix} = \begin{bmatrix} (\lambda + 2\mu)s_1^2 + \mu s_2^2 & -(\lambda + \mu)s_1 s_2 & i s_1 \\ -(\lambda + \mu)s_1 s_2 & \mu s_1^2 + (\lambda + 2\mu)s_2^2 & i s_2 \\ i s_1 & i s_2 & k(s_1^2 + s_2^2) \end{bmatrix},$$

where $\begin{cases} s_1 = \frac{2}{h}\sin\left(\frac{\theta_1}{2}\right) \\ s_2 = \frac{2}{h}\sin\left(\frac{\theta_2}{2}\right) \end{cases}$.

Most classical iterative methods can be expressed by means of a splitting of the operator $\mathbf{L}_h$ of the form $\mathbf{L}_h = \mathbf{L}_h^+ + \mathbf{L}_h^-$,where $\mathbf{L}_h^-$ relates the part of the operator corresponding to the unknowns which have been relaxed before the current approximation, and $\mathbf{L}_h^+$to those that are going to be updated in the current or in the following steps. In this way, the iteration matrix of the smoothing operator is given by$\mathbf{S_h} = -(\mathbf{L}_h^+)^{-1}\mathbf{L}_h^-$, and it is easy to prove that the Fourier modes are also its eigenvectors, satisfying a relation like in (4), with its Fourier symbol denoted as $\widetilde{\mathbf{S_h}}(\boldsymbol{\theta})$.

However, the overlapping block smoothers considered here do not come from such a decomposition of the discrete operator. The distinction with respect to other smoothers is that they update some variable more than once, due to the overlapping of the local subdomains which are simultaneously solved, and this fact has to be taken into account in the analysis because it causes that some intermediate errors appear apartfrom the initial and final errors. For this reason, Vanka smoothers require a special strategy to perform the local Fourier analysis. To our knowledge, there are only few papers dealing with LFA for multiplicative overlapping smoothers, see Sivaloganathan (1991), Molenaar (1991) andMaclachlan *et al* (2011) for discretizations on rectangular grids, and Rodrigo *et al* (2015) for triangular grids. Here, we apply this

analysis to the previously introduced Vanka smoother for the staggered grid discretization of the poroelasticity problem.

As shown in Figure 2, the considered Vanka smoother simultaneously solve the equations corresponding to unknowns $p_{i,j}$, $u_{i+1/2,j}$, $u_{i-1/2,j}$, $v_{i,j+1/2}$, $v_{i,j-1/2}$. Therefore, due to the overlapping of such blocks, the pressure unknowns are updated only once whereas the displacement unknowns are updated twice per relaxation step. More concretely, the states of the unknowns involved in the block are that $p_{i,j}$, $u_{i+1/2,j}$, and $v_{i,j+1/2}$ have not been updated yet and therefore will be updated for first time, whereas $u_{i-1/2,j}$ and $v_{i,j-1/2}$ have been already updated once and then will be completely updated after the relaxation of the block.

We can write the system to solve on each grid-point in terms of corrections and residuals as follows

