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Skew N-Normal Operators

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ABSTRACT

In this paper, we introduce a new class of operators acting on a complex Hilbert space H which is called skew n-normal operators. An operator T∈B(H) is called skew n-normal operator if (T^n T^*)T=T(T^* T^n), where n is positive integer number greater than 1 and T^* is the adjoint of the operator T., We investigate some basic properties of such operators and study the relation among the skew n-normal operators and some other operators.

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INTRODUCTION

Through this paper, $\mathcal{B}(\mathcal{H})$ denoted to the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be self-adjoint if $T^* = T$, isomtry if $T^*T = I$, unitary if $T^*T = TT^* = I$, where T^* is the adjoint of T (Berberian, 1976). The operator $T \in \mathcal{B}(\mathcal{H})$ is called normal if $TT^* = T^*T$ (Alzuraiqi and Patel , 2010) , quasi-normal if $T(T^*T) = (T^*T)T$ (Shqipe Lohaj, 2010) , n-normal if $T^nT^* = T^*T^n$ (Alzuraiqi and Patel , 2010) , quasi n-normal if $T(T^*T^n) = (T^*T^n)T$ and n power quasi normal if $T^n(T^*T) = (T^*T)T^n$ (Panayappan, 2012), where n is positive integer number.

Skew N-Normal Operators:

In this section, we will study some properties which are applied for the skew n-normal operators.

Definition (2.1):

The operator $T \in \mathcal{B}(\mathcal{H})$ is called skew n-normal operator if $(T^nT^*)T = T(T^*T^n)$, where T^* is the adjoint of the operator T.It is clear that every operator is skew 1- normal operator, therefore we assume in this paper that n is positive integer number greater than 1.

Example (2.2):

Let U be a unilateral shift operator on ℓ_2, (i.e. $U(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3 \dots)$ .Then U is skew n-normal operator for every n .

Example (2.3):

If $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ is an operator on $\mathbb{C}^2$, then T is not skew 2-normal operator.

Since $(T^2T^*)T = \begin{pmatrix} -1 & 2i \\ -2i & -5 \end{pmatrix} \neq \begin{pmatrix} -5 & -2i \\ 2i & -1 \end{pmatrix} = T(T^*T^2)$.

Proposition (2.4):

Every normal operator is skew n-normal operator for every positive integer number n.

Proof:

Let T be a normal operator, therefore $TT^* = T^*T$, so

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$(T^nT^*)T = (TT^*T^{n-1})T = T(T^*T^n)$

Thus T is skew n-normal operator for every positive integer number n .

The following example shows that the converse of Proposition (2.4) is not true.

Example (2.5):

If $T = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ is an operator on $\mathbb{C}^2$ where a is non-zero complex number , then T is skew n-normal operator for every n but it is not normal operator.

Remark (2.6):

skew n-normal operators and n-normal operators are independent as we saw in example (2.2) and (2.3).
In the following proposition, we give some properties of the skew n-normal operators.

Proposition (2.7):

If T is skew n-normal operator on a Hilbert space $\mathcal{H}$. Then:

1- αT is skew n-normal operator for every complex number α.
2- T^* is skew n-normal operator.
3- If T^{-1} exists, then T^{-1} is skew n-normal operator.
4- If S is unitarily equivalence to T, then S is skew n-normal operator.
5- If M is closed subspace of $\mathcal{H}$, then (T/M) is skew n-normal operator.

Proof: (1)

$((\alpha T)^n(\alpha T)^*)(\alpha T) = \alpha^n\bar{\alpha}\alpha(T^nT^*)T$

$= \alpha^n\bar{\alpha}\alpha T(T^*T^n) = (\alpha T)((\alpha T)^*(\alpha T)^n)$

So that (αT) is skew n-normal operator.

(2) since T is skew n-normal operator, then $(T^nT^*)T = T(T^*T^n)$ taking adjoint of two sides, we have $T^*(TT^{*n}) = (T^{*n}T)\,T^*$, therefore T^* is skew n-normal operator.

(3) since T is skew n-normal operator, then $(T^nT^*)T = T(T^*T^n)$ taking inverse of two side, we get $(T^{-1})[(T^{-1})^*(T^{-1})^n] = [(T^{-1})^n(T^{-1})^*](T^{-1})$

So that T^{-1} is skew n-normal operator.

(4) since S is unitarily equivalence to T. Then there exists unitary operator U such that $S = UTU^*$, so that $S^* = UT^*U^*$ and $S^n = UT^nU^*$

$(S^nS^*)S = (UT^nU^*UT^*U^*)UTU^* = U(T^nT^*)TU^*$

Since T is skew n-normal operator, then $(S^nS^*)S = UT(T^*T^n)U^*$

On the other hand $S(S^*S^n) = UTU^*(UT^*U^*UT^nU^*) = UT(T^*T^n)\,U^*$

Thus S is skew n-normal operator.

(5) $[(T/M)^n(T/M)^*](T/M) = [(T^n/M)(\,T^*/M)](T/M)$

$= (T^nT^*)T/M$

$= T(T^*T^n)/M$

$= (T/M)[(T/M)^*(T/M)^n]$

Therefore T/M is skew n-normal operator.

