A Reliable Computational Method for Solving First-order Periodic BVPs of Fredholm Integro-differential Equations

1Ghaleb Gumah, 2Asad Freihat, 3Mohammed Al-Smadi, 4Rasha Bani Ata, 5Mousa Ababneh

1Applied Science Department, Faculty of Engineering Technology, Al-Balqa Applied University, Amman 11942, Jordan
2Pioneer Center for Gifted Students, Ministry of Education, Jerash 26110, Jordan
3Applied Science Department, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan
4Department of Mathematics and Computer Science, Tafila Technical University, Tafila 66110, Jordan
5Finance and Administrative Department, Al-Balqa Applied University, Irbid 21510, Jordan

ABSTRACT

The object of this paper is to develop a novel adaptive reproducing kernel Hilbert space method (RKHSM) to solve a class of first-order periodic Fredholm integro-differential equations. The analytical and approximate solution is represented in the form of series in the reproducing kernel space. The n-term approximation is obtained and proved to converge to the analytical solution. Furthermore, the solution methodology is based on generating orthogonal basis from the obtained kernel functions; whilst the orthonormal basis is constructing in order to formulate and utilize the solutions with its series form. Finally, efficacious computational experiments are reported to illustrate the accuracy, generality, superiority and applicability of the presented method.

INTRODUCTION

Integro-differential equations (IDEs) arise in many branches of science, engineering and technology. They are effectively a combination of differential and integral equations that often involved in mathematical formulations of natural phenomena often as approximation to partial differential equations, which represent much of the continuum phenomena. In the recent years, they have played a very important role in a wide variety of practical fields such as fluid dynamics, unsteady aerodynamics, electrodynamics of complex medium, population growth, neural network modeling, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, elasticity, potential theory, compartmental systems, nuclear reactors, and mathematical modeling of a hereditary phenomena.

The IDEs are some time very difficult to solve analytically due to the complexities of these equation, so it is required to detect an efficient approximate and numerical methods to obtain its approximate solutions. Indeed, the solution of these equations can be easily approximated with a large variety of method. Therefore, they have been of great interest by several authors. For instance, the techniques that use quadrature rules based on the generalized exponential type of interpolation function, local Taylor expansions, collocation and interpolation have been studied extensively, for more details see (Aguilar and Brunner, 1988; Sidi, 1989; Arikoglu and Ozkol, 2005; Maleknejad and Mahmoudi, 2003; Ghasemi, 2007) and the references cited therein. Other authors use the Wavelet-Galerkin method (Maleknejad and Mirzaee, 2005), the Adomian decomposition method (ADM) (Deeba et al., 2000; Abdel-Aziz, 2001; El-Sayed and Abdel-Aziz, 2003), the reproducing kernel space method (RKHSM) (Abu Arqub and Al-Smadi, 2014; Altawallbeh et al., 2013; Al-Smadi et al., 2013; Komashynska and Al-Smadi, 2014; Abu Arqub et al., 2013; Abu-Gdairi et al., 2013; Al-Smadi et al., 2014), the variational iteration method (VIM) (Sweilam, 2007), the homotopy perturbation method (HPM) (Yıldırım, 2008; Ghasemi, 2007), and Tau method (Hosseini and Shahrmarad, 2005). The numerical solvability of other version of differential problems can be found in Al-Smadi, (2013), Komashynska et al. (2014a, 2014b), Al-Smadi and Gumah (2014), Momani et al. (2014), Freihat and Al-Smadi (2013) and references therein.

In this paper, we present a new reliable computational method to obtain the solution of the following first-order periodic Fredholm integro-differential equation

Corresponding Author: Mohammed H. Al-Smadi, Applied Science Department, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Ajloun-Jordan
E-mail: mhm.smadi@yahoo.com
\begin{equation}
\begin{cases}
u'(x) + f(x)u(x) = F(x, u(x), Tu(x)); & 0 \leq x \leq 1, \\
u(0) = u(1),
\end{cases}
\end{equation}

where \( Tu(x) = \int_{0}^{1} h(x, t)u(t)dt \), \( f(x), h(x, t) \) are continuous functions, and \( F(x, u(x), Tu(x)) \) is linear or non-linear function of \( u \) depending on the problem discussed. The existence and uniqueness results for this type of problems have been investigated by many authors. See e.g. (Fubao, 2000; Agarwal, 1986; Muresan, 2004). Throughout this paper, we assume that the conditions on the given functions of the Equation (1) are such that the existence and uniqueness results for the solution of (1) are satisfied.

This paper is arranged as follows. After a preliminary section, in Section 3, the representation of exact and approximate solution of Equation (1) based on the reproducing kernel Hilbert space theory under the assumption that the solution of Equation (1) is unique will be given. In Section 4, we propose numerical method for solving first-order periodic Fredholm integro-differential equations. Numerical examples are given to illustrate the accuracy, generality, superiority, capability and simplicity of proposed method. Finally, some concluding remarks are given in the last section.

