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On a First Order Nonlinear Integrodifferential Equations

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ABSTRACT

In the present paper ,we investigate the existence ,uniqueness and other properties of solutions of a certain nonlinear mixed Volterra – Fredholm integro – differential equations of first order. The main employed tools are the applications of the Banach fixed point theorem and integral inequality are used to establish the results.

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INTRODUCTION

Let us consider the nonlinear integrodifferential equation of the system of the first order

(1.1) $X'(t)=A(t)\,x(t) + f\left(t , x(t) , \int_0^t h\left(t,\sigma , x(\sigma)\right) d\sigma \, , \int_0^a k\left(t,\sigma ,x(\sigma)\right) d\sigma \right) , \qquad X(0) =x_0$,

as a perturbation of the linear system

(1.2) $y'(t)=A(t)\,y(t)$, $y(0)=x_0$,

for $t \epsilon [0\,,a]$ where x, y ,f are real n- vectors ; A is $n \times n$ matrix. Let R^n denotes the real n -dimensional Euclidean space of column vectors. The symbol $|\cdot|$ will denote the norm in R^n or $n \times n$ matrix norm depending on whether is applied to vector or matrix .

Let $J=[0,a]$, $R_+=[0,\infty)$ be the given subset of R , the set of real numbers and assume that $f \in C(J \times R^n \times R^n \times R^n, R^n)$, $h \in C(J^2 \times R^n , R^n)$, $k \in C(J^2 \times R^n , R^n)$ and A(t) is a continuous $n \times n$ matrix on R_+.

The problems of existence ,uniqueness of solution for integral equations ,Fredholm integro-differential equations ,Volterra integral equations and Volterra – Fredholm integral equations have much attentions in the recent years. There are many research results , see [(Aneta ,2001) ,(Muresan , 2004) , (Pachpatte , 2009)]

The main tools ,used in our analysis is based on an application ,the well known Banach fixed point theorem , see (Corduneanu ,1991) , coupled with the Bielecki type norm , see (Andras ,2003) and the integral inequality given by(Pachpatte ,1991)

This work may be considered as an extension of (Pachpatte , 2009) results.

Existence and Uniqueness:

Let S be the space of those functions $Z(t) \in R^n$ which are continuous for $t \in R_+$ and fulfil the condition

(2.1) $|z(t)| = O(\exp(\lambda t))$,

where λ is a positive constant. In the space S we define the norm ,see (Pachpatte ,2008)

(2.2) $|z|_S = \sup_{t \epsilon R_+} [\, |z(t)| \exp(-\lambda t) \,]$.

It is easy to see that S with norm defined in (2.2) is a Banach space . We note that the condition (2.1) implies that there exists a constant $N \geq 0$ such that

$|z(t)| \leq N \exp(\lambda t)$ *for $t \epsilon R_+$. Using this fact in (2.2), we observe that*

(2.3) $|z|_S \leq N$.

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Our approach and arguments led us to make use of the variation of constants formula , namely , any solution x(t) of equation (1.1) considered as a perturbation of equation (1.2) can be represented by the equivalent integral equation , see (Pachpatte , 2009)

(2.4) $X(t)= Y(t) Y^{-1}(0)x_0$
$+ \int_0^t Y(t)Y^{-1}(s) f\,(s , x(s) ,\int_0^s h(s,\sigma,x(\sigma))d\sigma,\int_0^a k\left(s,\sigma,\ x(\sigma)\right) d\sigma\,)\, ds,$

where Y(t) is the fundamental solution matrix of equation (1.2) such that Y(0) = I , the identity matrix. We need the following special version of the integral inequality given by (Pachpatte ,1991)

Lemma. Let $u(t) ,p(t) , f(t) , h(t) , g(t) \in C (J,R_+)$, $c \geq 0$ be a real constant. If

(2.5) $u(t) \leq c + \int_0^t f(s)\ [\ u(s) +\int_0^s g(\sigma)\, u(\sigma)d\sigma+\int_0^a h(\sigma)u(\sigma)d\sigma\ ]\, ds$,

for $t \in J$. $J = [0 , a]$, $a > 0$ .If

(2.6) $d = \int_0^a h(\sigma)\ \exp\,(\ [\ f(\tau) + g(\tau)\ ]d\tau\ )\ \ d\sigma < 1$,

then

(2.7) $u(t) \leq \frac{c}{1-d} \exp\,(\int_0^t [\, f(s) + g(s)]ds\,)$,

for $t \in J$.

