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An Alternative Proof of the Spectral Radius Inequality for Products of Hilbert Space Operators

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ABSTRACT

In this paper, we present a new alternative proof of the spectral radius inequality theorem for products of Hilbert space operators due to F. Kittaneh, in which the spectral radius of A obtained by using the formula $r(A) = \lim_{n\to\infty}\|A^n\|^{\frac{1}{n}}$, where $\|.\|$ represents any operator norm. Then, we use this alternative technique to prove some related results. Indeed, the spectral radius inequalities presented in this paper have diverse applications.

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INTRODUCTION

Recently, the Gelfand formula (Gelfand, 1941) is treated as a commonly known fact and is mentioned in practically all textbooks on linear analysis. This formula plays an important role in various applications such as quantitative economics, theory of operator algebras, weighted digraphs, sensor networks and many others. As is known, the spectral radius of any matrix A can be expressed in terms of the norms of its powers $\|A^n\|$ by the following Gelfand formula:

$$r(A) = \lim_{n\to\infty}\|A^n\|^{\frac{1}{n}}, \qquad (1)$$

where $\|\cdot\|$ is a matrix norm.

In other words, the Gelfand's formula gives us that the spectral radius of A represent the asymptotic growth rate of the normalized norm of A^n: $\|A^n\|^{\frac{1}{n}} \backsim r(A)$ as $n \to \infty$. Also, the normalized norm $\|A^n\|^{\frac{1}{n}}$ can be used to approximate the spectral radius and in the limit for $n \to \infty$ the two quantities coincide.

The paper is organized as follows. In Introduction, we presented the famous Gelfand formula for the spectral radius that has great importance in various mathematical constructions. In Section 2, basic facts, properties, inequalities and the relations between spectral radius and the other operator norms are introduced. In Section 3, the main result of the paper, Theorem 1, is formulated.

2. Preliminaries:

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and let $\lambda \in \mathbb{C}$ be the eigenvalue of A in $B(\mathcal{H})$. The spectrum is the set, collection, of all eigenvalues of A and is denoted by $\sigma(A)$. For $A \in B(\mathcal{H})$, the spectral radius of A is defined as the maximum of modulus of its eigenvalues, *i.e.* $r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. This is just a radius of the smallest disc centered at the origin in the complex plane that includes all the eigenvalues of A, so this formula yields a technique for estimating for the top eigenvalue.

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Let us now recall basic relations between spectral radius and the other operator norms. We start with the following observations. For every $A \in B(\mathcal{H})$ and any operator norm, it is well known that the spectral radius function is the greatest lower bound for the values of all operator norms of A, *i.e.*

$$r(A) \leq \|A\|, \tag{2}$$

and that the equality holds if A is normal (Halmos, 1982). On the other hand, let $A \in B(\mathcal{H})$, for every $\varepsilon > 0$ there is an operator norm $\|\cdot\|$ such that

$$\|A\| \leq r(A) + \varepsilon. \tag{3}$$

It is worth to mention that the spectral radius of A is not itself an operator norm, but if we let $\varepsilon \to 0$ in inequality (3) we have that r(A) is the greatest lower bound for the values of all operator norms of A, *i.e.*

$$r(A) = \inf_{\|.\| \in \mathcal{N}} \|A\|, \tag{4}$$

where $\mathcal{N}$ denotes the set of all possible induced operator norms.

Some further interesting and basic properties of the spectral radius are expressed in the following. From the spectral mapping theorem for polynomials (Kreyszig, 1978), we have

$$r(A^n) = (r(A))^n, \text{ for every positive integer } n. \tag{5}$$

Another consequence of $\sigma(AB) = \sigma(BA)$, the commutative property for every $A, B \in B(\mathcal{H})$ is given by

$$r(AB) = r(BA). \tag{6}$$

Thus, from Gelfand's formula (1), for every $A, B \in B(\mathcal{H})$ such that $AB = BA$, it is easy to see that

$$r(A + B) \leq r(A) + r(B), \tag{7}$$

and

$$r(AB) \leq r(A)r(B). \tag{8}$$

Furthermore, the fact that the spectral radius r(A) is contained in the spectrum implies

$$r(kI + A) = k + r(A), for\ all\ k \geq 0, \tag{9}$$

and

$$r(kA) = kr(A), for\ all\ k \geq 0, \tag{10}$$

where I denotes the identity operator on $\mathcal{H}$.

Now, we will present other facts and inequalities about the spectral radius for future use.

For every $A, B \in B(\mathcal{H})$, one can get

$$r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = max(r(A), r(B)), \tag{11}$$

and

$$r\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \sqrt{r(AB)}. \tag{12}$$

In 1995, Hou and Du proved that for every $A, B, C, D \in B(\mathcal{H})$, we have the following inequalities

$$\left\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right\| \leq \left\|\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right\|, \tag{13}$$

and

$$r\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq r\left(\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right). \tag{14}$$

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3. Main results:

The aim of this section is to obtain an alternative proof of the spectral radius inequality theorem for products of operators due to Kittaneh, 2005, where it is proved there based on the following inequality: if $A_1, A_2, B_1, B_2 \in B(\mathcal{H})$, then

$$r(A_1B_1 + A_2B_2) \leq \frac{1}{2}(\|B_1A_1\| + \|B_2A_2\| + \sqrt{(\|B_1A_1\| - \|B_2A_2\|)^2 + 4\|B_1A_2\|\|B_2A_1\|}).$$

Now, we present the main result and the proof of the theorem as follows.

