On Subclass of Harmonic Univalent Function Defined by Ruscheweyh Derivative

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ABSTRACT
In the present paper, we introduce a new subclass of harmonic function in the unit disc $U$ by the Ruscheweyh derivative. Also, we obtain coefficient condition, convolution condition, convex combinations, extreme point, $\delta$-neighborhood, integral operator.

INTRODUCTION
A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $D$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain $D \subset C$, we can write $f = h + \overline{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense preserving in $D$ is that $|h'(z)| > |g(z)|$ in $D$, see (Clunie, 1984). In 1984, Clunie and Sheil-Small (Clunie, 1984) investigated the class $AH_S$ and studied some sufficient bounds. Since then the-re have been several paper published related to $AH_S$ and its subclasses. In fact by introducing new subclasses Sheil-Small (Sheil-Small, 1990), Silverman (Silverman, 1998), Silverman and Silvia (Silverman, 1999), Jahangiri (Jahangiri, 1999) and Ahuja (Ahuja, 2005) presented a systematic and unified study of harmonic univalent functions. Furthermore we refer to Duren (Duren, 2004), Ponnusamy (Ponnusamy, 2007) and references there in for basic results on the subject.

Denoted by $AH_S$, the class of function $f = h + \overline{g}$ that are harmonic, univalent and sense-preserving in the unit disk $U = \{z: |z| < 1\}$ with normalization $f(0) = h(0) = f_1(0) - 1 = 0$. Then for $f = h + \overline{g} \in AH_S$, we may express the analytic functions $h$ and $g$ as $h(z) = z + \sum_{k=2}^{\infty} a_k z^k, g(z) = z + \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1$.

Observe that $AH_S$ reduces to , the class of normalized univalent functions, if the co-analytic part of $f$ is zero. Also, denoted by $AH_S$ the subclass of $AH_S$, consisting of function $f$ that map $U$ onto a starlike domain.

For $f = h + \overline{g}$ given by (1) and $D^3 f(z)$ is the Ruscheweyh derivative of $f$ and defined by

$$D^3 f(z) = \sum_{k=1}^{\infty} B_k(\lambda) c_k \; z^k, \lambda > -1, \lambda \in \mathbb{C},$$

$$B_k(\lambda) = \frac{(\lambda + 1)(\lambda + 2)\ldots(\lambda + n - 1)}{(n - 1)!}$$

also $D^3 f(z) = D^3 h(z) + D^3 g(z)$.

Recently Rosy et al. (Rosy, 2001) defined the subclass $G_S \subseteq AH_S$ consisting of a harmonic univalent function $f(z)$ satisfying the condition

$$D^3 f(z) = \sum_{k=1}^{\infty} B_k(\lambda) c_k \; z^k, \lambda > -1, B_k(\lambda) = \frac{(\lambda + 1)(\lambda + 2)\ldots(\lambda + n - 1)}{(n - 1)!}$$

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\[ Re \left\{ (1 + e^{ia}) \frac{zf'(z)}{f(z)} - e^{ia} \right\} \geq \gamma, 0 \leq \gamma < 1, a \in R. \]

They proved that if \( f = h + \bar{g} \) is given by (1) and if
\[ \sum_{k=1}^{\infty} \left[ \frac{2k+1-\gamma}{(1-\gamma)} |a_k| + \frac{2k+1+\gamma}{(1-\gamma)} |a_k| \right] \leq 2, 0 \leq \gamma < 1, \]
then \( f \) is in \( G_{\alpha}(\gamma) \).

This condition is proved to be also necessary by Rosy et al. if \( h \) and \( g \) are of the form
\[ h(z) = z - \sum_{k=2}^{\infty} a_k z^k, g(z) = -\sum_{k=1}^{\infty} |b_k| z^k \]
Motivated by this aforementioned work, new we introduce the class \( G_{\alpha}(\lambda, \alpha, p) \) as the subclass of functions of the form (1) satisfy the following condition
\[ Re \left\{ (1 + pe^{ia}) \frac{D^q f(z)}{D^q f(z)} - pe^{ia} \right\} \geq \gamma, 0 \leq \gamma < 1, a \in R, p \geq 0, q \in N, \]
where \( D^q f(z) \) is defined by (2).