Now we are in a position to formulate the main result of this section.

Theorem 1. Let Y(t) be the fundamental solution matrix of equation (1.2) such that

(2.8) $|Y(t) Y^{-1} (s)| \leq M$,

for $s \leq t$; $s , t \in J$, where M is a positive constant . Suppose that

(i) the functions f , h and k in equation (1.1) satisfy the conditions

(2.9) $|f (t ,x ,u ,v) - f (t, \bar{x} , \bar{u} ,\bar{v}) | \leq g(t) [\ | x - \bar{x} |+| u - \bar{u} |+ | v - \bar{v} |\ ]$,

(2.10) $|h (t , \sigma , x) - h(t, \sigma , \bar{x}) | \leq L_1 (t , \sigma) | x - \bar{x} |$,

(2.11) $| k(t, \sigma ,x) - k(t, \sigma , \bar{x}) | \leq L_2 (t , \sigma) |x - \bar{x} |$,

where $g \in C (J, R_+)$, $L_1(t, \sigma)$, $L_2 (t ,\sigma) \in C (J^2 ,R_+)$,

(ii) for λ as in (2.1)

(a) there exists a nonnegative constant $\propto$ such that $\propto < 1$ and

$\int_0^t M\, g(s) [\exp(\lambda s) + \int_0^s L_1(s ,\sigma) \exp(\lambda \sigma)\, d \sigma + \int_0^a L_2 (s, \sigma) \exp(\lambda \sigma)\, d\sigma]\, ds$

(2 . 12) $\leq \propto \exp(\lambda t)$,

(b) there exists a nonnegative constant β such that

(2.13) $M |x_0| + \int_0^t M\ | f (s ,0 , \int_0^s h (s ,\sigma ,0)\, d \sigma , \int_0^a k(s , \sigma, 0)\, d \sigma) |\, d s$
$\leq \beta \exp(\lambda t)$,

for $t \in J$ .Then the equation (1.1) has a unique solution on J .

Proof. Let $x(t) \in S$ and define the operator T by

(2.14) $(Tx) (t) = Y(t) Y^{-1} (0)x_0$
$+ \int_0^t Y(t)Y^{-1}(s) f\, (s , x(s) ,\int_0^s h(s,\sigma,x(\sigma))d\sigma,\int_0^a k\left(s,\sigma,\ x(\sigma)\right) d\sigma\,)\, ds$.

First we shall show that Tx maps S into itself . Evidently , Tx is continuous on J and $Tx \in R^n$. We verify that (2.3) is fulfilled . From (2 .14) and using the hypotheses and (2.3) we have