$$\begin{pmatrix} 2\left(\frac{\lambda+3\mu}{h^2}\right) & \frac{\lambda+\mu}{h^2} & -\frac{\lambda+2\mu}{h^2} & -\frac{\lambda+\mu}{h^2} & -\frac{1}{h} \\ \frac{\lambda+\mu}{h^2} & 2\left(\frac{\lambda+3\mu}{h^2}\right) & -\frac{\lambda+\mu}{h^2} & -\frac{\lambda+2\mu}{h^2} & -\frac{1}{h} \\ -\frac{\lambda+2\mu}{h^2} & -\frac{\lambda+\mu}{h^2} & 2\left(\frac{\lambda+3\mu}{h^2}\right) & \frac{\lambda+\mu}{h^2} & \frac{1}{h} \\ -\frac{\lambda+\mu}{h^2} & -\frac{\lambda+2\mu}{h^2} & \frac{\lambda+\mu}{h^2} & 2\left(\frac{\lambda+3\mu}{h^2}\right) & \frac{1}{h} \\ \frac{1}{h} & \frac{1}{h} & -\frac{1}{h} & -\frac{1}{h} & \frac{4k}{h^2} \end{pmatrix} \begin{pmatrix} \delta u_{i+1/2,j} \\ \delta v_{i,j+1/2} \\ \delta u_{i-1/2,j} \\ \delta v_{i,j-1/2} \\ \delta p_{i,j} \end{pmatrix} = \begin{pmatrix} r^u_{i+\frac{1}{2},j} \\ r^v_{i,j+\frac{1}{2}} \\ r^u_{i-\frac{1}{2},j} \\ r^v_{i,j-\frac{1}{2}} \\ r^p_{i,j} \end{pmatrix}, \quad (5)$$

where

$$\begin{aligned} \delta u_{i+1/2,j} &= e_h^{u,j+\frac{1}{2}}(x^1_{i+1/2,j}) - e_h^{u,j}(x^1_{i+1/2,j}) \\ \delta v_{i,j+1/2} &= e_h^{v,j+\frac{1}{2}}(x^2_{i,j+1/2}) - e_h^{v,j}(x^2_{i,j+1/2}) \\ \delta u_{i-1/2,j} &= e_h^{u,j+1}(x^1_{i-1/2,j}) - e_h^{u,j+\frac{1}{2}}(x^1_{i-1/2,j}) \\ \delta v_{i,j-1/2} &= e_h^{v,j+1}(x^2_{i,j-1/2}) - e_h^{v,j+\frac{1}{2}}(x^2_{i,j-1/2}) \\ \delta p_{i,j} &= e_h^{p,j+1}(x^3_{i,j}) - e_h^{p,j}(x^3_{i,j}) \end{aligned} \quad (6)$$

being $e_h^{u,j}$, $e_h^{v,j}$ and $e_h^{p,j}$ the initial errors at j-iteration for u, v, and p, respectively, $e_h^{u,j+1}$, $e_h^{v,j+1}$ and $e_h^{p,j+1}$ the final errors, and $e_h^{u,j+\frac{1}{2}}$ and $e_h^{v,j+\frac{1}{2}}$ the intermediate errors that appear once that u or v have just been updated the first time. As stated the LFA assumptions, errors can be written as a formal linear combination of Fourier modes. Then, without loss of generality, we can write the initial, intermediate and final errors as a single Fourier mode multiplied by a coefficient $\alpha_u^{(k)}(\boldsymbol{\theta})$, $\alpha_v^{(k)}(\boldsymbol{\theta})$, or $\alpha_p^{(k)}(\boldsymbol{\theta})$, where the superscript$(k)$denotes that the variable (indicated in the subscript) has been updated k-times in this iteration (k = 0,1,2). In this way, we can write the corrections appearing in (6) and the residuals in the right-hand side of (5) in terms of these coefficients, and we can rewrite system (5) as a system for the updated Fourier coefficients, which can be written in terms of the non-updated ones:

$$P\begin{pmatrix} \alpha_u^{(1)}(\boldsymbol{\theta}) \\ \alpha_v^{(1)}(\boldsymbol{\theta}) \\ \alpha_u^{(2)}(\boldsymbol{\theta}) \\ \alpha_v^{(2)}(\boldsymbol{\theta}) \\ \alpha_p^{(1)}(\boldsymbol{\theta}) \end{pmatrix} = Q\begin{pmatrix} \alpha_u^{(0)}(\boldsymbol{\theta}) \\ \alpha_v^{(0)}(\boldsymbol{\theta}) \\ \alpha_p^{(0)}(\boldsymbol{\theta}) \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha_u^{(1)}(\boldsymbol{\theta}) \\ \alpha_v^{(1)}(\boldsymbol{\theta}) \\ \alpha_u^{(2)}(\boldsymbol{\theta}) \\ \alpha_v^{(2)}(\boldsymbol{\theta}) \\ \alpha_p^{(1)}(\boldsymbol{\theta}) \end{pmatrix} = (P^{-1}Q)\begin{pmatrix} \alpha_u^{(0)}(\boldsymbol{\theta}) \\ \alpha_v^{(0)}(\boldsymbol{\theta}) \\ \alpha_p^{(0)}(\boldsymbol{\theta}) \end{pmatrix}.$$