2. Preliminaries:
In this section, we present a review of the notations, definitions and preliminary of the reproducing kernel Hilbert space theory, according to the references (Geng and Cui, 2007a, 2007b, 2008; Geng, 2009a, 2009b).

Definition 1:
Let \( E \) be a nonempty abstract set. A function \( K : E \times E \rightarrow \mathbb{R} \) is a reproducing kernel of the Hilbert space \( \mathcal{H} \) if
\begin{enumerate}
\item for each \( x \in E \), \( K(\cdot, x) \in \mathcal{H} \).
\item for each \( x \in E \) and \( \varphi \in \mathcal{H} \), \( \varphi, K(\cdot, x) = \varphi(x) \).
\end{enumerate}
The last condition is called "the reproducing property": the value of the function \( \varphi \) at the point \( x \) is reproducing by the inner product of \( \varphi \) with \( K(\cdot, x) \).

Definition 2:
A Hilbert space \( \mathcal{H} \) of functions on a set \( E \) is called a reproducing kernel Hilbert space (RKHS) if there exists a reproducing kernel \( K \) of \( \mathcal{H} \).

Next, we firstly construct the space \( W_{2}^{2}[0,1] \) in which every function satisfies the periodic condition (2) and then utilize the space \( W_{2}^{1}[0,1] \).

Definition 3:
The inner product space \( W_{2}^{2}[0,1] \) is defined as follows:
\begin{equation} W_{2}^{2}[0,1] = \left\{ u(x) \left| \begin{array}{c}
u'(x) \text{ is absolutely continuous real valued function} \\
u'(x) \in L^{2}[0,1], \quad u(0) = u(1)\end{array} \right. \right\}. \end{equation}

Whilst the inner product and norm in \( W_{2}^{2}[0,1] \) are defined, respectively, by
\begin{equation}
\langle u(x), v(x) \rangle_{W_{2}^{2}} = \sum_{i=0}^{1} u^{i}(0)v^{i}(0) + \int_{0}^{1} u^{'}(t)v^{'}(t)dt,
\end{equation}
and \( \|u\| = \langle u, u \rangle_{W_{2}^{2}}^{\frac{1}{2}} \), where \( u, v \in W_{2}^{2}[0,1] \). It is easy to prove that \( W_{2}^{2}[0,1] \) is an inner product space.

Definition 4:
The inner product space \( W_{2}^{2}[0,1] \) is called a reproducing kernel space if and only if for any \( x \in [0,1] \), \( l: u \rightarrow u(x) \) is a bounded functional in \( W_{2}^{2}[0,1] \).

Theorem 1:
The inner product space \( W_{2}^{2}[0,1] \) is a reproducing kernel Hilbert space.

Proof:
The proof of the Theorem can be found similarly in (Geng and Cui, 2007a, 2007b) and references therein.
Theorem 2:
The space $W^2_2[0,1]$ is a complete reproducing kernel space. That is, for each fixed $x \in [0,1]$, there exists $R_x(y) \in W^2_2[0,1]$ such that $(u(y), R_x(y))_{W^2_2} = u(x)$ for any $u(y) \in W^2_2[0,1]$ and $y \in [0,1]$. As well as the reproducing kernel function $R_x(y)$ can be written as

$$R_x(y) = \begin{cases} 
  p_1(x) + p_2(x)y + p_3(x)y^2 + p_4(x)y^3, & y \leq x, \\
  q_1(x) + q_2(x)y + q_3(x)y^2 + q_4(x)y^3, & y > x,
\end{cases}$$

(3)

where $p_i(x)$ and $q_i(x)$, $i = 1,2,3,4$, will be given in the following proof.

Proof:
The proof of the completeness and reproducing property of $W^2_2[0,1]$ is similar to the proof in (Al-Smadi et al., 2012; Abu Arqub et al., 2012).

Now, let us find out the expression form of the reproducing kernel function $R_x(y)$ in $W^2_2[0,1]$. Through several integration by parts to Equation (2), we have

$$\int_0^1 u'(y)R_x'(y)dy = \sum_{i=0}^{1} (-1)^{1-i}u^{(i)}(y)\frac{\partial^3 R_x}{\partial y^3} \bigg|_{y=0}^{y=1} + \int_0^1 u(y)\frac{\partial^4 R_x}{\partial y^4}dy.$$ 

Thus

$$\langle u(y), R_x(y) \rangle_{W^2_2} = \sum_{i=0}^{1} \frac{u^{(i)}(0)}{(-1)^{1-i}}[\partial^i R_x(0) + (-1)^i \partial^3 R_x(0)]$$

$$+ \sum_{i=0}^{1} (-1)^{1-i}u^{(i)}(1)\partial^3 R_x(1) + \int_0^1 u(y)\frac{\partial^4 R_x}{\partial y^4}dy.$$ 