(2.15) $| (Tx)(t) | \leq | Y(t) Y^{-1}(0) |.|X_0|$
$+\int_0^t |Y(t)Y^{-1}(s)|.\ |f\,(s , x(s) ,\int_0^s h(s,\sigma,x(\sigma))d\sigma,\int_0^a k\left(s,\sigma,\ x(\sigma)\right) d\sigma\,) -$
$f (s ,0 , \int_0^s h (s ,\sigma ,0)\, d \sigma , \int_0^a k(s , \sigma, 0)\, d \sigma) |ds$
$+ \int_0^t |Y(t)Y^{-1}(s)|.\ | f (s ,0 , \int_0^s h (s ,\sigma ,0)\, d \sigma , \int_0^a k(s , \sigma, 0)\, d \sigma) |ds$
$\leq M|X_0| + \int_0^t M|\ f (s ,0 , \int_0^s h (s ,\sigma ,0)\, d \sigma , \int_0^a k(s , \sigma, 0)\, d \sigma) |ds$
$+ \int_0^t Mg(s)\, [\, |x(s)| + \int_0^s L_1(s ,\sigma)\, |x(\sigma)|\, d \sigma + \int_0^a L_2 (s, \sigma)\, |x(\sigma)|\, d \sigma\,]\, ds$
$\leq \beta \exp(\lambda t)$
$+ |x|_S \int_0^t M\, g(s) [\exp(\lambda s) + \int_0^s L_1(s ,\sigma) \exp(\lambda \sigma)\, d \sigma + \int_0^a L_2 (s, \sigma) \exp(\lambda \sigma)\, d\sigma]\, ds$
$\leq [\beta + N \propto] \exp(\lambda t)$.

From (2.15) it follows that $Tx \in S$.

This proves that T maps S in to itself . Now , we verify that the operator T is a contraction map . Let $x(t)$, $z(t) \in S$. From (2.14) and using the hypotheses we have

(2.16) $| (Tx)(t) - (Tz)(t) |$
$\leq \int_0^t |Y(t)Y^{-1}(s)|.\ |f\,(s , x(s) ,\int_0^s h(s,\sigma,x(\sigma))d\sigma,\int_0^a k\left(s,\sigma,\ x(\sigma)\right) d\sigma\,)$

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$- \ f\ (s, z(s), \int_0^s h(s,\sigma,z(\sigma))d\sigma, \int_0^a k\left(s,\sigma,\ z(\sigma)\right)d\sigma\ )\ \big|\, ds$

$\leq \int_0^t Mg(s)\,[\,|x(s) - z(s)| + \int_0^s L_1(s,\sigma)\,|x(\sigma) - z(\sigma)|\, d\sigma + \int_0^a L_2(s,\sigma)\,|x(\sigma) - z(\sigma)|\, d\sigma\,]\, ds$

$\leq |x - z|_s \int_0^t M\, g(s)\,[\,\exp(\lambda s) + \int_0^s L_1(s,\sigma)\exp(\lambda\sigma)\, d\sigma + \int_0^a L_2(s,\sigma)\exp(\lambda\sigma)\, d\sigma\,]\, ds$

$\leq |x - z|_s \ \propto \exp(\lambda t)$

From (2.16) we obtain

$|Tx - Tz|_s \leq \ \propto |x - z|_s$.

Since $\propto\ < 1$ *, it follows from Banach fixed point theorem ,see (Corduneanu ,1991) that T has a unique fixed point in S . The fixed point of T is however a solution of IVP (1.1 -1.2) . The proof is complete .* □

Remark 1. The norm $|.|_s$ *defined in (2.2) was first used by Bielecki ,See (Andras ,2003) for proving global existence and uniqueness of solution of ordinary differential equation. For a detailed discussion related to this topic, see (Corduneanu ,1984)*

The following theorem deals with the uniqueness of solutions of IVP (1.1) in R^n without existence part .

Theorem 2. Let the fundamental solution matrix Y(t) of equation (1.2) be as in Theorem 1. Assume that the functions f, h, and k in equation (1.1) satisfy the conditions (2.9), (2.10) and (2.11) with $g \in C(J,R_+)$ and

$L_i(t,\sigma) = e_i(t)\, r_i(\sigma)$; $e_i, r_i \in C(J,R_+)$ (i= 1 , 2)

$e(t) = \max\{ e_1(t), e_2(t) \}$ ……………. (2.17)

and $e(t) \geq 1$. Suppose that

(2.18) $\bar{d} = \int_0^a r_2(\sigma)\exp(\int_0^\sigma [\, M\, g(\tau)\, e(\tau) + r_1(\tau)\,]\, d\tau\ )\, d\sigma < 1$.

Then the equation (1. 1) has at most one solution on J.