In this way, by taking the bottom 3×3 block of $(P^{-1}Q)$, we obtain the relation between the initial and fully corrected errors given by $\widetilde{\mathbf{S_h}}(\boldsymbol{\theta}) = (P^{-1}Q)_{\{3:5,1:3\}}$, that is,

$$\begin{pmatrix} \alpha_u^{(2)}(\boldsymbol{\theta}) \\ \alpha_v^{(2)}(\boldsymbol{\theta}) \\ \alpha_p^{(1)}(\boldsymbol{\theta}) \end{pmatrix} = \widetilde{\mathbf{S_h}}(\boldsymbol{\theta})\begin{pmatrix} \alpha_u^{(0)}(\boldsymbol{\theta}) \\ \alpha_v^{(0)}(\boldsymbol{\theta}) \\ \alpha_p^{(0)}(\boldsymbol{\theta}) \end{pmatrix}.$$

Once we have computed the Fourier representation of the smoother, we can perform any of the variants of the local Fourier analysis. Notice that the only change necessary to perform the LFA for the diagonal version of the Vanka smoother is when we build system (5), because the first 4×4 block of the coefficient matrix would become diagonal since the displacement unknowns are assumed to be only coupled with the pressure unknown but not with the rest of displacements anymore.

As previously commented one of the basis of multigrid methods is the so-called smoothing property of some iterative methods, that is, how

efficiently they reduce the high oscillating components of the error. This property is measured by the so-called smoothing factor, which is defined as the factor by which the oscillatory modes are damped, at worst, with each relaxation sweep. In order to investigate the influence of a smoothing operator on the high-frequency error components, Fourier smoothing analysis is used, and for this, low- and high-frequency Fourier components have to be distinguished. This distinction depends on the coarsening strategy under consideration, since it is based on the fact that some Fourier components are "not visible" on the coarsest grid. As stated before, here standard coarsening is considered, and therefore the subset of low frequencies is defined as $\left(-\frac{\pi}{2},\frac{\pi}{2}\right]^2$, whereas the subset of high frequencies is given by $(-\pi,\pi]^2 \setminus \left(-\frac{\pi}{2},\frac{\pi}{2}\right]^2$. More concretely, each low frequency $\boldsymbol{\theta}^{\mathbf{00}} = (\boldsymbol{\theta}_1^{\mathbf{00}}, \boldsymbol{\theta}_2^{\mathbf{00}}) \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right]^2$ coincides on the coarse grid with three high frequencies $\boldsymbol{\theta}^{\mathbf{10}}, \boldsymbol{\theta}^{\mathbf{01}}, \boldsymbol{\theta}^{\mathbf{11}}$, given by

$\boldsymbol{\theta}^{\mathbf{10}} = \boldsymbol{\theta}^{\mathbf{00}} - (sign(\theta_1^{00})\pi, 0)$,
$\boldsymbol{\theta}^{\mathbf{01}} = \boldsymbol{\theta}^{\mathbf{00}} - (0, sign(\theta_2^{00})\pi)$,
$\boldsymbol{\theta}^{\mathbf{11}} = \boldsymbol{\theta}^{\mathbf{00}} - (sign(\theta_1^{00})\pi, sign(\theta_2^{00})\pi)$.