Let \( G_{\alpha}(\lambda, \alpha, p) \) denoted that the subclass of \( G_{\alpha}(\lambda, \alpha, p) \) which consists of harmonic function \( f_k = h + \bar{g}_k \) such that \( h \) and \( g \) are the form
\[ h(z) = z - \sum_{k=2}^{\infty} a_k z^k, g(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k. \]

In this paper, we will give sufficient condition for function \( f = h + \bar{g} \), where \( h \) and \( g \) are gven by (1) to be in the class \( G_{\alpha}(\lambda, \alpha, p) \) it is shown that is coefficient condition is also necessary for function in the class \( G_{\alpha}(\lambda, \alpha, p) \). Also we obtain distortion theorem and characterize the extreme point and convolution conditions for functions in \( G_{\alpha}(\lambda, \alpha, p) \).

Closure theorems and application of neighborhood also obtain.

**Coefficient Inequality:**

We being with a sufficient condition for in \( G_{\alpha}(\lambda, \alpha, p) \).

**Theorem 2.1:** Let \( f = h + \bar{g} \) be given by (2.1). If
\[ \sum_{k=1}^{\infty} \left[ \{k(1 + \rho) - (\alpha + \rho)\} |a_k| + \{k(1 + \rho) + (\alpha + \rho)\} |b_k| \right] B_{\lambda}(\lambda) \leq 2(1 - \alpha), \]
where \( a_1 = 1, \lambda \in N_0, B_{\lambda}(\lambda) = \frac{(\lambda + 1) \cdots (\lambda + n - 1)}{(n - 1)!}, \rho \geq 0 \) and \( 0 \leq \alpha < 1 \), then \( f \) is sense preserving harmonic in \( U \) and \( f \in G_{\alpha}(\lambda, \alpha, p) \).

**Proof:** If \( Z_1 \neq Z_2 \), then
\[ \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \geq 1 - \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}. \]

\[ = 1 - \left| \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right| \]
\[ \geq 1 - \frac{\sum_{k=1}^{\infty} k |b_k|}{1 - \sum_{k=1}^{\infty} k |a_k|} \]
\[ \geq 1 - \frac{\sum_{k=1}^{\infty} [k(1 + \rho) + (\alpha + \rho)] B_{\lambda}(\lambda) |b_k|}{1 - \alpha} \]
\[ \geq 1 - \frac{\sum_{k=2}^{\infty} [k(1 + \rho) + (\alpha + \rho)] B_{\lambda}(\lambda) |a_k|}{1 - \alpha} \]
\[ \geq 0, \]
which proves univalence. Not that \( f \) is sense preserving in \( U \). This is because
\[ |h'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \]

\[ \geq 1 - \sum_{k=2}^{\infty} \frac{[k(1 + \rho) - (\alpha + \rho)B_k(\lambda)]a_k}{1 - \alpha} \]

\[ \geq 1 - \sum_{k=1}^{\infty} \frac{[k(1 + \rho) + (\alpha + \rho)B_k(\lambda)]b_k}{1 - \alpha} \]

\[ \geq 1 - \sum_{k=1}^{\infty} \frac{[k(1 + \rho) + (\alpha + \rho)B_k(\lambda)]b_k ||z||^{k-1}}{1 - \alpha} \geq \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \]

\[ \geq |g'(z)|. \]

Using the fact that \( Re w > \alpha \) if and only if \([1 - \alpha + w] \geq [1 + \alpha - w] \) it suffice to show that
\[ (1 - \alpha) + (1 + \rho e^{ir}) \frac{D^{k+1}f(z)}{D^{k}f(z)} - \rho e^{ir} \geq 0. \] (10)

Substituting the value of \( D^{k}f(z) \) in (10) yields, by (7),
\[ (1 - \alpha) + (1 + \rho e^{ir})D^{k}f(z) + (1 + \rho e^{ir})D^{k+1}f(z) \]

\[ = (2 - \alpha)z + \sum_{k=2}^{\infty} \{(1 + \rho e^{ir}) + (1 - \alpha - \rho e^{ir})\}B_k(\lambda) \times a_kz^k \]

\[ = (-1)^k \sum_{k=1}^{\infty} \{(1 + \rho e^{ir}) - (1 - \alpha - \rho e^{ir})\}B_k(\lambda) \times b_kz^k \]

\[ = -az + \sum_{k=2}^{\infty} \{(1 + \rho e^{ir}) + (1 + \alpha + \rho e^{ir})\}B_k(\lambda) \times a_kz^k \]

\[ \geq 2(1 - \alpha)|z| \left[ 1 - \sum_{k=2}^{\infty} \frac{[k(1 + \rho) - (\alpha + \rho)]B_k(\lambda)||z||^k}{1 - \alpha} \right] - \sum_{k=1}^{\infty} \frac{[k(1 + \rho) + (\alpha + \rho)]B_k(\lambda)||b_k||^k}{1 - \alpha} \]