By using the fact $R_x(y) \in W^2_2[0,1]$, it follows that $R_x(0) = R_x(1)$. Also, since $u(x) \in W^2_2[0,1]$, it follows that $u(0) = u(1)$. Then

$$\langle u(y), R_x(y) \rangle_{W^2_2} = \sum_{i=0}^{1} \frac{u^{(i)}(0)}{(-1)^{1-i}}[\partial^i R_x(0) + (-1)^i \partial^3 R_x(0)]$$

$$+ \int_0^1 u(y)\frac{\partial^4 R_x}{\partial y^4}dy + c_1(u(0) - u(1)).$$

(4)

If

$$\begin{align*}
\partial^2 R_x(1) &= 0, \\
R_x(0) + \partial^3 R_x(0) + c_1 &= 0, \\
\partial^1 R_x(0) - \partial^2 R_x(0) &= 0, \\
\partial^3 R_x(1) + c_1 &= 0,
\end{align*}$$

then Equation (4) implies that

$$\langle u(y), R_x(y) \rangle_{W^2_2} = \int_0^1 u(y)\frac{\partial^2 R_x}{\partial y^2}dy.$$ 

Now, for any $x \in [0,1]$, if $R_x(y)$ satisfies

$$\frac{\partial^2 R_x}{\partial y^2}(y) = -\delta(x - y),$$

(5)

then $\langle u(y), R_x(y) \rangle_{W^2_2} = u(x)$. Obviously, $R_x(y)$ is the reproducing kernel of the space $W^2_2[0,1]$. 


The characteristic equation of Equation (5) is given by \(\lambda^4 = 0\), then the characteristic values are \(\lambda = 0\) with multiplicity 4. So, let the expression of the reproducing kernel function \(R_x(y)\) be as defined in Equation (3). On the other hand, for Equation (5), let \(R_x(y)\) satisfy the following

\[
\frac{\partial^m}{\partial y^m} R_x(x + 0) = \frac{\partial^m}{\partial y^m} R_x(x - 0), \quad m = 0, 1, 2.
\]

Integrating \(\frac{\partial^2}{\partial y^2} R_x(y)\) from \(x - \epsilon\) to \(x + \epsilon\) with respect to \(y\) and let \(\epsilon \to 0\), we have the jump degree of \(\frac{\partial^2}{\partial y^2} R_x(y)\) at \(y = x\) given by

\[
\frac{\partial^2}{\partial y^2} R_x(x + 0) - \frac{\partial^2}{\partial y^2} R_x(x - 0) = -1.
\]

Through the last descriptions the unknown coefficients of Equation (3) can be obtained. This completes the proof.

However, the representation of the reproducing kernel function \(R_x(y)\) in \(W_2^2[0,1]\), using Mathematica 7.0 software package, is provided by

\[
R_x(y) = \begin{cases} 
\frac{1}{48} \left[ x^2 y (6 + 3 y - y^2) + 3 x y (-6 - 3 y + y^2) + 6 x y (2 + y + y^2) - 8 (-6 + y^3) \right], \\
\frac{1}{48} \left[ 48 + 6 x y (2 - 3 y + y^2) + 3 x^2 y (2 - 3 y + y^2) - x^3 (8 - 6 y - 3 y^2 + y^3) \right].
\end{cases}
\]

**Theorem 3:**

If a Hilbert space \(H\) of functions on a set \(X\) admits a reproducing kernel \(R_x(y)\), then the reproducing kernel \(R_x(y)\) is uniquely determined by the Hilbert space \(H\).

**Proof:** 
Let \(R_x(y)\) be a reproducing kernel of \(H\). Suppose that there exists another kernel \(K_x(y)\) of \(H\).

Applying the reproducing property for \(R_x(y)\) and \(K_x(y)\), we get

\[
\|R_x - K_x\|^2 = \langle R_x - K_x, R_x - K_x \rangle = \langle R_x - K_x, R_x \rangle - \langle R_x - K_x, K_x \rangle \\
= \langle R_x - K_x \rangle(x) - \langle R_x - K_x \rangle(x) = 0.
\]

Hence, \(R_x = K_x\), that is, \(R_x(y) = K_x(y)\) for all \(x, y \in X\). The proof is complete.

3. Description of the RKHS Method:

In this section, we present a relatively new numeric-analytic method for solving first-order periodic Fredholm integro-differential equations subject to given periodic initial conditions based on the generalized Fourier series formula together with the reproducing kernel theory (Li, 2008; Lu et al., 2007; Li and Cui, 2003).

