On Growth of Entire Special Monogenic Function

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Abstract

In this paper we study the growth of entire functions represented by special monogenic functions. The characterizations of their order and type have obtained.

INTRODUCTION

If m is a positive integer, The Clifford algebra \( A_m \) can defined as the ring extension of the field \( \mathbb{R} \) with symbols \( e_1,e_2,...,e_m \). Satisfying the fundamental multiplication rule:

\[
e_i e_j + e_j e_i = -2 \delta_{ij}
\]  

(1.1)

Clearly, \( A_m \) is then a non-commutative algebra. There are two special cases: \( A_1 \) is isomorphic to the complex field \( \mathbb{C} \) and \( A_2 \) is isomorphic to quaternion skew field \( H \). If \( m \geq 2 \), \( A_m \) contains zero divisors for instance,(1+\( e_1 e_2 \)) (1-\( e_1 e_2 \)) = 0. A general element of \( A_m \) can be written as: A general element of \( A_m \) can be written as:

\[
a = \sum A_M a_M e_M
\]  

(1.2)

Where \( M \) stands for \( \{1,2,...,m\} \), \( a_M \in \mathbb{R} \) and if \( A = \{i_1,i_2,...,i_k\} \) and \( i_1 < i_2 < ... < i_k \), \( e_A \) stands for \( e_{i_1} e_{i_2} ... e_{i_k} \). This expansion of \( a \) in terms of the \( e_A \) is unique, so \( A_m \) is a \( 2^m \) dimensional real vector space. An importantinvolution \( (\cdot)^\star \) is defined on \( A_m \) by the rules \( \lambda = \lambda \) if \( \lambda \in \mathbb{R} \), \( a + b = a + b \), \( a b = -b a \) and \( \overline{e_i} = -e_i \).

\[
|a| = \sqrt{\sum_{A M} a_M^2}
\]  

(1.3)

Some care must be taken when using this norm to estimate products, we will always use the formals \( |ab| \leq 2^{2m}|a||b| \) in general and \( |ab| = |a||b| \) if \( a \in \mathbb{R} \) or \( b \in \mathbb{R} \).

The space \( \mathbb{R}^{m+1} \) is identified with a subset \( A_m \), associated to \( (x_0,x_1,...,x_m) \) the element \( x_0 + x_1 e_1 + ... + x_m e_m \) of \( A_m \). The elements of this subset will be referred to as vectors. One easily see that \( \mathbb{R}^{m+1} \) is isomorphic to \( \mathbb{R}^m \). As a consequence, one can divide through nonzero vectors, since \( x^\star = \frac{1}{x} \). Clifford analysis is a generalization of complex analysis to functions defined on open sets in \( \mathbb{R}^{m+1} \). The generalized Cauchy-Riemann operator \( D \) is

\[
D = \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + ... + e_m \frac{\partial}{\partial x_m}
\]  

(1.4)

It can act both the left and the right on functions in \( \mathbb{C}(\mathbb{R}^{m+1};A_m) \). The generalization of holomorphic functions in \( \mathbb{C} \) are called monogenic functions. That is to say if \( U \subseteq \mathbb{R}^{m+1} \) is an open set, Then a function \( g:U \rightarrow A_m \) is called left (right) monogenic at a point \( z \in U \) if \( D g(z) = 0 \) or \( (g D(z)) = 0 \). The functions which are monogenic in the whole complex plane are called entire monogenic functions.

Following Abul-Ez and Constales (1990), we consider the class of monogenic polynomials \( P_m \) of degree \( m \), where \( m = (m_1,m_2,...,m_n) \) and \( |m| = m_1 + m_2 + ... + m_n \) defined as:

\[
P_m(z) = \sum |z|^m \frac{(n-1/2)(n+1)!}{i!j!} (\overline{z})^i (z)^j
\]  

(1.5)

2-Preliminaries:

Now following Abul-Ez and De Almeida (2013), we give some definitions which will be used in the next section.

Definition 1:

Let \( \Omega \) be a connected open subset of \( \mathbb{R}^{m+1} \) containing the origin and let \( g(z) \) be monogenic...
Definition 2:
The radius of regularity $R_g$ of special monogenic function is defined by:

$$R_g=1/\lim_{|m|\to\infty} \sup |c_{m}|^{\frac{1}{|m|}}.$$ 

(2.2)

Then the special monogenic function (2.1) is entire if $R_g=\infty$.

Definition 3:
Let $g(z)=\sum_{m=0}^{\infty} p_m(z)c_m$ be the power series expansion of an entire special monogenic function near zero. The order of $g$ is then defined by:

$$\rho=\lim_{|m|\to\infty} \sup |m|^{\frac{1}{\log |c_m|}}.$$ 

(2.3)

And if $0<\rho<\infty$, then Type $\rho$ of $g$ defined by:

$$(\rho T)^{\rho}=\lim_{|m|\to\infty} a \sup |m|^{\frac{1}{\log |c_m|}}.$$ 

(2.4)

In this paper, we shall obtain the relations between two or more entire special monogenic functions and study the relations between the coefficients in the Taylor expansion of entire functions and their orders and types.