Theorem 1*:*

If $A, B \in B(\mathcal{H})$, then

$$r(AB) \leq \frac{1}{4}\left(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4\min(\|A\|\|BAB\|, \|B\|\|ABA\|)}\right), \tag{15}$$

Proof:

We have

$$\begin{aligned} 2r(AB) &= r\left(\begin{bmatrix} 2AB & 0 \\ 0 & 0 \end{bmatrix}\right) \quad (by\ property\ (11)) \\ &= r\left(\begin{bmatrix} AB & A \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ B & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} I & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} AB & A \\ 0 & 0 \end{bmatrix}\right) (by\ property\ (6)) \\ &= r\left(\begin{bmatrix} AB & A \\ BAB & BA \end{bmatrix}\right) \\ &\leq r\left(\begin{bmatrix} \|AB\| & \|A\| \\ \|BAB\| & \|BA\| \end{bmatrix}\right) (by\ inequality\ (14)) \\ &= \frac{1}{2}\left(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4\|A\|\|BAB\|}\right). \end{aligned}$$

Hence,

$$r(AB) \leq \frac{1}{4}\left(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4\|A\|\|BAB\|}\right). \tag{16}$$

Meanwhile, we have

$$\begin{aligned} 2r(AB) &= 2r(BA) = r\left(\begin{bmatrix} 2BA & 0 \\ 0 & 0 \end{bmatrix}\right) \quad (by\ property\ (11)) \\ &= r\left(\begin{bmatrix} BA & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ A & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} I & 0 \\ A & 0 \end{bmatrix}\begin{bmatrix} BA & B \\ 0 & 0 \end{bmatrix}\right) (by\ property\ (6)) \\ &= r\left(\begin{bmatrix} BA & B \\ ABA & AB \end{bmatrix}\right) \\ &\leq r\left(\begin{bmatrix} \|BA\| & \|B\| \\ \|ABA\| & \|AB\| \end{bmatrix}\right) \quad (by\ inequality\ (14)) \\ &= \frac{1}{2}\left(\|BA\| + \|AB\| + \sqrt{(\|BA\| - \|AB\|)^2 + 4\|B\|\|ABA\|}\right), \end{aligned}$$

and

$$r(AB) \leq \frac{1}{4}\left(\|BA\| + \|AB\| + \sqrt{(\|BA\| - \|AB\|)^2 + 4\|B\|\|ABA\|}\right). \tag{17}$$

Thus, from the inequalities (16) and (17), we have inequality (15) directly. This completes the proof of the theorem.

Remark 1:

We remark that Kittaneh in 2005 used the facts

$$2r(AB) = r\left(\begin{bmatrix} 2AB & 0 \\ B & 0 \end{bmatrix}\right), \tag{18}$$

and

$$2r(BA) = r\left(\begin{bmatrix} 2BA & 0 \\ A & 0 \end{bmatrix}\right), \tag{19}$$

to prove the main result, for more details see "Problem 71" in (Halmos, 1982). Here, we give another easier facts to prove the same result.

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Corollary 1:

If $A_1, A_2, \dots, A_n \in B(\mathcal{H})$, then

$$r(A_1A_2 \dots A_n) \le \frac{1}{4}(\|A_1A_2 \dots A_n\| + \|A_2A_3 \dots A_nA_1\|) + \frac{1}{4}((\|A_1A_2 \dots A_n\| - \|A_2A_3 \dots A_nA_1\|)^2 + 4\min(\|A_1\|\|A_2A_3 \dots A_nA_1A_2A_3 \dots A_n\|, \|A_2A_3 \dots A_n\|\|A_1A_2 \dots A_nA_1\|))^{\frac{1}{2}}. \quad (20)$$

Proof:

The inequality (20) follows from Theorem 1 by letting $A_1 = A \ and \ A_2A_3 \dots A_n = B$.

Corollary 2:

If $A, B \in B(\mathcal{H})$and $AB = BA$, then

$$r(AB) \le \frac{1}{2}\left(\|AB\| + \sqrt{min(\|A\|\|AB^2\|, \|B\|\|BA^2\|)}\right). \quad (21)$$

It is pertinent to note that the spectral radius inequality for products of operators presented in this paper has widely applications. For instance, it can be useful to obtain new bounds for the zeros of monic polynomials and to other bound, for more details see (Kittaneh, 2003 a and b; Kittaneh and Shebrawi, 2007; Horn and Zhang, 2010) and the references therein. It is hoped that further research will actually bear this out.

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