Let \(L: W_2^2[0,1] \to W_2^2[0,1]\) such that \(Lu(x) = u'(x) + f(x)u(x)\), then Equation (1) can be converted into the form as follows:

\[
Lu(x) = F(x, u(x), Tu(x)), \quad 0 \leq x \leq 1
\]

\[
u(0) = u(1),
\]

where \(u(x) \in W_2^2[0,1]\) and \(F(x, u(x), Tu(x)) \in W_2^2[0,1]\).

**Lemma 1:**

The operator \(L: W_2^2[0,1] \to W_2^2[0,1]\) is a bounded linear operator.

**Proof:**

It is so easy to see that \(L\) is a linear operator. Thus, it is enough to show that \(L\) is bounded operator. From the Equation (2), we have

\[
\|Lu\|_{W_2^2} = \|Lu\|_{W_2^2} = [(Lu)(0)]^2 + \int_0^1 [(Lu)(x)]^2 dx.
\]

By reproducing property of \(R_x(y)\), we have
\[ \begin{align*}
\langle u(x), (Lu)(x) \rangle_{W^2_2}, \\
(Lu)(x) &= \langle u, (L^*R_x) \rangle_{W^2_2}, \\
(Lu)'(x) &= \langle u, (L^*R_x) \rangle_{W^2_2}.
\end{align*} \]

By Schwarz inequality, we get
\[
||(Lu)(x)|| = \left| \langle u, (L^*R_x) \rangle_{W^2_2} \right| \leq ||L^*R_x||_{W^2_2}||u||_{W^2_2} = M_1||u||_{W^2_2},
\]
and
\[
||(Lu)'(x)|| = \left| \langle u, (L^*R_x) \rangle_{W^2_2} \right| \leq ||(L^*R_x)||_{W^2_2}||u||_{W^2_2} = M_2||u||_{W^2_2},
\]
where \( M_1, M_2 > 0 \) are positive constants.

Thus \( [(Lu)(0)]^2 \leq M_1^2||u||_{W^2_2}^2 \), \( [(Lu)'(x)]^2 \leq M_2^2||u||_{W^2_2}^2 \), and \( \int_0^1 [(Lu)'(x)]^2 dx \leq (M_1^2 + M_2^2)||u||_{W^2_2}^2 \). That is,
\[
|| (Lu)(x) ||_{W^2_2}^2 = [(Lu)(0)]^2 + \int_0^1 [(Lu)'(x)]^2 dx \leq (M_1^2 + M_2^2)||u||_{W^2_2}^2 = M||u||_{W^2_2}^2.
\]

where \( M = M_1^2 + M_2^2 > 0 \). The proof is complete.

The solution methodology in this paper is based on generating the orthogonal basis from the obtained kernel functions. Whilst the orthonormal basis is constructing in order to formulate and utilize the solutions with series form. Now, we construct an orthogonal system of functions as follows:

Let \( \Phi_i(x) = R_x(x) \) and \( \Psi_i(x) = L^* \Phi_i(x) \), where \( L^* \) is the conjugate operator of \( L \). Consequently, in terms of the properties of reproducing kernel \( R_x(y) \), one obtains
\[
\langle u(x), \Psi_i(x) \rangle_{W^2_2} = \langle u(x), L^* \Phi_i(x) \rangle_{W^2_2} = \langle (Lu)(x), \Phi_i(x) \rangle_{W^2_2} = Lu(x_i), i = 1, 2, \ldots
\]

Next, we collect an important lemma in order to construct the reproducing kernel Hilbert space (RKHS) method.

**Lemma 2:**

If \( \{x_i\}_{i=1}^\infty \) is dense on \([0,1]\), then \( \{\Psi_i(x)\}_{i=1}^\infty \) is a complete system of \( W^2_2[0,1] \) and \( \Psi_i(x) = L_yR_x(y) |_{y=x_i} \).

**Proof:**

First of all, one can see that
\[
\Psi_i(x) = L^* \Phi_i(x) = \langle L^* \Phi_i(x), R_x(y) \rangle_{W^2_2} = \langle \Phi_i(x), L^*R_x(y) \rangle_{W^2_2} = L_yR_x(y) |_{y=x_i}.
\]

Now, for each fixed \( u(x) \in W^2_2[0,1] \). If \( \langle u(x), \Psi_i(x) \rangle_{W^2_2} = 0, i = 1, 2, \ldots \), then
\[
\langle u(x), \Psi_i(x) \rangle_{W^2_2} = \langle u(x), (L^* \Phi_i(x)) \rangle_{W^2_2} = \langle Lu(x), \Phi_i(x) \rangle_{W^2_2} = Lu(x_i) = 0.
\]

By using the fact that \( \{x_i\}_{i=1}^\infty \) is dense on \([0,1] \), therefore, \( Lu(x) = 0 \). It follows that \( u(x) = 0 \) from the existence of \( L^{-1} \) and the continuity of \( u(x) \). The proof is complete.