This coupling gives rise to the definition of the subspaces of 2h-harmonics, which are generated by the Fourier modes associated with coupled frequencies as follows

$$\mathcal{F}^4(\boldsymbol{\theta}^{\mathbf{00}}) := \text{span}\{\boldsymbol{\varphi}_h(\boldsymbol{\theta}^{\mathbf{00}},\cdot), \boldsymbol{\varphi}_h(\boldsymbol{\theta}^{\mathbf{11}},\cdot), \boldsymbol{\varphi}_h(\boldsymbol{\theta}^{\mathbf{10}},\cdot), \boldsymbol{\varphi}_h(\boldsymbol{\theta}^{\mathbf{01}},\cdot)\}, \boldsymbol{\theta}^{\mathbf{00}} \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right]^2,$$

Which form a decomposition of the Fourier space into four-dimensional subspaces.

After this distinction between low and high frequencies, we can define the smoothing factor after ν iterations in the following way

$$\mu^{\nu} = \sup_{\boldsymbol{\theta}\in(-\pi,\pi]^2\setminus\left(-\frac{\pi}{2},\frac{\pi}{2}\right]^2} |\rho(\widetilde{\mathbf{S}_h}^{\nu}(\boldsymbol{\theta}))|.$$

To get more insight into the interplay between the relaxation and the coarse-grid correction, including the effect of inter-grid transfer operators in the analysis, we can perform a two-grid local Fourier analysis, which is based on the study of how the two-grid operator given in (3) acts on the Fourier components. The key for this analysis is that the 2h-harmonic subspaces remain invariant under the application of $M_{h,2h}$, and therefore we can obtain its block representation on the Fourier space. In this way, the symbol of $M_{h,2h}$, can be computed by using the Fourier representations of the involved operators as follows

$$\widetilde{M_{h,2h}}(\boldsymbol{\theta}^{\mathbf{00}}) = \widetilde{S_h^{\nu_2}}(\boldsymbol{\theta}^{\mathbf{00}})\left(\widetilde{I_h}(\boldsymbol{\theta}^{\mathbf{00}}) - \widetilde{P_{2h,h}}(\boldsymbol{\theta}^{\mathbf{00}})\left(\widetilde{L_{2h}}(\boldsymbol{\theta}^{\mathbf{00}})\right)^{-1}\widetilde{R_{h,2h}}(\boldsymbol{\theta}^{\mathbf{00}})\widetilde{L_h}(\boldsymbol{\theta}^{\mathbf{00}})\right)\widetilde{S_h^{\nu_1}}(\boldsymbol{\theta}^{\mathbf{00}}),$$

where $\boldsymbol{\theta}^{\mathbf{00}} \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right]^2$, becoming a block-diagonal matrix, consisting of $(4n\times 4n)$ −blocks (being n the number of variables of our problem). As a consequence, the spectral radius of the two-grid operator, that indicates the asymptotic convergence factor of the two-grid method, can be computed by means of the spectral radius of $(4n\times 4n)$ −matrices, and it is given as

$$\rho_{2g} = \rho\left(M_{h,2h}\right) = \sup_{\boldsymbol{\theta}^{\mathbf{00}}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right]^2} \rho(\widetilde{M_{h,2h}}(\boldsymbol{\theta}^{\mathbf{00}})).$$

4. *Numerical Results:*

This section is devoted to show the results from the developed local Fourier analysis in order to illustrate its accuracy and utility, as well as to show the robustness and efficiency of the considered Vanka smoothers.

For all the presented results, we will consider some realistic values for the Lamé parameters: $\lambda = 12500$ and $\mu = 8333$, and we will let parameter k to vary in some of the tests to see the robustness of the proposed multigrid with respect to the variation of the permeability. For the experiments with the real code, we will consider a unit square domain $\Omega = [0,1]\times[0,1]$, and in order to obtain real asymptotic convergence factors we will assume a zero right-hand side, homogeneous boundary conditions and a random initial guess to avoid round-off errors.