The following example shows that if unitarily equivalence in Proposition (2.7)(4) is replaced by similarity then the result is need not be true.

Example (2.8):

Consider the two operators $T = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acting on the two dimensional Hilbert space $\mathbb{C}^2$, then T is skew 2-normal operator, but $S = XTX^{-1} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ is not skew 2-normal operator.

Remark (2.9):

Not every invertible operator is skew n-normal operator for example the operator $T = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ on the two dimensional Hilbert space $\mathbb{C}^2$ is invertible but not skew 2-normal operator .

The following example shows that the product of two skew n-normal operator is not necessary skew n-normal operator.

Example (2.10):

The operators $S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acting on two dimensional Hilbert space $\mathbb{C}^2$ are skew 2-normal operator, but

$[(ST)^2(ST)^*](ST) = \begin{pmatrix}1 & -1\\3 & -2\end{pmatrix} \neq \begin{pmatrix}1 & -3\\1 & -2\end{pmatrix} = (ST)[(ST)^*(ST)^2]$

Therefore ST is not skew 2-normal operator.

Proposition (2.11):

Let S be a normal operator and T is skew n-normal operator. If S and T are commute. Then (ST) is skew n-normal operator.

Proof:

Since S is a normal operator commute with T, then by Fuglede-Putnam theorem, S commute with T^*. Hence

$$[(ST)^n(ST)^*](ST) = [S^nT^nT^*S^*]ST = S^n(T^nT^*)TS^*S = S^nT(T^*T^n)S^*S = STT^*S^{n-1}T^nS^*S$$
$$= (ST)[T^*S^*S^{n-1}T^nS] = (ST)[T^*S^*S^nT^n] = (ST)[(ST)^*(ST)^n]$$

So that (ST) is skew n-normal operator.

The following example shows that if T and S are skew n-normal operator, then $T + S$ is not necessary skew n-normal operator.

Example (2.12):

The operators $T = I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ and $S = \begin{pmatrix}0 & a\\0 & 0\end{pmatrix}$ are skew 2-normal operators , where a is non-zero complex number, but

$$((S+T)^2(S+T)^*)(S+T) = \begin{pmatrix}1+2a^2 & 3a+2a^3\\a & 1+a^2\end{pmatrix} \neq \begin{pmatrix}1+a^2 & 3a+2a^3\\a & 1+2a^2\end{pmatrix} = (S+T)((S+T)^*(S+T)^2)$$

Theorem (2.13):

If T is skew n-normal operator, then T is skew n+k(n-1)-normal operator, for every positive integer number k.

Proof: since T is skew n-normal operator then $(T^nT^*)T = T(T^*T^n)$

We prove by induction that T is skew n+k(n-1)-normal operator for every positive integer k.

(Base case) : when k=1

$$\left(T^{n+(n-1)}\ T^*\right)T = T^{n-1}(T^n\ T^*)T = T^{n-1}\ T(T^*T^n) = (T^n\ T^*)TT^{n-1} = T(T^*T^n)T^{n-1} = T\left(T^*T^{n+(n-1)}\right)$$

(Inductive step) : suppose the result is true for n=k.

$$\left(T^{n+(k+1)(n-1)}\ T^*\right)T = T^{n-1}\left[\left(T^{n+k(n-1)}\ T^*\right)T\right]$$
$$= T^{n-1}[T(T^*T^{n+k(n-1)})] = [\ (T^n\ T^*)T]\ T^{n+k(n-1)-1}$$
$$= T(T^*T^n)T^{(k+1)(n-1)} = T\left(T^*T^{n+(k+1)(n-1)}\right)$$

Therefore T is skew $n + (k+1)(n-1)$ -normal operator.

The proof of the following corollary consequence from theorem (2.13).

Corollary (2.14): If T is skew 2-normal operator ,then T is skew n-normal operator $\forall\, n \geq 2$.

The following example shows if T is skew 3-normal operator, then not necessary T is skew 2-normal operator.

Example (2.15):

Let $T = \begin{pmatrix}1 & 1 & 1\\0 & 0 & 0\\-1 & 0 & -1\end{pmatrix}$ be an operator on Hilbert space $\mathbb{C}^3$.Then T is skew 3-normal operator, but not skew 2-normal operator.

Proposition (2.16):

If $T = -T^*$, then T is skew n-normal operator for every n.

Proof:

we show T is skew 2-normal operator ,hence by corollary (2.14) T is skew n-normal operator for every n .

$(T^2T^*)T = T(-T^*)T^*T = TT^*(-T^*)T = T(T^*T^2)$

Thus T is skew 2-normal operator.

Remark (2.17):

1- If T is isometry operator, then T is skew n-normal operator.
2- If T is self-adjoint, then T is skew n-normal operator.
3- If T is unitary operator, then T is skew n-normal operator.

Theorem (2.18): Let $T_1, T_2, \ldots, T_m$ be skew n-normal operators in $\mathcal{B}(\mathcal{H})$. Then $(T_1 \oplus T_2 \oplus \ldots \oplus T_m)$ and $(T_1 \otimes T_2 \otimes \ldots \otimes T_m)$ are skew n-normal operator.