In contrast, the orthonormal system of functions \( \{\overline{\Psi}_i(x)\}_{i=1}^\infty \) in \( W^2_2[0,1] \) can be derived from Gram-Schmidt orthogonalization process of \( \{\Psi_i(x)\}_{i=1}^\infty \).

\[
\overline{\Psi}_i(x) = \sum_{k=1}^i \beta_{ik} \Psi_k(x),
\]

where \( \beta_{ik} \) are orthogonalization coefficients, \( \beta_{ii} > 0, i = 1, 2, \ldots, n \), and
\[
c_i = \langle \Psi_i(x), \Psi_j(x) \rangle = \sum_{r=0}^{2} \Psi_i^{(r)}(0) \Psi_j^{(r)}(0) + \int_{0}^{1} \Psi_i^{(r)}(t) \Psi_j^{(r)}(t) dt, \tag{8}
\]

\[
\overline{c}_{ij} = \langle \Psi_i, \overline{\Psi_j} \rangle = \sum_{k=1}^{j} \beta_{jk} c_{ik}, \tag{9}
\]

\[
\beta_{i1} = \frac{1}{\sqrt{c_{11}}}, \tag{10}
\]

\[
\beta_{ik} = -\frac{\sum_{j=k+1}^{i} \overline{c}_{ij} \beta_{jk}}{\sqrt{c_{ii} - \sum_{k=1}^{i-1} (\overline{c}_{ik})^2}} (k < i), \tag{11}
\]

\[
\beta_{ii} = \frac{1}{\sqrt{c_{ii} - \sum_{k=1}^{i-1} (\overline{c}_{ik})^2}} (i > 1). \tag{12}
\]

In the next theorem, we will give the presentation of the exact solution of Equation (6) in the reproducing kernel Hilbert space \(W_2^2[0,1]\).

**Theorem 4:**

If \([x_i]_{i=1}^{\infty}\) is dense on \([0, 1]\) and \(u(x) \in W_2^2[0,1]\) is the solution of Equation (6), then \(u(x)\) satisfy the following form

\[
u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k)) \overline{\Psi_i}(x), \tag{13}\]

and the approximate solution can be obtained by

\[
u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} F(x_k, u_{n-1}(x_k), Tu_{n-1}(x_k)) \overline{\Psi_i}(x), \tag{14}\]

where \(u_0(x) \in W_2^2[0,1] (u_0\ fixed)\).

**Proof:**

Clear that \((\overline{\Psi_i}(x))_{i=1}^{\infty}\ is the complete orthonormal basis in \(W_2^2[0,1]\). Since \(u(x) \in W_2^2[0,1]\, u(x)\ can be expanded in the form of Fourier series about \((\overline{\Psi_i}(x))_{i=1}^{\infty}\ as

\[
u(x) = \sum_{i=1}^{\infty} \langle u(x), \overline{\Psi_i}(x) \rangle \overline{\Psi_i}(x),
\]

and since the space \(W_2^2[0,1]\ is the Hilbert space, then the series \(u(x) = \sum_{i=1}^{\infty} \langle u(x), \overline{\Psi_i}(x) \rangle \overline{\Psi_i}(x)\) is convergent in the norm \(\|\|_{W_2^2}\). From the Fourier series expansion and by Equation (7), \(u(x)\ can be written

\[
u(x) = \sum_{i=1}^{\infty} \langle u(x), \overline{\Psi_i}(x) \rangle_{W_2^2} \overline{\Psi_i}(x)
\]

\[
= \sum_{i=1}^{\infty} \langle u(x), \sum_{k=1}^{i} \beta_{ik} \Psi_k(x) \rangle_{W_2^2} \overline{\Psi_i}(x)
\]

\[
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), \Psi_k(x) \rangle_{W_2^2} \overline{\Psi_i}(x)
\]
\[
\begin{align*}
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (u(x), L' \phi_k(x))_{W^2} \overline{\varphi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (Lu(x), \phi_k(x))_{W^2} \overline{\varphi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} L u(x_k) \overline{\varphi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k)) \overline{\varphi}_i(x).
\end{align*}
\]

The approximate solution can be obtained by the \( n \)-term intercept of the exact solution \( u(x) \) as

\[
u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} F(x_k, u_{k-1}(x_k), Tu_{k-1}(x_k)) \overline{\varphi}_i(x).
\]

Therefore, from the equations (8) to (14), we obtain the reproducing kernel Hilbert space method to solve Equation (6). The proof is complete.

