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Dominating Sets and Domination Polynomials of Stars

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ABSTRACT

Background: Let $G = (V, E)$be a simple graph. A set$D \subseteq V$ is a dominating set of$G$, if every vertex in $V - D$is adjacent to at least one vertex in D. Let $S_n$be star graph with order n. LetS_n^i be the family of dominating sets of a star $S_n$with cardinality i, and let $d(S_n, i) = |S_n^i|$. **Results.** In this paper, we construct S_n, and obtain a recursive formula for $d(S_n, i)$. **Conclusion.** Using this recursive formula, we consider the polynomial $D(S_n, x) = \sum_{i=1}^{n} d(S_n, i)x^i$, which we call domination polynomial of star graphs and obtain some properties of this polynomial.

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INTRODUCTION

Let G=(V,E) be a simple graph of order |V|= n. A set D⊆V is a dominating set of G, if every vertex in V-D is adjacent to at least one vertex in D. The domination number γ(G) is the minimum cardinality of a dominating set in G. For a detailed treatment of this parameter, the reader is referred to (Haynes, Hedetniemi, Slater, 1998). It is well known and generally accepted that the problem of determining the dominating sets of an arbitrary graph is a difficult one (Garey, Johnson,1979). Alikhani and Peng found the dominating set and domination polynomial of cycles and paths and certain graphs (Alikhani, Peng, 2008), (Alikhani, Peng, 2009), (Alikhani, Peng, 2010). Let Gn be graph with order n and let G_i^n be the family of dominating sets of a graph G_n with cardinality i and let d(G_n, i) = $|G_i^n|$. We call the polynomial $D(G_n, x) = \sum_{i=\gamma(G)}^{n} d(G_n, i)x^i$, the domination polynomial of graph G (Alikhani, Peng, 2010). Let S_i^n be the family of dominating sets of a star S_n with cardinality i and let $d(S_n, i)$ =$|S_i^n|$..We call the polynomial $D(S_n, x) = \sum_{i=1}^{n} d(S_n, i)x^i$, the domination polynomial of star.

In the next section we construct the families of dominating sets of $S_n$with cardinality i by combination $n-1$ to $i-1$, and the families of dominating sets of S_{n-1} with cardinality i and $i-1$. We also investigate the domination polynomial of stars. As usual we use $\binom{n}{i}$ for the combination n to i, and we denote the set{1,2,…,n}simply by [n].

Dominating Sets of Stars:

Let $S_n, n \geq 3$, be the star with n vertices $V(S_n)$ = [n] and $E(S_n)$ = {(1,2),(1,3),...,(1, n)}. Let S_i^n be the family of dominating sets of S_n with cardinality i. We shall investigate dominating sets of stars. To prove our main results we need the following lemmas.

Lemma 1:

The following properties hold for all graph G.

(i) $|G_n^n|$= 1
(ii) $|G_n^{n-1}|= n$
(iii) $|G_n^i|$= 0 if $i > n$.
(iv) $|G_n^0|$= 0

Proof:

Let $G = (V, E)$ be a simple graph of order n, then
(i) $G_n^n = \{G\}$therefore,$|G_n^n|$= 1
(ii) G_n^{n-1}=$\{\{G - v\}$:$\forall v \in G\}$, therefore $|G_n^{n-1}|= n$
(iii) $\nexists H \subseteq G$such that $|V(H)| > |V(G)|$.

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(iv) $\nexists H \subseteq G$ such that $|V\ (H)| = 0$, $\emptyset$ is not dominating set of G.

Lemma 2:

The following properties are hold by definition of combination $\left(\binom{n}{i} = \frac{n!}{i!(n-i)!}\right)$ for all n $\in Z^{+}$.