This last expression is non-negative by (8), and so the proof is complete. ■

The harmonic function
\[ f(z) = z + \sum_{k=2}^{\infty} \frac{1 - \alpha}{(k + 1 + \rho)^{1 - \alpha}} x_kz^k + \sum_{k=1}^{\infty} \frac{1 - \alpha}{(k + 1 + \rho)^{1 - \alpha}} y_kz^k \]

where \( \lambda \in \mathbb{N}_0, 0 \leq \rho \leq 1 \) and \( \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1 \), shows that the coefficient bound given by (7) is sharp. The functions of the form (11) are in \( G_\alpha(\lambda, \alpha, \rho) \) because
\[ \sum_{k=1}^{\infty} \left[ \frac{1 + \alpha + \rho}{1 - \alpha} a_k + \frac{1 + \alpha}{1 - \alpha} b_k \right] B_k(\lambda) = \frac{1}{\lambda + \alpha} \sum_{k=1}^{\infty} \left[ (k + 1 + \rho) - (\alpha + \rho) \right] a_k + \frac{1 - \alpha}{1 - \alpha} \]

\[ = 1 + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 2. \] (12)

In the following theorem, it is shown that the condition (7) is also necessary for functions \( f_n = h + g_n \),

where \( h \) and \( g_n \) are of the form (6).

Theorem 2.2: Let \( f_n = h + g_n \) be given by (7). Then \( f_n \in \overline{G_\alpha}(\lambda, \alpha, \rho) \) if and only if

\[ \sum_{k=1}^{\infty} \left[ (k + 1 + \rho) - (\alpha + \rho) \right] a_k + \frac{1 - \alpha}{1 - \alpha} \sum_{k=1}^{\infty} b_k |B_k(\lambda)| \leq 2(1 - \alpha) \]

where \( a_1 = 1, \lambda \in \mathbb{N}_0, B_k(\lambda) = \frac{(\lambda + 1)(\lambda + 2) \ldots (\lambda + n - 1)}{(n - 1)!}, \rho \geq 0 \) and \( 0 \leq \alpha < 1 \).

Proof: Since \( \overline{G_\alpha}(\lambda, \alpha, \rho) \subset G_\alpha(\lambda, \alpha, \rho) \) we only need to prove the "only if" part of Theorem (2.2). To this end, for functions \( f_n \) of the form (6), we notice that the condition (5) is equation to

\[ Re \left\{ (1 + \rho e^{i\alpha}) \frac{D^{k+1}f(z)}{D^{k}f(z)} - (\rho e^{i\alpha} + \alpha) \right\} \geq 0 \]

\[ \Rightarrow \]
\[
Re \left\{ (1 + pe^{iz})D^{x+1}f(z) - (\rho e^{iz} + \alpha)D^x f(z) \right\} \geq 0
\]

\[
\Rightarrow 
\left( 1 + pe^{iz} \right) \left( z - \sum_{k=2}^{\infty} kB_k(\lambda)a_k|z^k + \sum_{k=1}^{\infty} (-1)^{2k+1} k|b_k|B_k(\lambda)\bar{z}^k \right)
\]

\[
\left( \rho e^{iz} + \alpha \right) \left( z - \sum_{k=2}^{\infty} kB_k(\lambda)a_k|z^k + \sum_{k=1}^{\infty} (-1)^{2k} k|b_k|B_k(\lambda)\bar{z}^k \right)
\]

\[
\left\{ z - \sum_{k=2}^{\infty} B_k(\lambda)a_k|z^k + \sum_{k=1}^{\infty} (-1)^{2k+1} k^2|b_k|^2B_k(\lambda)\bar{z}^k \right\} \geq 0
\]

\[
\Rightarrow 
\left( 1 - \frac{1}{2} \right) \left( k(1 + pe^{iz}) - (\rho e^{iz} + \alpha)B_k(\lambda)a_k|z^{k-1} \right)
\]

\[
1 - \sum_{k=2}^{\infty} B_k(\lambda)a_k|z^{k-1} + \sum_{k=1}^{\infty} (-1)^{2k+1} k|b_k|^2B_k(\lambda)\bar{z}^{k-1}
\]

\[
\left\{ z - \sum_{k=2}^{\infty} B_k(\lambda)a_k|z^{k-1} + \sum_{k=1}^{\infty} (-1)^{2k+1} k|b_k|^2B_k(\lambda)\bar{z}^{k-1} \right\} \geq 0
\]