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Proof:$[(T_1\oplus T_2\oplus \ldots\oplus T_m)^n(T_1\oplus T_2\oplus \ldots\oplus T_m)^*](T_1\oplus T_2\oplus \ldots\oplus T_m)$
$= [(T_1{}^n\oplus T_2{}^n\oplus \ldots\oplus T_m^n)(T_1^*\oplus T_2{}^*\oplus \ldots\oplus T_m^*)](T_1\oplus T_2\oplus \ldots\oplus T_m)$
$= (T_1{}^nT_1^*\oplus T_2{}^nT_2{}^*\oplus \ldots\oplus T_m^nT_m^*)(T_1\oplus T_2\oplus \ldots\oplus T_m)$
$= ((T_1{}^nT_1^*)T_1\oplus (T_2{}^nT_2{}^*)T_2\oplus \ldots\oplus(T_m^nT_m^*)T_m)$
Since $T_1, T_2, \ldots, T_m$ are skew n-normal operators. Then
$= (T_1(T_1^*T_1{}^n)\oplus T_2(T_2{}^*T_2{}^n)\oplus \ldots\oplus T_m(T_m^*T_m^n))$
$= (T_1\oplus T_2\oplus \ldots\oplus T_m)[(T_1\oplus T_2\oplus \ldots\oplus T_m)^*(T_1\oplus T_2\oplus \ldots\oplus T_m)^n]$
Also $[(T_1\otimes T_2\otimes \ldots\otimes T_m)^n(T_1\otimes T_2\otimes \ldots\otimes T_m)^*](T_1\otimes T_2\otimes \ldots\otimes T_m)$
$= [(T_1{}^n\otimes T_2{}^n\otimes \ldots\otimes T_m^n)(T_1^*\otimes T_2{}^*\otimes \ldots\otimes T_m^*)](T_1\otimes T_2\otimes \ldots\otimes T_m)$
$= (T_1{}^nT_1^*\otimes T_2{}^nT_2{}^*\otimes \ldots\otimes T_m^nT_m^*)(T_1\otimes T_2\otimes \ldots\otimes T_m)$
$= ((T_1{}^nT_1^*)T_1\otimes (T_2{}^nT_2{}^*)T_2\otimes \ldots\otimes(T_m^nT_m^*)T_m)$
Since $T_1, T_2, \ldots, T_m$ are skew n-normal operators. Then
$= (T_1(T_1^*T_1{}^n)\otimes T_2(T_2{}^*T_2{}^n)\otimes \ldots\otimes T_m(T_m^*T_m^n))$
$= (T_1\otimes T_2\otimes \ldots\otimes T_m)[(T_1\otimes T_2\otimes \ldots\otimes T_m)^*(T_1\otimes T_2\otimes \ldots\otimes T_m)^n]$.

Proposition (2.19):
Every quasi normal operator is skew n-normal operator.

Proof:
Let T be a quasi normal operator. Then T^{n-1} commute with T^*T for every n ,so that

$$(T^n T^*)T = T\,T^{n-1}(T^*T) = T(T^*T)T^{n-1} = T(T^*T^n)$$

Therefore T is skew n-normal operator.

Remark (2.20):
The converse of Proposition (2.19) is not necessary true for example the operator $T = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ on two dimensional Hilbert space $\mathbb{C}^2$ is skew n-normal operator but not quasi normal operator.

The following two examples shows that the skew n-normal operator and n-power quasi normal operator are independent.

Example (2.21):
Let U^* be the adjoint of the unilateral shift operator on ℓ_2. (i.e. $U^*(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4 \ldots))$
Since $(U^{*n}U)U^* = U^{*n-1}U^* = U^{*n} = U^*(UU^{*n})$. Then U^* is skew n-normal operator.
But $U^{*^n}(UU^*) = U^{*^n} \neq (UU^*)U^{*^n}$ so that U^* is not n-power quasi- normal operator.

Example (2.22):
The operator $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ on $\mathbb{C}^2$ is 2-power quasi- normal operator, but not skew 2-normal operator.

Proposition (2.23):
If T is n-normal operator and quasi n-normal operator. Then T is skew n-normal operator.

Proof:
since $T^nT^* = T^*T^n$ and $T(T^*T^n) = (T^*T^n)T$.
Then $T(T^*T^n) = (T^*T^n)T = (T^nT^*)T$
Thus T is skew n-normal operator.
The following example shows that skew n-normal operator not necessary be quasi n-normal operator.

Example (2.24):
Let U^* be the adjoint of the unilateral shift operator then U^* is skew n-normal operator, but $U^*(UU^{*n}) = U^{*n} \neq (UU^{*n})U^*$
Thus U^* is not quasi n-normal operator.

Remark (2.25):
If T is nilpotent operator with index k. Then
1- T is skew n-normal operator for every $n \geq k$.
2- If $n < k, then$ T is not necessary skew n-normal operator as we saw in Example (2.15).

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