**Lemma 3:**

If \( u(x) \in W^2_2[0,1] \), then there exists \( M > 0 \), such that \( \|u\|_{C^2[0,1]} \leq \|u\|_{W^2_2[0,1]} \) where

\[
\|u\|_{C^2[0,1]} = \max_{x \in [0,1]} |u(x)| + \max_{x \in [0,1]} |u'(x)| + \max_{x \in [0,1]} |u''(x)|.
\]

**Lemma 4:**

If \( \|u_n - u\|_{W^2_2} \to 0 \), \( x_n \to x \), \( n \to \infty \) and \( F(x, y, z) \) for \( x \in [0,1], y, z \in (-\infty, \infty) \) is continuous with respect to \( x, y, z \), then

\[
F(x_n, u_{n-1}(x_n), Tu_{n-1}(x_n)) \to F(x, u(x), Tu(x)) \text{ as } n \to \infty.
\]

Next, we present the iterative formula of \( u_n(x) \) in Equation (14), which is converging to the exact solution to Equation (6).

**Theorem 5:**

Suppose the following conditions are satisfied:

(i) \( \|u_n\|_{W^2_2} \) is bounded.

(ii) \( \{x_i\}_{i=1}^{\infty} \) is dense on \( [0,1] \).

(iii) \( u(x) \in W^2_2[0,1] \) and \( F \in W^1_2[0,1] \).

Then, the approximate solution \( u_n(x) \) in formula (14) is convergent to the exact solution \( u(x) \) of Equation (6) in \( W^2_2[0,1] \), where as \( u(x) \) is given by

\[
u(x) = \sum_{i=1}^{\infty} A_i \overline{\varphi}_i(x),
\]

where

\[
A_i = \sum_{k=1}^{i} \beta_{ik} F(x_k, u_{k-1}(x_k), Tu_{k-1}(x_k)).
\]

**Proof:**

Firstly, we will prove the convergence of the approximate solution \( u_n(x) \). By using Equation (14), we have
\[ u_{n+1}(x) = u_n(x) + A_{n+1} \Psi_{n+1}(x). \]

From the orthogonality of \( \{ \Psi_i(x) \}_{i=1}^\infty \), it follows that
\[
\| u_{n+1} \|_{W_2^2}^2 = \| u_n \|_{W_2^2}^2 + (A_{n+1})^2 = \| u_{n-1} \|_{W_2^2}^2 + (A_n)^2 + (A_{n+1})^2 \\
= \| u_0 \|_{W_2^2}^2 + \sum_{i=1}^{n+1} (A_i)^2.
\]

From boundedness of \( \| u_n \|_{W_2^2} \), we have \( \sum_{i=1}^{\infty} (A_i)^2 < \infty \), that is, \( \{ A_i \}_{i=1}^{\infty} \in l^2 \).

Let \( m > n \), then \( (u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n) \), it follows that
\[
\| u_m(x) - u_n(x) \|_{W_2^2}^2 = \| u_m(x) - u_{m-1}(x) + u_{m-1}(x) - \cdots + u_{n+1}(x) - u_n(x) \|_{W_2^2}^2 \\
= \| u_m(x) - u_{m-1}(x) \|_{W_2^2}^2 + \cdots + \| u_{n+1}(x) - u_n(x) \|_{W_2^2}^2 = \sum_{i=n+1}^{m} (A_i)^2 \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

Considering the completeness of \( W_2^2[0,1] \), there exists \( u(x) \in W_2^2[0,1] \) such that \( u_n(x) \rightarrow u(x) \) as \( n \rightarrow \infty \) in the sense of the norm of \( W_2^2[0,1] \).

Secondly, we will prove that \( u(x) \) is the solution of Equation (6). By lemma (3) and since \( \{ x_i \}_{i=1}^{\infty} \) is dense on \([0, 1]\), we know \( u_n(x) \) converge uniformly to \( u(x) \). It follows that, on taking limit in Equation (14), we have
\[
u(x) = \sum_{i=1}^{\infty} A_i \Psi_i(x).
\]

Since
\[
(Lu)(x_j) = \sum_{i=1}^{\infty} A_i (L \Psi_i(x), \Phi_j(x))_{W_2^2} = \sum_{i=1}^{\infty} A_i (\Psi_i(x), L^* \Phi_j(x))_{W_2^2} = \sum_{i=1}^{\infty} A_i (\Psi_i(x), \Psi_j(x))_{W_2^2}.
\]

It follows that
\[
\sum_{j=1}^{n} \beta_{nj} (Lu)(x_j) = \sum_{i=1}^{\infty} A_i (\Psi_i(x), \sum_{j=1}^{n} \beta_{nj} \Psi_j(x))_{W_2^2} = \sum_{i=1}^{\infty} A_i (\Psi_i(x), \Psi_n(x))_{W_2^2} = A_n.
\]

If \( n = 1 \), then
\[
(Lu)(x_1) = F(x_1, u_0(x_1), Tu_0(x_1)).
\]

If \( n = 2 \), then
\[
(Lu)(x_2) = F(x_2, u_1(x_2), Tu_1(x_2)).
\]

Furthermore, it is easy to see by induction that
\[
(Lu)(x_j) = F(x_j, u_{j-1}(x_j), Tu_{j-1}(x_j)).
\]

Since \( \{ x_j \}_{j=1}^{\infty} \) is dense on \([0, 1]\), for any \( y \in [0,1] \), there exists subsequence \( \{ x_{nj} \} \) such that \( x_{nj} \rightarrow y \), as \( j \rightarrow \infty \).