3. Main Results:

We now prove the following:

**Theorem 3.1:**
Let $g_1(z)=\sum_{m=0}^{\infty} p_m(z)a_m$ and $g_2(z)=\sum_{m=0}^{\infty} p_m(z)b_m$ be two entire special monogenic functions of non-zero finite orders $\rho_1$, $\rho_2$, respectively. Then the function:

$$g(z)=\sum_{m=0}^{\infty} p_m(z)c_m$$

(3.1)

is an entire function such that:

$$\frac{1}{\rho} \geq \frac{1}{\rho_1}+\frac{1}{\rho_2},$$

where $\rho$ is the order of $g(z)$.

**Proof:**
Since $g_1(z)$ and $g_2(z)$ are entire functions. Then using (2.2) we have

$$\lim_{|m|\to\infty} \sup |a_m|^{\frac{1}{|m|}} = \lim_{|m|\to\infty} \sup |b_m|^{\frac{1}{|m|}} = 0.$$ 

Also

$$|c_m| \geq |a_m||b_m|,$$ 

therefore

$$\lim_{|m|\to\infty} \sup |c_m|^{\frac{1}{|m|}} \leq $$

$$\lim_{|m|\to\infty} \sup |a_m|^{\frac{1}{|m|}} \lim_{|m|\to\infty} \sup |b_m|^{\frac{1}{|m|}},$$

Hence $g(z)$ is an entire function.

Now using (2.3) for the function $g_1(z)$ and $g_2(z)$ we have

$$\lim_{|m|\to\infty} \sup |m|^{\log |m|} = \rho_1$$

and

$$\lim_{|m|\to\infty} \sup |m|^{\log \log |m|} = \rho_2.$$

Therefore, for an arbitrary $\varepsilon > 0$, we get

$$\log |a_m| \geq \left(\frac{1}{\rho_1} - \frac{\varepsilon}{2}\right) |m| \log |m|,$$

for $|m| > k_1$.

**Corollary 3.1.1:**
Let $g_1(z)=\sum_{m=0}^{\infty} p_m(z)a_m$, where $s=1,2,\ldots,p$ be $(p)$ entire special monogenic functions of non-zero finite orders $\rho_1$, $\rho_2$, respectively. Then the function:

$$g(z)=g_1(z)g_2(z)$$

is an entire function such that

$$\rho \geq \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$ 

**Theorem 3.2:**
Let $g_1(z)=\sum_{m=0}^{\infty} p_m(z)a_m$ and $g_2(z)=\sum_{m=0}^{\infty} p_m(z)b_m$ be two entire special monogenic functions of non-zero finite orders $\rho_1$, $\rho_2$, respectively. Then the function:

$$g(z)=\sum_{m=0}^{\infty} p_m(z)c_m$$

(3.4)

is an entire function such that

$$\rho \leq (\rho_1+\rho_2)^{1/2},$$

where $\rho$ is the order of $g(z)$.

**Proof:**
Since $g_1(z)$ and $g_2(z)$ are entire functions therefore using (2.2) we have for an arbitrary $\varepsilon > 0$ and large $R$

$$\frac{1}{|a_m|} > (R-\varepsilon)^{|m|}$$

For $|m| > k_1$

and

$$\frac{1}{|b_n|} > (R-\varepsilon)^{|n|}$$

For $|n| > k_2$.

Therefore $|m| > k_{max}(k_1,k_2)$

$$\left|\log \left(\frac{1}{|a_m|}\right)\right| \leq |m|\log(R-\varepsilon).$$

Thus, if

$$\log \left(\frac{1}{|a_m|}\right) \sim \left|\log \left(\frac{1}{|b_m|}\right)\right|,$$

Then for large $|m|$ we get

$$\log \left(\frac{1}{|a_m|}\right) \sim \left|\log \left(\frac{1}{|b_m|}\right)\right|.$$ 

Or

$$\lim_{|m|\to\infty} \sup |c_m|^{\frac{1}{|m|}} = 0.$$ 

Hence $g(z)$ is an entire function, now from (3.2) and (3.3) we have for sufficiently large $|m|$

$$\left|\log \left(\frac{1}{|a_m|}\right)\right| \sim \left|\log \left(\frac{1}{|b_m|}\right)\right|,$$

and

$$\left|\log \left(\frac{1}{|a_m|}\right)\right| \sim \left|\log \left(\frac{1}{|b_m|}\right)\right|.$$
\[ \frac{1}{\rho} = \lim \inf_{|m| \to \infty} \frac{\log (|a_m|)}{|m| \log |m|} \geq \left( \frac{1}{\rho_1} \right)^{1/2}. \]

Or
\[ \rho \leq (\rho_1 \rho_2)^{1/2}. \]

**Corollary 3.2.2:**

Let \( g(z) = \sum_{|m|=0}^{\infty} p_m(z) a_m^{(g)} \) \( s=1,2,\ldots,p \) be \( (p) \) entire special monogenic functions of non-zero finite orders \( \rho_1, \rho_2, \ldots, \rho_p \) respectively. Then the function
\[ g(z) = \sum_{|m|=0}^{\infty} p_m(z) a_m. \]

Where
\[ \log \left( \frac{1}{|c_m|} \right) \sim \left( \prod_{s=1}^{p} (|a_m|)^{1/p_s} \right) \]

is an entire monogenic function such that:
\[ \rho \leq \left( \prod_{s=1}^{p} \rho_s \right)^{1/p}, \]
Where \( \rho \) is the order of \( g(z) \).