The above condition (14) must hold for all values of z on the positive real axes, where, \(0 \leq |z| \leq \gamma < 1\), we must have

\[
\left\{ \left( 1 - a \right) - \sum_{k=2}^{\infty} (k - \alpha)B_k(\lambda)a_k|y^{k-1} \right\}
\]

\[
1 - \sum_{k=2}^{\infty} B_k(\lambda)a_k|y^{k-1} + \sum_{k=1}^{\infty} (-1)^{2k+1} k|b_k|^2B_k(\lambda)\bar{y}^{k-1}
\]

\[
\left\{ \left( -1 \right)^{2k} \sum_{k=2}^{\infty} (k + \alpha)B_k(\lambda)b_k|y^{k-1} - \rho e^{iz} \sum_{k=2}^{\infty} (k - 1)B_k(\lambda)a_k|y^{k-1} \right\}
\]

\[
1 - \sum_{k=2}^{\infty} B_k(\lambda)a_k|y^{k-1} + \sum_{k=1}^{\infty} (-1)^{2k+1} k|b_k|^2B_k(\lambda)\bar{y}^{k-1}
\]

Since \(Re(-e^{iz}) \geq -|e^{iz}| = -1\), the above inequality reduce to

\[
\left( 1 - a \right) - \sum_{k=2}^{\infty} (k + \rho + \alpha)B_k(\lambda)a_k|y^{k-1}
\]

\[
1 - \sum_{k=2}^{\infty} B_k(\lambda)a_k|y^{k-1} + \sum_{k=1}^{\infty} B_k(\lambda)b_k|y^{k-1}
\]
\[
\sum_{k=2}^{\infty} \frac{(1+\rho)(\rho+\alpha)B_k(\lambda)\beta_k^{k-1}}{1-\sum_{k=2}^{\infty} B_k(\lambda)\beta_k^{k-1} + \sum_{k=1}^{\infty} B_k(\lambda)\beta_k^{k-1}} \geq 0.
\]

If the condition (13) does not hold, then the numerator in (15) is negative for \(\gamma\) sufficiently close to 1. Hence there exists a \(z_0 = y_0\in (0, 1)\) for which the quotient in (15) is negative. This contradicts the condition for \(f \in \overline{G}_S(\lambda, \alpha, p)\) and so proof is complete.

3. Distortion Bounds:

In this section, we will obtain distortion bounds for function in \(\overline{G}_S(\lambda, \alpha, p)\).

Theorem 3.1: Let \(f_n \in \overline{G}_S(\lambda, \alpha, p)\). Then for \(|z| = \gamma < 1\), we have

\[
|f_n(z)| \leq (1 + |b_1|)\gamma + \frac{(1-\alpha)}{2(1+\rho) - (\rho + \alpha)(\lambda + 1)} \left[ 1 - \frac{1 + 2\rho + \alpha}{1 - \alpha} |b_1| \right] \gamma^2.
\]

Proof: We only prove the lift -hand inequality. The proof for the right inequality is similar and is thus omitted. Let \(f_n \in \overline{G}_S(\lambda, \alpha, p)\). Taking the absolute value of \(f_n\), we obtain

\[
|f_n(z)| = \left| z - \sum_{k=2}^{\infty} a_kz^k + (-1)^{2n} \sum_{k=1}^{\infty} b_k\bar{z}^k \right|
\]

\[
\leq (1 + |b_1|)\gamma + \sum_{k=2}^{\infty} (|a_k| + |b_k|)\gamma^k
\]

\[
\leq (1 + |b_1|)\gamma + \sum_{k=2}^{\infty} (|a_k| + |b_k|)\gamma^2
\]