(i) $\binom{n}{n} = 1$ (ii) $\binom{n}{in-1}$ =n (iii) $\binom{n}{1}$ =n (iv) $\binom{n}{0} = 1$ (v) $\binom{n}{i} = 0$ if $i > n$ or $i < 0$

Theorem 1:

Let S_n be star with order n, then

$d(S_n, i) = \binom{n}{i} - \binom{n-1}{i}\ \forall\ i < n-1, n \geq 3$

Proof:

Let S_n be a star and $v \in V(S_n)$ such that v is center of S_n, i.e. ($\rho(v) = n-1$), and let H=S_n-v, then $H_n^i = \emptyset\ \ \forall i < n$, since $|V\ (H)| = n-1$, then $\binom{n-1}{i}$ number of subset of H with cardinality i, and $\binom{n}{i}$ number of subset of S_n with cardinality i, and since dominating set of S_n is every subset of S_n has vertex v, and since H is dominating set of S_n with cardinality $n-1$, therefore $d(S_n, i) = \binom{n}{i} - \binom{n-1}{i}\ \forall\ i < n-1$.

∎

Theorem 2:

Let S_n be star with order $n \geq 3$, then

(i) $d(S_n, i) = \binom{n-1}{i-1}\ \forall\ i < n-1$.

(ii) $d(S_n, i) = d(S_{n-1}, i) + d(S_{n-1}, i-1)\ \forall i \neq n-2$.

(iii) $d(S_n, i) = d(S_{n-1}, i) + d(S_{n-1}, i-1)$-1 1f $i = n-2$.

Proof:

(i) By Theorem 1, $d(S_n, i) = \binom{n}{i} - \binom{n-1}{i} = \frac{n(n-1)\dots(n-i+1)}{i!} - \frac{(n-1)(n-2)\dots(n-i)!}{i!} = \frac{(n-1)(n-2)\dots(n-i+1)}{(i-1)!} . \frac{n-(n-1)}{i} = \frac{(n-1)(n-2)\dots(n-i+1)}{(i-1)!} = \binom{n-1}{i-1}$.

(ii)We have $\binom{n}{i} = \frac{n(n-1)\dots(n-i+1)(n-i)!}{i!(n-i)!} = \frac{n(n-1)\dots(n-i+1)}{i!} = \frac{(n-1)(n-2)\dots(n-i+1)}{i!} . n$,

$$\binom{n-1}{i} = \frac{(n-1)(n-2)\dots(n-i)}{i!} = \frac{(n-1)(n-2)\dots(n-i+1)}{i!} . (n-i),$$

$$\binom{n-2}{i} = \frac{(n-2)(n-3)\dots(n-i-1)}{i!} = \frac{(n-1)(n-2)\dots(n-i+1)}{i!} . \frac{(n-i)(n-i-1)}{(n-1)},$$

$\binom{n-1}{i-1} = \frac{(n-1)(n-2)\dots(n-i)}{(i-1)!} = \frac{(n-1)(n-2)\dots(n-i+1)}{i!} . i$ and

$$\binom{n-2}{i-1} = \frac{(n-2)(n-3)\dots(n-i)}{(i-1)!} = \frac{(n-1)(n-2)\dots(n-i+1)}{i!} . \frac{(n-i)}{(n-1)} i$$

Let $m = \frac{(n-1)(n-2)\dots(n-i+1)}{i!}$, then $d(S_n, i) = \binom{n}{i} - \binom{n-1}{i} = mn - m(n-i) = mi$, and $d(S_{n-1}, i) + d(S_{n-1}, i-$ *1=n−1i−n−2i+n−1i−1−n−2i−1=m.n−i−m.n−in−i−1n−1+m.i−m.(n−i)(n−1)i=m(n−1)n−in−1−n−in −i−1+in−1−i(n−i)=m(n−1)in−i+i(n−1)=m(n−1)in−1=m.i=d(Sn,i).*

(iii) If $i = n-2$, then

* $d(S_n, i) = d(S_n, n-2) = \binom{n}{n-2} - \binom{n-1}{n-2} = \binom{n}{2} - \binom{n-1}{2} = \frac{n(n-1)}{2} - (n-1) = \frac{(n^2-3n+2)}{2} = \frac{(n-1)(n-2)}{2}$

* $d(S_{n-1}, i) = d(S_{n-1}, n-2) = (n-1)$ (by Lemma 1),

* $d(S_{n-1}, i-1) = d(S_{n-1}, n-3) = \binom{n-1}{n-3} - \binom{n-2}{n-3} = \frac{(n-1)(n-2)}{2} - (n-2)$

We have $d(S_{n-1}, i) + d(S_{n-1}, i-1) = (n-1) + \frac{(n-1)(n-2)}{2} - (n-2) = \frac{(n-1)(n-2)}{2}$ $= (S_n, i)$+1. Therefore $d(S_n, i) = d(S_{n-1}, i) + d(S_{n-1}, i-1)$-1.