**Theorem 3.3:**

Let \( g_1(z) = \sum_{|m|=0}^{\infty} p_m(z) a_m \) and \( g_2(z) = \sum_{|m|=0}^{\infty} p_m(z) b_m \) be two entire monogenic functions of non-zero finite orders \( \rho_1, \rho_2 \) respectively and finite non-zero type \( T_1, T_2 \) respectively, then the function
\[ g(z) = \sum_{|m|=0}^{\infty} p_m(z) c_m, \]

Where
\[ |c_m|^{-1} \left[ |a_m|^{1/|m|} b_m \right]^{1/|m|} \]

is an entire function such that
\[ (\rho T)^{2/p} \leq \left( \frac{\rho_1 T_1}{\rho_2 T_2} \right)^{1/\rho_1} \left( \frac{\rho_2 T_2}{\rho_1 T_1} \right)^{1/\rho_2}, \]

where \( \rho, T \) are the order and type of \( g(z) \) respectively and
\[ \frac{2}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}. \]

**Proof:**

We can prove as in the proof of theorem 3.1 that \( g(z) \) is an entire function, where
\[ |c_m|^{-1} \left[ |a_m|^{1/|m|} b_m \right]^{1/|m|} \]

Further, using (2.4) for the functions \( g_1(z), g_2(z) \) we have
\[ (\rho_1 T_1)^{1/\rho_1} \sim \lim |m|^{-1/\rho_1} \left[ |a_m|^{1/|m|} \right] \]
(3.5)

\[ (\rho_2 T_2)^{1/\rho_2} \sim \lim |m|^{-1/\rho_2} \left[ |b_m|^{1/|m|} \right] \]
(3.6)

From (3.5) and (3.6) we get for arbitrary \( \epsilon > 0 \)
\[ |m|^{1/\rho_1} \left[ |a_m|^{1/|m|} \right] < \left( \rho_1 T_1 + \epsilon \right)^{1/\rho_1}, \text{for} \ |m| > k_1 \]
\[ |m|^{1/\rho_2} \left[ |b_m|^{1/|m|} \right] < \left( \rho_2 T_2 + \epsilon \right)^{1/\rho_2}, \text{for} \ |m| > k_2. \]

Thus, for \( |m| > k = \max (k_1, k_2) \) and
\[ \frac{2}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}. \]

\[ \left( |m|^{1/\rho} \left[ |a_m|^{1/|m|} b_m \right]^{1/|m|} \right)^{1/\rho} \]
\[ < \left( \rho_1 T_1 + \epsilon \right)^{1/\rho_1} < \left( \rho_2 T_2 + \epsilon \right)^{1/\rho_2} \]

Therefore if \( |c_m|^{-1} \left[ |a_m|^{1/|m|} b_m \right]^{1/|m|} \)
we obtain
\[ \lim \sup \left[ |m|^{1/\rho} \left[ |c_m| \right]^{1/|m|} \right] \]
\[ \leq \left( \rho_1 T_1 \right)^{2/p_1} \times \left( \rho_2 T_2 \right)^{2/p_2} \]

Or
\[ (\rho T)^{2/p} \leq \left( \rho_1 T_1 \right)^{2/p_1} \times \left( \rho_2 T_2 \right)^{2/p_2} \]

Where \( \rho \) and \( T \) are the order and type of \( g(z) \) respectively, hence,
\[ (\rho T)^{2/p} \leq \left( \rho_1 T_1 \right)^{1/\rho_1} \left( \rho_2 T_2 \right)^{1/\rho_2} \]

**Corollary 3.3.3:**

Let \( g_s(z) = \sum_{|m|=0}^{\infty} p_m(z) a_m^{(g)} \), \( s=1,2,\ldots, p \) where \( \rho_1, \rho_2, \ldots, \rho_p \) and finite non-zero types \( T_1, T_2, \ldots, T_p \) respectively. Then the function
\[ g(z) = \sum_{|m|=0}^{\infty} p_m(z) a_m. \]

Where
\[ |a_m|^{1/|m|} \left( \prod_{s=1}^{p} |a_m|^{1/|m|} \right)^{1/p_s} \]

is an entire monogenic function such that
\[ (\rho T)^{2/p} \leq \left( \rho_1 T_1 \right)^{1/\rho_1} \left( \rho_2 T_2 \right)^{1/\rho_2} \]

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**Reference**


\[ (3.5) \]