\[
\leq (1 + |b_1|)\gamma + \frac{1 - \alpha}{2(1+\rho) - (\rho + \alpha)(\lambda + 1)} \sum_{k=2}^{\infty} \left[ \frac{k(1+\rho) - (\rho + \alpha)B_k(\lambda)}{1 - \alpha} |a_k| + \frac{k(1+\rho) + (\rho + \alpha)B_k(\lambda)}{1 - \alpha} |b_k| \right] \gamma^2.
\]

\[
\leq (1 + |b_1|)\gamma + \frac{1 - \alpha}{2(1+\rho) - (\rho + \alpha)(\lambda + 1)} \left[ 1 - \frac{(1 + \rho) + (\rho + \alpha)}{1 - \alpha} |b_1| \right] \gamma^2
\]

\[
\leq (1 + |b_1|)\gamma + \frac{1 - \alpha}{2(1+\rho) - (\rho + \alpha)(\lambda + 1)} \left[ 1 - \frac{2 + \rho + \alpha}{1 - \alpha} |b_1| \right] \gamma^2.
\]

The functions

\[
f(z) = z + |b_1|\bar{z} + \frac{1 - \alpha}{\lambda + 1} \left[ \frac{1 - \alpha}{2(1+\rho) - (\rho + \alpha)} - \frac{2 + \rho + \alpha}{2(1+\rho) - (\rho + \alpha)} \right] \bar{z}^2
\]

\[
f(z) = (1 - |b_1|)z - \frac{1 - \alpha}{\lambda + 1} \left[ \frac{1 - \alpha}{2(1+\rho) - (\rho + \alpha)} - \frac{2 + \rho + \alpha}{2(1+\rho) - (\rho + \alpha)} \right] z^2
\]

for \(|b_1| \leq \frac{1 - \alpha}{1 + 2\rho + \alpha}\). Show that the bounds given in the Theorem 3.1 are sharp.

The following covering result follows form the left -hand inequality in Theorem 3.1.

4. Convex Combination And Extreme Points:

Let the function \(f_{n,j}(z)\) be defined, for \(j=1, 2 \ldots m\), by
\( f_{n,j}(z) = z - \sum_{k=0}^{\infty} |a_{k,j}| z^k + (-1)^n \sum_{k=0}^{\infty} |b_{k,j}| z^k. \)  

(16)

**Theorem 4.1:** Let the function \( f_{n,j}(z) \) defined by (16) be in the class \( \overline{G}_S(\lambda, \alpha, p) \) for every \( j = 1, 2, \ldots, m \). Then the functions \( t_j(z) \) defined by

\[ t_j(z) = \sum_{j=1}^{m} c_j f_{n,j}(z), \quad o \leq c_j < 1, \]

are also in the class \( \overline{G}_S(\lambda, \alpha, p) \), where \( \sum_{j=1}^{m} c_j = 1 \).

**Proof:** A corollary to the definition of \( t_j \), we can write

\[ t_j(z) = z - \sum_{k=2}^{\infty} \left( \sum_{j=1}^{m} c_j |a_{k,j}| \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left( \sum_{j=1}^{m} |b_{k,j}| \right) z^k. \]

Further, since \( f_{n,j}(z) \) are in \( \overline{G}_S(\lambda, \alpha, p) \) for every \( j = 1, 2, \ldots, m \), then

\[ \sum_{k=1}^{\infty} \left\{ \left( k(1 + \rho) - (\alpha + \rho) \right) \left( \sum_{j=1}^{m} c_j |a_{k,j}| \right) + (k(1 + \rho) + (\alpha + \rho)) \left( \sum_{j=1}^{m} c_j |b_{k,j}| \right) \right\} B_k(\lambda) \]

\[ = \sum_{j=1}^{m} c_j \left\{ k(1 + \rho) - (\alpha + \rho) |a_{k,j}| + (k(1 + \rho) + (\alpha + \rho)) |b_{k,j}| \right\} B_k(\lambda) \]

\[ \leq \sum_{j=1}^{m} c_j 2(1 - \alpha) \leq (1 - \alpha). \]

Hence theorem 4.1 follows.

