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Estimation of P[*Y*<*X*] for the class of Kumaraswamy-*G* distributions

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ABSTRACT

This paper deals with the estimation of $P[Y<X]$ when X and Y are two independent random variables from the Kumaraswamy generalized distributions (*Kw-G*) where G is a baseline distribution function. The maximum likelihood (ML) estimator and Bayes estimation of R are obtained. Exact and asymptotic distributions based on ML estimators are obtained together with the corresponding confidence intervals of $P[Y<X]$. Assuming that one of the shape parameters is known, ML estimator, Bayes estimation of R and confidence interval are obtained. The ML estimator of R, asymptotic distribution and Bayes estimation of R in the general case is also studied. Monte Carlo simulations are performed to compare the different methods when the baseline distribution G is distributed as uniform and Weibull distributions.

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INTRODUCTION

In the reliability literature, the stress-strength term refers to a component which has a random strength X and is subject to a random stress Y. The component fails if the stress applied to it exceeds the strength, while the component works whenever $Y < X$. Thus, $R = P(Y<X)$ is a measure of component reliability. In general, it seems natural to use the expression of the type $P(Y < X)$ for examining the probability of inequality of variables, so R is recognized as a general measure of the difference between two populations. Some examples of the interpretation of $P(Y<X)$ come from the literature on the inequality measures between income distributions (Dagum, 1980), and that relative to the evaluation of the area under the receiver operating characteristic (ROC) curve for diagnostic tests with continuous outcomes (Adimari and Chiogna, 2006).

The estimation of R has been extensively investigated in the literature when X and Y are independent variables belonging to the same univariate family of distributions. A comprehensive account of this topic is given by Kotz *et al*. (2003). By the end of the seventies, inference on R was carried out for the majority of common distribution families. Here, some recent contributions can be mentioned as examples: Rezaei *et al*. (2010) and Wong (2012) made inferences about R for generalized Pareto distributions; Baklizi (2008 a) and Sengupta (2011) estimated R when both stress and strength variables have two-parameter exponential distribution. The inference about R for X and Y being right truncated exponential variables is conducted by Jiang and Wong (2008). Different forms of the generalized exponential distribution have been used by Baklizi (2008 b), Raqab *et al*. (2008), Saraçoğlu *et al*. (2012). Stress and strength are gamma distributed in the papers by Krishnamoorthy *et al*. (2008) and Huang *et al*. (2012), while Weibull distributed in Krishnamoorthy and Lin (2010). Gupta and Peng (2009) and Gupta *et al*. (2010) drew inferences on reliability in proportional odds models based on the family of tilted survival functions. The case when the stress and strength are distributed as bivariate log-normal was discussed by Gupta *et al*. (2013) while Domma and Giordano (2012) discussed the case when stress-strength variables model are dependent to measure household financial fragility.

In this article, the reliability, R, when X and Y are independent but not identically distributed Kumaraswamy-generalized class of continuous distributions (*Kw-G*) random variables is considered, where $G(x)$ is a parent cumulative distribution with probability density function $g(x)$. The estimation is made under the assumption that the associated parameters of the cumulative distribution $G(x)$ are assumed known. The *Kw-G* has the cumulative distribution function (cdf) and probability density function (pdf) for $X > 0$:

$$F(x) = 1-(1-G(x)^{\alpha})^{\beta}, \quad x>0, \tag{1.1}$$

and

$$f(x) = \alpha\beta g(x)G(x)^{\alpha-1}(1-G(x)^{\alpha})^{\beta-1}, \qquad x>0 \tag{1.2}$$

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where $\alpha > 0$ and $\beta > 0$ are two shape parameters whose rule is to skewness and to vary tail weights (Cordeiro and Castro, 2011).

In probability and statistics, Special Kumaraswamy generalized (*Kw-G*) distributions are obtained based on the choice of the cdf $G(x)$. For example, The *Kw* distribution is a special case of the *Kw-G* distribution with *G* being the uniform distribution on [0;1]. When $\alpha = 1$, the *Kw-G* distribution coincides with the beta-*G* distribution generated by the Beta(1;β) distribution. Furthermore, for $\beta = 1$ and α being an integer, the *Kw-G* is the distribution of the maximum of a random sample of size α from *G*. Hence, each new *Kw-G* distribution can be obtained from a specified *G* distribution. Taking $G(x)$ in formula to be the distribution function of the normal distribution will create the *Kw*-normal (*Kw-N*) distribution. Analogously, the *Kw*-Weibull (*Kw-W*), the *Kw*-InvWeibull (*Kw-W*), *Kw*-gamma (*Kw-Ga*) and *Kw*-Gumbel (*Kw-Gu*) distributions are obtained by taking $G(x)$ to be the cdf of the Weibull, inverse Weibull, gamma and Gumbel distributions, respectively, among several others. One major benefit of the *Kw* family of generalized distributions is its ability of fitting skewed data that cannot be properly fitted by existing distributions (Cordeiro and Castro, 2011).