$u(t) = |X_1(t) - X_2(t)|$. Then by hypotheses , we have

(2.19)

$u(t) \leq \int_0^t |Y(t)Y^{-1}(s)|.\ |f\,(s, x_1(s), \int_0^s h(s,\sigma,x_1(\sigma))\, d\sigma, \int_0^a k(s,\sigma,x_1(\sigma))\, d\sigma\,)$

$- \ \ f\,(s, x_2(s), \int_0^s h(s,\sigma,x_2(\sigma))\, d\sigma, \int_0^a k(s,\sigma,x_2(\sigma))\, d\sigma)|\, ds$

$\leq \int_0^t Mg(s)\,[\,|x_1(s) - x_2(s)| + \int_0^s L_1(s,\sigma)\,|x_1(\sigma) - x_2(\sigma)|\, d\sigma$

$+ \int_0^a L_2(s,\sigma)\,|x_1\, x(\sigma) - x_2(\sigma)|\, d\sigma\,]\, ds$

$\leq \int_0^t Mg(s)\,[u(s) + \int_0^s e_1(s)\, r_1(\sigma)\, u(\sigma)\, d\sigma + \int_0^a e_2(s)\, r_2(\sigma)\, u(\sigma)\, d\sigma]\, ds$

By using (2.17) , we have

$\leq \int_0^t Mg(s)\, e(s)\,[\,u(s) + \int_0^s r_1(\sigma)\, u(\sigma)\, d\sigma + \int_0^a r_2(\sigma)\, u(\sigma)\, d\sigma]\, ds$.

Now a suitable application of lemma to (2.19) yields

$$|X_1(t) - X_2(t)| \leq 0 ,$$

and hence $X_1(t) = X_2(t)$. Thus there is at most one solution to equation (1 .1) on J. □

Properties of Solutions:

In this section we shall study some fundamental properties of solutions of equation (1.1) by using some suitable conditions on the functions involved therein .

First , we shall give the following theorem concerning an estimate on the solution of equation (1.1) .

Theorem 3. Suppose that the functions f , h , k in equation (11) satisfy the conditions

(3.1) $|f(t,x,u,v)| \leq \bar{g}(t)\ [\,|x| + |u| + |v|\,]$,

(3.2) $|h(t,\sigma,u)| \leq \bar{e}_1(t)\ \bar{r}_1(\sigma)\,|u|$,

(3.3) $|k(t,\sigma,u)| \leq \bar{e}_2(t)\ \bar{r}_2(\sigma)\,|u|$,

where $\bar{g}, \bar{e}_i, \bar{r}_i \in C(J,R_+)$, (i= 1 , 2) , and $\bar{e}(t) \geq 1$. Assume that

(3.4) $\bar{d}_0 = \int_0^a \bar{r}_2(\sigma)\exp(\int_0^\sigma [\, M\, \bar{g}(\tau)\, \bar{e}(\tau) + \bar{r}_1(\tau)\,]\, d\tau\ )\, d\sigma < 1$.

If x(t), $t \in J$ is any solution of equation (1.1) ,then

(3.5) $|x(t)| \leq \frac{|Mx0|}{1-d0}\exp(\int_0^t [\, M\, \bar{g}(s)\, \bar{e}(s) + \bar{r}_1(s)\,]\, ds\,)$,

for $t \in J$.

Proof. Using the fact that x(t) , $t \in J$ is a solution of equation (1.1) and the hypotheses we have

(3.6) $|x(t)| \leq |Y(t)\, Y^{-1}(0)|.|X_0|$

$+ \int_0^t |Y(t)Y^{-1}(s)|.\ \ |f\,(s, x(s), \int_0^s h(s,\sigma,x(\sigma))d\sigma, \int_0^a k\left(s,\sigma,\ x(\sigma)\right)d\sigma\,)|\, ds$

$\leq M|X_0| + \int_0^t M\bar{g}(s)\,[\,|x(s)| + \int_0^s \bar{e}_1(s)\,\bar{r}_1(s)\,|x(\sigma)|\, d\sigma + \int_0^a \bar{e}_2(s)\,\bar{r}_2(s)\,|x(\sigma)|\, d\sigma]\, ds$

$\leq \quad M|X_0| +$

$\int_0^t M\bar{g}(s)\,\bar{e}(s)\,[\,|x(s)| + \int_0^s \bar{r}_1(\sigma)\,|x(\sigma)|\,d\sigma + \int_0^a \bar{r}_2(\sigma)\,|x(\sigma)|\,d\sigma]\,ds$.