First, in order to show the accuracy of the predictions of the local Fourier analysis, in Table 1, we show the smoothing and two-grid convergence factors predicted by LFA, μ^{ν} and ρ_{2g} respectively, together with the asymptotic convergence factors, ρ_h, computed by using the multigrid W-cycle version of the algorithm with a fine grid of size $h = \frac{1}{256}$, and considering $k = 10^{-3}$. These results are displayed for different numbers of smoothing steps, and show that the analysis predicts very accurately the real convergence factors of the multigrid method. We also performed the LFA for the diagonal Vanka smoother, however, the two grid convergence factor provided by using this relaxation becomes two or three times bigger than the factors presented in Table 1 for the full Vanka version. For example, with three smoothing steps a two-grid convergence factor around 0.35 is obtained. This makes that in this case the use of diagonal Vanka smoother is not competitive against that of the full version.

Table 1: Comparison between smoothing μ^{ν} and two-grid ρ_{2g} convergence factors predicted by LFA and the real asymptotic convergence factors ρ_h, for different numbers of smoothing steps ν.

$\nu = \nu_1 + \nu_2$	μ^{ν}	ρ_{2g}	ρ_h
1	0.535	0.439	0.420
2	0.286	0.192	0.190
3	0.153	0.118	0.119
4	0.082	0.082	0.082

Next, in order to show the robustness of the Vanka smoother, and as a consequence of the multigrid algorithm, with respect to parameter k, in Table 2, we display the two-grid convergence factors provided by LFA for different values of this parameter ranging from 1 to 10^{-8}, by using three smoothing steps. Also we show in the second raw of the table the number of W-cycle multigrid iterations necessary to reduce the maximum initial residual in a factor of 10^{-10}. We can observe that the convergence is independent of the value of k, even for very small values of this parameter.

Finally, in order to show the independence of the convergence of the multigrid method with respect to the spatial discretization parameter, in Figure 4 we show the history of the convergence of the multigrid algorithm by using different numbers of spatial mesh points, ranging from 32×32 to 512×512. These results are shown for both W- and V-cycles, since it is interesting to consider these latter due to their lower computational cost. Apart from the h-independent convergence of both multigrid methods, we can observe that the use of V-cycles provides very similar results to those given by the W-cycle. Therefore, from the practical point of view, it is advisable the use of V-cycles for the real computations.

Table 2: Two-grid convergence factors predicted by LFA for different values of parameter k, and number of iterations necessary to reduce the maximum initial residual in a factor of 10^{-10}.

k	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}
ρ_{2g}	0.118	0.118	0.118	0.118	0.116
it	9	9	9	9	9

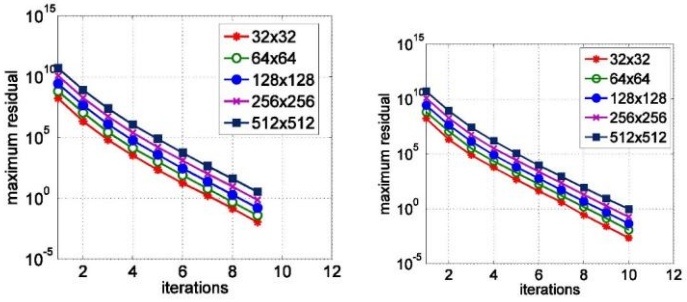


Fig. 4: History of the multigrid convergence for different numbers of spatial mesh points, by using both W-cycles (left picture) and V-cycles (right picture).

Conclusion:

In this work we present a local Fourier analysis technique to analyze a Vanka smoother for the discretization of poroelasticity problem on staggered grids. This analysis is developed in the framework of multigrid methods and allows to accurately predict the asymptotic convergence factors of these algorithms. Its practical utility is shown, since the analysis gives advices for an adequate choice of the multigrid components.

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