Using Theorem 1 and Theorem 2, we obtain the coefficients of $D(S_n, x)$ for $1 \leq n \leq 15$ in Table 1. Let $d(S_n, i) = |S_n^i|$. There are interesting relationships between the numbers $d(S_n, i)$ $(1 \leq i \leq n)$ in the table.

In the following theorem, we obtain some properties of $d(S_n, i)$

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Theorem 3:

The following properties hold for coefficients of $D(S_n, x)$, For every $n \in Z^+, n \geq 3$.

(i) $d(S_n, 1) = 1$.
(ii) $d(S_n, 2) = n$-1, $\forall n > 3$.
(iii) $d(S_n, i) = d(S_n, n - i + 1), \forall 3 \leq i \leq n - 2$.
(iv) $\gamma(S_n) = 1$

Table 1: $d(S_n, i)$ The number of dominating sets of S_n with cardinality i.

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n															
1	1														
2	2	1													
3	1	3	1												
4	1	3	4	1											
5	1	4	6	5	1										
6	1	5	10	10	6	1									
7	1	6	15	20	15	7	1								
8	1	7	21	35	35	21	8	1							
9	1	8	28	56	70	56	28	9	1						
10	1	9	36	84	126	126	84	36	10	1					
11	1	10	45	120	210	252	210	120	45	11	1				
12	1	11	55	165	330	462	462	330	165	55	12	1			
13	1	12	66	220	495	792	924	792	495	220	66	13	1		
14	1	13	78	286	715	1287	1716	1716	1287	715	286	78	14	1	
15	1	14	91	364	1001	2002	3003	3532	3003	2002	1001	364	91	15	1

Proof:

Let S be a star and $v \in V(S)$ such that v is center of S, then
(i) $S_n^i = \{\{v\}\} \; \forall n \geq 3$, then $d(S_n, 1) = 1$
(ii)Let 1 be center of S_n then $S_n^i = \{\{1, j\}: \forall j \in S_n, \; j \neq 1\}$, therefore $d(S_n, 2) = n - 1$
(iii) We have $\binom{n-1}{i} = \frac{(n-1)(n-2)\dots(n-i)}{i!} = \frac{(n-1)(n-2)\dots(n-i+1)}{i!}.(n - i)$, and

$$\binom{n}{i-i+1} = \binom{n}{i-1} = \frac{n(n-1)\dots(n-i+2)}{(i-1)!} = \frac{(n-1)(n-2)\dots(n-i+1)}{i!}.\frac{ni}{n-i+1},$$

$$\binom{n-1}{i-i+1} = \binom{n-1}{i-1} = \frac{(n-1)(n-2)\dots(n-i+2)}{(i-1)!} = \frac{(n-1)(n-2)\dots(n-i+1)}{i!}.\frac{i(i-1)}{n-i+1},$$

Let $m = \frac{(n-1)(n-2)\dots(n-i+1)}{i!}$, then d($S_n$, i)=$\binom{n}{i} - \binom{n-1}{i} = mn - m(n - i) = mi$, and
$d(S_n, n - i + 1) = \binom{n}{i-i+1} - \binom{n-1}{i-i+1} = m.\frac{ni}{n-i+1} - m.\frac{i(i-1)}{n-i+1} = mi.\frac{(n-i+1)}{n-i+1} = mi$.Therefore
$d(S_n, i) = d(S_n, n - i + 1)$
(iv) since $\{v\}$ is dominating set of S_n, then $\gamma(S_n) = 1$.

Domination Polynomial of a Star:

In this section we introduce and investigate the domination polynomial of stars.