Hence, let \( j \rightarrow \infty \) in the last equation, by the convergence of \( u_n(x) \) and lemma (4), we have
\[
(Lu)(y) = F(y, u(y), Tu(y)).
\]

That is, \( u(x) \) is the solution of Equation (6). The proof is complete.
Theorem 6:
Assume that \( u(x) \in W^2_2[0,1] \) is the solution of Equation (6) and \( r_n(x) \) is the error in the approximate solution \( u_n(x) \), where \( u_n(x) \) is given by Equation (14). Then the error is monotone decreasing in the sense of \( \| \cdot \|_{W^2_2} \).

Proof:
Suppose that \( u(x) \) and \( u_n(x) \) are given by Equations (13) and (14), respectively. We have

\[
\| r_n(x) \|_{W^2_2}^2 = \| u(x) - u_n(x) \|_{W^2_2}^2 = \left\| \sum_{i=m+1}^{\infty} \sum_{k=1}^{i} \beta_k F(x_k, u_{k-1}(x_k), T u_{k-1}(x_k)) W_i(x) \right\|_{W^2_2}^2
\]

and \( \| r_{n-1}(x) \|_{W^2_2}^2 = \sum_{i=m}^{n} (A_i)^2 \). Thus, \( \| r_n(x) \|_{W^2_2} \leq \| r_{n-1}(x) \|_{W^2_2} \). Consequently, the error \( r_n \) is monotone decreasing in the sense of \( \| \cdot \|_{W^2_2} \). The proof is complete.

4. Numerical Examples:
In this section, we have applied the reproducing kernel Hilbert space method (RKHSM) to a class of linear and nonlinear Fredholm IDEs subject to given initial condition of periodic type. Efficacious computational experiments are studied to guarantee the procedure and to demonstrate the accuracy of the present method. The results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other. The examples are computed using Mathematica 7.0.

Example 1:
Consider the following linear periodic Fredholm integro-differential equation

\[
\begin{cases}
u'(x) + u(x) + \int_0^1 (2t - 1) \cos(t) u(t) dt = f(x), & 0 \leq x \leq 1, \\
u(0) = u(1),
\end{cases}
\]

where \( f(x) = e^{-(x^2-x)} + 2x \sinh(x^2 - x) \).

Table 1: Numerical results for Example 1 when \( n = 101 \).

<table>
<thead>
<tr>
<th>Node</th>
<th>Exact Solution</th>
<th>Approximate Solution</th>
<th>Absolute Error</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>1.00904528339575</td>
<td>1.00903826731733</td>
<td>7.01608 \times 10^{-6}</td>
<td>6.95318 \times 10^{-6}</td>
</tr>
<tr>
<td>32</td>
<td>1.02376844422378</td>
<td>1.0237577043935</td>
<td>1.07368 \times 10^{-5}</td>
<td>1.04875 \times 10^{-5}</td>
</tr>
<tr>
<td>48</td>
<td>1.03131213746320</td>
<td>1.0313002516722</td>
<td>1.21123 \times 10^{-5}</td>
<td>1.17445 \times 10^{-5}</td>
</tr>
<tr>
<td>64</td>
<td>1.02665970162571</td>
<td>1.0266485431210</td>
<td>1.11585 \times 10^{-5}</td>
<td>1.08687 \times 10^{-5}</td>
</tr>
<tr>
<td>80</td>
<td>1.01282732997901</td>
<td>1.01281937456327</td>
<td>7.95542 \times 10^{-6}</td>
<td>7.85666 \times 10^{-6}</td>
</tr>
<tr>
<td>96</td>
<td>1.000737370601420</td>
<td>1.00073481042444</td>
<td>2.56018 \times 10^{-6}</td>
<td>2.55829 \times 10^{-6}</td>
</tr>
</tbody>
</table>

The exact solution is \( u(x) = \cosh(\sqrt{x^2 - x}) \). Using RKHS method, taking \( x_i = \frac{i-1}{n-1}, \ i = 1, 2, \ldots, n \). The numerical results at some selected grid points for \( n = 101 \) are given in Table 1.