Now an application of lemma to (3.6) yields (3.5). □

Remark 2. We note that, if the estimate obtained in (3.5) is bounded, then the solution x(t) of equation (1.1) is bounded on J.

The next theorem deals with the dependency of solution of equation (1.1) on given initial values.

Theorem 4. Suppose that Y(t), f, h, k and $\bar{d}$ be as in Theorem 2. Let x(t) and z(t) be the solutions of equation (1.1) with the given initial conditions

(3.7) $x(0) = x_0$,

and

(3.8) $z(0) = z_0$.

Then

(3.9) $|x(t) - z(t)| \le \frac{M|x0 - z0|}{1-d0} \exp(\int_0^t [M\, g(s)\, e(s) + r_1(s)]\, ds)$,

for $t \in J$.

Proof. Since x(t) and z(t) are the solutions of equation (1.1) with initial values (3.7) and (3.8) we have

(3.10) $|x(t) - z(t)| \le |Y(t)\,Y^{-1}(0)|\,.\,|X_0 - z_0|$

$+\int_0^t |Y(t)Y^{-1}(s)|\,.\,|f(s, x(s), \int_0^s h(s,\sigma,x(\sigma))d\sigma, \int_0^a k(s,\sigma,x(\sigma))\,d\sigma)$

$-f(s, z(s), \int_0^s h(s,\sigma,z(\sigma))d\sigma, \int_0^a k(s,\sigma,z(\sigma))\,d\sigma)|\,ds$

$\le M|X_0 - z_0| +$

$\int_0^t Mg(s)\,[\,|x(s) - z(s)| + \int_0^s e_1(s)\,r_1(\sigma)\,|x(\sigma) - z(\sigma)|\,d\sigma$

$+\int_0^a e_2(s)\,r_2(\sigma)\,|x(\sigma) - z(\sigma)|\,d\sigma]\,ds$

$\le M|X_0 - z_0| +$

$\int_0^t Mg(s)\,e(s)\,[\,|x(s) - z(s)| + \int_0^s r_1(\sigma)\,|x(\sigma) - z(\sigma)|\,d\sigma + \int_0^a r_2(\sigma)\,|x(\sigma) - z(\sigma)|\,d\sigma]\,ds$.

Now an application of Lemma to (3.10) yields the estimate (3.9), Which shows the dependency of solutions of equation (1.1) on given initial values.

We next consider the following initial value problems

(3.11)

$x'(t) = A(t)\,x(t) +$

$F(t, x(t), \int_0^t h(t,\sigma,x(\sigma))\,d\sigma, \int_0^a k(t,\sigma,x(\sigma))\,d\sigma, \mu)$, $X(0) = x_0$,

(3.12) $x'(t) = A(t)\,x(t) +$

$F(t, x(t), \int_0^t h(t,\sigma,x(\sigma))\,d\sigma, \int_0^a k(t,\sigma,x(\sigma))\,d\sigma, \mu_0)$, $X(0) = x_0$,

for $t \in J$, considered as a perturbation of the linear system (1.2),

where $F \in C(J \times R^n \times R^n \times R^n \times R, R^n)$, $h \in C(J^2 \times R^n, R^n)$,

$k \in C(J^2 \times R^n, R^n)$ and μ, μ_0 are real parameters.

The following theorem shows the dependency of solution of equations (3.11) and (3.12) on parameters.