**Corollary 4.2:** The class \( \overline{G}_S(\lambda, \alpha, p) \) is closed under convex linear combinations.

**Proof:** Let the functions \( f_{n,j}(z) = (j=1, 2, \ldots, m) \) defined by (16) be in the class \( \overline{G}_S(\lambda, \alpha, p) \). Then the function \( \Psi(z) \) defined by

\[ \Psi(z) = \mu f_{n,j}(z) + (1 - \mu) f_{n,j}(z), \quad o \leq \mu < 1 \]

is in the class \( \overline{G}_S(\lambda, \alpha, p) \).

Next we determine the extreme point of closed convex hulls of \( \overline{G}_S(\lambda, \alpha, p) \), denoted by \( c\overline{cloco} \overline{G}_S(\lambda, \alpha, p) \).

**Theorem 4.3:** Let \( f_n \) be given by (7). Then \( f_n \in \overline{G}_S(\lambda, \alpha, p) \) if and only if

\[ f_n(z) = \sum_{k=1}^{m} \left( X_n h_k(z) + Y_n g_{n,k}(z) \right), \]

where

\[ h_k(z) = z, \quad h_k(z) = z - \frac{1 - \alpha}{k(1 + \rho) - (\alpha + \rho) B_k(\lambda)} z^k, \quad z = 2, 3, \ldots, \]

\[ g_{n,k}(z) = z - (-1)^n \frac{1 - \alpha}{k(1 + \rho) + (\alpha + \rho) B_k(\lambda)} \tilde{z}^k \]

and \( \sum_{k=1}^{\infty} (X_k + Y_k) = 1, X_k \geq 0, Y_k \geq 0. \) In particular, the extreme points of \( \overline{G}_S(\lambda, \alpha, p) \) are

\( \{ h_k \} \) and \( \{ g_{n,k} \} \).

**Proof:** For the function \( f_n \) of the form (4.3), we have

\[ f_n(z) = \sum_{k=1}^{m} \left( X_n h_k(z) + Y_n g_{n,k}(z) \right) \]

\[ f_n(z) = \sum_{k=1}^{m} \left( X_n h_k(z) + Y_n g_{n,k}(z) \right) \]

Then

\[ \sum_{k=1}^{\infty} \frac{k(1 + \rho) - (\rho + a) B_k(\lambda)}{1 - \alpha} |a_k| \]

\[ = \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \left( k(1 + \rho) - (\beta + \alpha) B_k(\lambda) \right) \left( \frac{1 - \alpha}{k(1 + \rho) + (\alpha + \rho) B_k(\lambda)} \right) Y_n \tilde{z}^k. \]

(18)

Conversely, suppose that \( f_n \in c\overline{cloco} \overline{G}_S(\lambda, \alpha, p) \). Setting

\[ X_k = \frac{k(1 + \rho) - (\rho + a) B_k(\lambda)}{1 - \alpha} |a_k|, \quad 0 \leq X_k < 1, \quad k = 2, 3, \]

\[ Y_k = \frac{k(1 + \rho) - (\rho + a) B_k(\lambda)}{1 - \alpha} |b_k|, \quad 0 \leq Y_k < 1, \quad k = 2, 3, \ldots, \]

(19)
and $X_1 = 1 - \sum_{k=1}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$ then $f_n$ can be written as

$$f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k \bar{z}^k$$

$$= z - \sum_{k=2}^{\infty} \frac{(1-\alpha)X_k}{k(1+\rho)-(\alpha+\rho)B_k(\lambda)} z^k + (-1)^n \sum_{k=1}^{\infty} \frac{(1-\alpha)Y_k}{k(1+\rho)+(\alpha+\rho)B_k(\lambda)} \bar{z}^k$$

$$= z + \sum_{k=2}^{\infty} (h(z) - z)X_k + \sum_{k=1}^{\infty} (g_n(z) - z)Y_k$$

$$= \sum_{k=2}^{\infty} h_k(z)X_k + \sum_{k=1}^{\infty} g_n(z)Y_k + z(1 - \sum_{k=1}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k)$$

$$= \sum_{k=2}^{\infty} (h_k(z)X_k + g_n(z)Y_k), \text{as required}. \quad (20)$$

Using corollary (4.2) we have $clco \overline{G_0}(\lambda, \alpha, p) = \overline{G_5}(\lambda, \alpha, p)$. Then the statement of Theorem (4.3) is true for $f \in \overline{G_5}(\lambda, \alpha, p)$.