Definition:

Let S_n^i be the family of dominating sets of a star S_n with cardinality i and le $d(S_n, i)$= $|S_n^i|$ and since $\gamma(S_n)$ = 1. Then the domination polynomial $D(S_n, x)$ of S_n is defined as

$$D(S_n, x) = \sum_{i=1}^{n} d(S_n, i) x^i$$

Theorem 4:

The following properties hold for all $D(S_n, x) \forall \; n \geq 3$

(i) $D(S_n,x) = D(S_{n-1},x) + xD(S_{n-1},x) - x^{n-2}$
(ii)$D(S_n,x) = \sum_{i=1}^{n}\binom{n}{i}x^i - \sum_{i=1}^{n-2}\binom{n-1}{i}x^i$
(iii) $D(S_n,x) = \sum_{i=1}^{n}\binom{n-1}{i-1}x^i + x^{n-1}$

Proof:
(i) From definition of the domination polynomial and Theorem 2, we have
$D(S_n,x) = \sum_{i=1}^{n} d(S_n,i)x^i = \sum_{i=1}^{n}[d(S_{n-1},i) + d(S_{n-1},i-1)]x^i = \sum_{i=1}^{n} d(S_{n-1},i)x^i + \sum_{i=1}^{n} d(S_{n-1},i -$ *1)xi*, we have *d(Sn, i) = 0* if *i > n* or *i = 0* (Lemma1), then
(1) $\sum_{i=1}^{n}[d(S_{n-1},i)x^i = \sum_{i=1}^{n-1}[d(S_{n-1},i)x^i = D(S_{n-1},i)$ and $\sum_{i=1}^{n} d(S_{n-1},i-1)x^i = x\sum_{i=2}^{n}[d(S_{n-1},i)x^{i-1} = x\,D(S_{n-1},x)$. By Theorem 1
(2) $d(S_n,n-2)x^{n-2} = [d(S_{n-1},n-2) + d(S_{n-1},n-3) - 1]x^{n-2} = d(S_{n-1},n-2)x^{n-2} + d(S_{n-1},n -$ *3xn−2−xn−2*
From (1) and (2), we get $D(S_n,x) = D(S_{n-1},x) + xD(S_{n-1},x) - x^{n-2}$
(ii) $D(S_n,x) = \sum_{i=1}^{n} d(S_n,i)x^i = \sum_{i=1}^{n}[\binom{n}{i} - \binom{n-1}{i}]x^i$(by Theorem1)
$=\sum_{i=1}^{n}\binom{n}{i}x^i - \sum_{i=1}^{n-2}\binom{n-1}{i}x^i \;\forall\; i < n-1$ and (by Lemma 1).
(iii) From definition of the domination polynomial and Theorem 2, we have
$D(S_n,x) = \sum_{i=1}^{n} d(S_n,i)x^i = \sum_{i=1}^{n}\binom{n-1}{i-1}x^i$and since $\binom{n-1}{n-2} = n-1$ but $d(S_{n-1},n-2) = n$, hence $d(S_{n-1},n -$ *2 = n−1i−1+1*, then *dSn−1, n−2xn−1* = *[n−1i−1+1]xn−1.*
Therefore $D(S_n,x) = \sum_{i=1}^{n}\binom{n-1}{i-1}x^i + x^{n-1}$.

Example 1:
Let S_7be star with order 7, then (by Theorem 4) we have $D(S_7,x) = \sum_{i=1}^{7}\binom{6}{i-1}x^i + x^6 =$ $x + 6x^2 + 15x^3 + 20x^4 + 15x^5 + 7x^6 + x^7$.(see Fig-1)

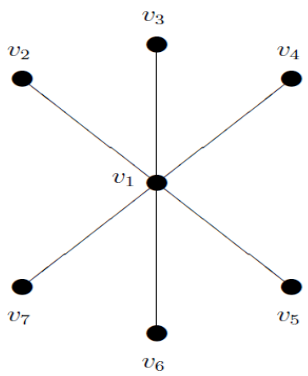

Fig. 1: G=S_7has $\binom{6}{i-1}$ dominating sets with cardinality i

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