Theorem 5. Let the fundamental solution matrix Y(t) of equation (1.2) be as in Theorem 1. Assume that the functions F, h and K in (3.11), (3.12) satisfy the conditions

(3.13) $|F(t, x, u, v, \mu) - F(t, \bar{x}, \bar{u}, \bar{v}, \mu)|$

$\le G(t)\,[\,|x - \bar{x}| + |u - \bar{u}| + |v - \bar{v}|\,]$,

(3.14) $|F(t, x, u, v, \mu) - F(t, x, u, v, \mu_0)| \le H(t)\,|\mu - \mu_0|$,

(3.15) $|h(t, \sigma, x) - h(t, \sigma, \bar{x})| \le E_1(t)\,Q_1(\sigma)\,|x - \bar{x}|$,

(3.16) $|k(t, \sigma, x) - k(t, \sigma, \bar{x})| \le E_2(t)\,Q_2(\sigma)\,|x - \bar{x}|$,

Where G, H, E_i, $Q_i \in C(J, R_+)$, $E(t) \ge 1$, (i = 1, 2) and

(3.17) $\int_0^t H(s)\,ds \le \varepsilon$,

$E(t) = \max\{E_1(t), E_2(t)\}$

for $t \in J$, where $\varepsilon > 0$ is an arbitrary small constant. Assume that

(3.18) $d_1 = \int_0^a Q_2(\sigma) \exp(\int_0^\sigma [M\, G(\tau)\, E(\tau) + Q_1(\tau)]\, d\tau)\, d\sigma < 1$.

Let $x_1(t)$ and $x_2(t)$ be the solutions of equations (3.11) and (3.12) respectively, then

(3.19) $|x_1(t) - x_2(t)| \le \frac{M\varepsilon|\mu - \mu0|}{1-d1} \exp(\int_0^t [M\, G(s)\, E(s) + Q_1(s)]\, ds)$,

for $t \in J$.

Proof. Let $x(t) = x_1(t) - x_2(t)$, where $x_1(t)$ and $x_2(t)$ are the solutions of equations (3.11) and (3.12). From the hypotheses we have

(3.20) $|x(t)| \le$

$\int_0^t |Y(t)Y^{-1}(s)|\,|\,F(s, x_1(s), \int_0^s h(s,\sigma, x_1(\sigma))\,d\sigma, \int_0^a k(s,\sigma, x_1(\sigma))\,d\sigma, \mu)$

$- F(s, x_2(s), \int_0^s h(s,\sigma, x_2(\sigma))\,d\sigma, \int_0^a k(s,\sigma, x_2(\sigma))\,d\sigma, \mu)$

$+ F(s, x_2(s), \int_0^s h(s,\sigma, x_2(\sigma))\,d\sigma, \int_0^a k(s,\sigma, x_2(\sigma))\,d\sigma, \mu)$

$-F(s, x_2(s), \int_0^s h(s,\sigma, x_2(\sigma))\,d\sigma, \int_0^a k(s,\sigma, x_2(\sigma))\,d\sigma, \mu_0)\,|\,ds$

$\le \int_0^t M\,H(s)\,|\mu - \mu_0|\,ds$

$+ \int_0^t MG(s)\,[\,|x_1(s) - x_2(s)| + \int_0^s E_1(s)\,Q_1(\sigma)\,|x_1(\sigma) - x_2(\sigma)|\,d\sigma$

$+ \int_0^a E_2(s)\,Q_2(\sigma)\,|x_1(\sigma) - x_2(\sigma)|\,d\sigma\,]\,ds$

$\le M\varepsilon\,|\mu - \mu_0| + \int_0^t MG(s)\,E(s)\,[\,|x(s)| + \int_0^s Q_1(\sigma)\,|x(\sigma)|\,d\sigma$

$+ \int_0^a Q_2(\sigma)\,|x(\sigma)|\,d\sigma\,]\,ds.$

Now an application of Lemma to (3.20) yields (3.19) , which shows the dependency of solutions of equations (3.11) and (3.12) on parameters . □

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