5. Neighborhoods:

The $\delta - \text{neighborhood } N_\delta(f)$ off is the set (see Atintas, 2000) and (Rucsheweyh, 1981):

$$N_\delta(f) = \{ F : \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k| + |b_1 - B_1| \leq \delta \}, \quad (21)$$

where the function $F(z)$ is given by

$$F(z) = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k \bar{z}^k \quad (22)$$

In [8], ÖZTÜRK and YALCIN defined the generalized $\delta$-neighborhood of $f$ to be the set:

$$N_\delta(f) = \left\{ F : \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k| + |b_1 - B_1| \leq \delta(1-\alpha) \right\}$$

Theorem 5.1: Let $f(z) = z + \overline{b_1 z} + \sum_{k=2}^{\infty} (a_k z^k + \overline{b_k z^k})$ be a member of $\overline{G_5}(\lambda, \alpha, p)$. If $\delta \leq \frac{2\rho(1-\alpha)}{\rho + 1} + |b_1|$, then $N_\delta(f) \subset \overline{G_5}(\lambda, \alpha, p)$.

Proof: Let $f \in \overline{G_5}(\lambda, \alpha, p)$.

$$F(z) = z + \overline{b_1 z} + \sum_{k=2}^{\infty} (A_k z^k + \overline{B_k z^k})$$

belong to $N_\delta(f)$. We have

$$(1-\alpha)|b_1| + \sum_{k=1}^{\infty} (k-\alpha)(|A_k| + |B_k|)$$

\leq (1-\alpha)|b_1 - B_1| + \sum_{k=1}^{\infty} (k-\alpha)(|A_k - a_k| + |B_k - b_k|) + (1-\alpha)|b_1| + \sum_{k=1}^{\infty} (k-\alpha)(|a_k| + |b_k|)$$

\leq (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{1}{\rho + 1} \sum_{k=1}^{\infty} ((k(1+\rho) - (\alpha+\rho) + (1+\rho) + \rho + 1(1-\alpha)B_k(\lambda)(a_k + |b_k|)$$

\leq (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{2(1-\alpha)}{\rho + 1} \leq 1 - \alpha,$$

if $\delta \leq \frac{2\rho(1-\alpha)}{\rho + 1} + |b_1|$.

Thus $F \in \overline{G_5}(\lambda, \alpha, p)$.

6. Integral Operators:

Now, we examine a closure property of class $\overline{G_5}(\lambda, \alpha, p)$ under the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ which is defined by

$$L_c(f) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) \, dt, \ c > -1.$$ 

Theorem 6.1: Let $f \in \overline{G_5}(\lambda, \alpha, p)$ Then $L_c(f) \in \overline{G_5}(\lambda, \alpha, p)$.
Proof: From the representation of $L_c(f(z))$, it follows that

$$L_c(f) = \frac{c + 1}{z^c} \int_0^z t^{c-1} [h(z) + g(t)] \, dt.$$  

$$= \frac{c + 1}{z^c} \int_0^z t^{c-1} (t - \sum_{k=2}^\infty a_k t^k) \, dt - \int_0^z t^{c-1} (t - \sum_{k=1}^\infty b_k t^k) \, dt.$$  

$$= z - \sum_{k=2}^\infty A_k z^k - \sum_{k=1}^\infty B_k z^k$$  

where $A_k = \frac{c + 1}{k + 1} a_k$; $B_k = \frac{c + 1}{k + 1} b_k$.

Therefore,

$$\sum_{k=1}^\infty \left( \frac{k(1 + \rho) - (\rho + \alpha)}{1 - \alpha} \left( \frac{c + 1}{k + 1} \right) |a_k| + \frac{k(1 + \rho) + (\rho + \alpha)}{1 - \alpha} \left( \frac{c + 1}{k + 1} \right) |b_k| \right) B_k(\lambda)$$  

$$\leq \sum_{k=1}^\infty \left( \frac{k(1 + \rho) - (\rho + \alpha)}{1 - \alpha} |a_k| + \frac{k(1 + \rho) + (\rho + \alpha)}{1 - \alpha} |b_k| \right) B_k(\lambda)$$  

$$\leq 2(1 - \alpha).$$  

Since $f \in \mathcal{G}_S(\alpha, \rho, \alpha)$, therefor by Theorem (2.2), $L_c(f) \in \mathcal{G}_S(\lambda, \alpha, \rho)$.

REFERENCES