Example 2:
Consider the following linear periodic Fredholm integro-differential equation
\[
\begin{align*}
\left\{ \begin{array}{l}
u'(x) + xu(x) + \int_0^1 t(x - t^2)u(t)dt = f(x), \quad 0 \leq x \leq 1, \\
u(0) = u(1),
\end{array} \right.
\]

where \( f(x) = 2x^2(x^2 - x + 3) - \frac{41}{10}x + \frac{1}{21}. \)

The exact solution is \( u(x) = 2x^2(x - 1). \) The numerical results at some selected grid points, taking \( x_i = \frac{i-1}{n-1} \) for \( i = 1, 2, ..., n, \) for \( n = 101 \) are given in Table 2.

**Example 3:**

Consider the following nonlinear periodic Fredholm integro-differential equation

\[
\begin{align*}
\left\{ \begin{array}{l}
u'(x) - 2x(2x - 1)(u(x))^2 - \int_0^1 \frac{2t}{u(t)}dt = f(x), \quad 0 \leq x \leq 1, \\
u(0) = u(1),
\end{array} \right.
\]

where \( f(x) = \frac{x^2}{(x^2 - x^2 + 1)^2} - \frac{9}{10}. \)

The exact solution is \( u(x) = \frac{1}{x^2(1-x) + 1}. \) The numerical results at some selected grid points, taking \( x_i = \frac{i-1}{n-1} \) for \( i = 1, 2, ..., N, \) for \( N = 51 \) and \( n = 3 \) are given in Table 3.

**Example 4:**

Consider the following nonlinear periodic Fredholm integro-differential equation

\[
\begin{align*}
\left\{ \begin{array}{l}
u'(x) + (u(x))^2 - \int_0^1 xt(3t - 2)u(t)dt = f(x), \quad 0 \leq x \leq 1, \\
u(0) = u(1),
\end{array} \right.
\]

where \( f(x) = x(2 - 3x) \cos(x^2(1 - x)) + (1 + \sin(x^2(1 - x)))^2. \)

The exact solution is \( u(x) = \sin(x^2(1 - x)) + 1. \) The numerical results at some selected grid points, reveal that the method is fully compatible with the complexity of such problems, by taking \( x_i = \frac{i-1}{N-1}, \) for \( N = 51 \) and \( n = 3 \) are given in Table 4.
Table 4: Numerical results for Example 4 when N = 51, n = 3.

<table>
<thead>
<tr>
<th>Node</th>
<th>Exact Solution</th>
<th>Approximate Solution</th>
<th>Absolute Error</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.16</td>
<td>1.0215023427178134</td>
<td>1.0215006417717398</td>
<td>1.70095×10⁻⁶</td>
<td>1.66514×10⁻⁶</td>
</tr>
<tr>
<td>0.32</td>
<td>1.0695757438417552</td>
<td>1.0695729798603852</td>
<td>2.76398×10⁻⁶</td>
<td>2.58418×10⁻⁶</td>
</tr>
<tr>
<td>0.48</td>
<td>1.1195215858254712</td>
<td>1.119519266469458</td>
<td>2.31936×10⁻⁶</td>
<td>2.07174×10⁻⁶</td>
</tr>
<tr>
<td>0.64</td>
<td>1.146922179876829</td>
<td>1.146927771516749</td>
<td>4.4036×10⁻⁷</td>
<td>3.84365×10⁻⁷</td>
</tr>
<tr>
<td>0.80</td>
<td>1.127650760861488</td>
<td>1.1276512832377876</td>
<td>5.22352×10⁻⁷</td>
<td>4.63221×10⁻⁷</td>
</tr>
<tr>
<td>0.96</td>
<td>1.0368556511508822</td>
<td>1.0368552015863124</td>
<td>4.49565×10⁻⁷</td>
<td>4.33585×10⁻⁷</td>
</tr>
</tbody>
</table>

Conclusion:
In this research, a computational method is applied to a special class of Fredholm integro-differential equations subject to given initial condition. Efficacious computational experiments are studied to guarantee the procedure and to demonstrate the accuracy of the present method. The proposed method possess some of well known advantages: Firstly, it is accurate, needless effort to achieve the results, and is developed especially for the nonlinear case. Secondly, it used in a direct way without using linearization, perturbation or restrictive assumptions, and it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time. Thirdly, it provides the solutions in terms of convergent series with easily computable components and the results have shown remarkable performance. Finally, it is possible to pick any point in the interval of integration and as well the approximate solutions and its derivatives will be applicable. However, with the availability of this methodology, it will be possible to investigate the approximate solution of some applicable integro-differential equations. So, this method can be used to solve different types of integro-differential equations. Also, the accuracy of the solution can be improved by selecting large values of N and n. The examples are computed by using Mathematica 7.0 software package.

REFERENCES