The rest of the paper is organized as follows. In section 2, the maximum likelihood estimation and Bayesian estimation of R are discussed with common shape parameter α and further when α is assumed known. Section 3 is devoted to estimation of R in the general case when both parameters are assumed unknown. In section 4, two special cases for the baseline cumulative distribution $G(x)$ are studied regarding the results that have been obtained. Section 5 is devoted to simulation studies and finally the paper is concluded.

Estimation of R with common shape parameter α:

Let *X* and *Y* be two independent *Kw-G* random variables with parameters (α, β_1) and (α, β_2) respectively. Therefore, the reliability of the system

$$\begin{aligned} R &= P(Y < X) \\ &= \int_0^\infty \int_0^x \alpha\,\beta_1\, g(x) G(x)^{\alpha-1} (1-G(x)^\alpha)^{\beta_1-1}\, \alpha\,\beta_2\, g(y) G(y)^{\alpha-1} (1-G(y)^\alpha)^{\beta_2-1}\, dy\, dx \\ &= 1 - \frac{\beta_1}{\beta_1+\beta_2} = \frac{\beta_2}{\beta_1+\beta_2}\,. \end{aligned} \tag{2.1}$$

The maximum likelihood estimator of R:

Let $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_m)$ be two independent random samples from *Kw-G*(α, β_1), and *Kw-G*(α, β_2) respectively. The likelihood function of α, β_1 and β_2 for the observed samples is

$$\begin{aligned} L(data;\lambda,\alpha,\beta) &= \alpha^n \beta_1^{\,n} \prod_{i=1}^n g(x_i) \prod_{i=1}^n G(x_i)^{\alpha-1} \prod_{i=1}^n (1-G(x_i)^\alpha)^{\beta_1-1} \\ &\times\ \alpha^m \beta_2^{\,m} \prod_{j=1}^m g(y_j) \prod_{j=1}^m G(y_j)^{\alpha-1} \prod_{j=1}^m (1-G(y_j)^\alpha)^{\beta_1-1} \end{aligned} \tag{2.2}$$

Therefore, the log-likelihood function of α, β_1 and β_2 will be

$$\begin{aligned} \log L &= (m+n)\log\alpha + n\log\beta_1 + m\log\beta_2 + \sum_{i=1}^n \log g(x_i) + \sum_{j=1}^m \log g(y_j) \\ &+ (\alpha-1)\{\sum_{i=1}^n \log G(x_i) + \sum_{j=1}^m \log G(y_j)\} + (\beta_1-1)\sum_{i=1}^n \log[1-G(x_i)^\alpha] \\ &+ (\beta_2-1)\sum_{j=1}^m \log[1-G(y_j)^\alpha] \end{aligned} \tag{2.3}$$

The estimators $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ of the parameters of α, β_1 and β_2 respectively can then be obtained as the solution of the likelihood equations

$$\begin{aligned} \frac{\partial \log L}{\partial \alpha} &= \frac{(n+m)}{\alpha} + \sum_{i=1}^n \log G(x_i) + \sum_{j=1}^m \log G(y_j) - (\beta_1\text{-}1)\sum_{i=1}^n \frac{G(x_i)^\alpha \log G(x_i)}{1-G(x_i)^\alpha} \\ &-(\beta_2\text{-}1)\sum_{j=1}^m \frac{G(y_j)^\alpha \log G(y_j)}{1-G(y_j)^\alpha}\,, \end{aligned} \tag{2.4}$$

$$\frac{\partial \log L}{\partial \beta_1} = \frac{n}{\beta_1} + \sum_{i=1}^{n} \log[1 - G(x_i)^{\alpha}] = 0 \quad , \tag{2.5}$$

$$\frac{\partial \log L}{\partial \beta_2} = \frac{m}{\beta_2} + \sum_{j=1}^{m} \log[1 - G(y_j)^{\alpha}] = 0 \quad , \tag{2.6}$$

Eqs (2.5) and (2.6) leads to

$$\hat{\beta}_1 = \frac{-n}{\sum_{i=1}^{n} \log[1 - G(x_i)^{\alpha}]} \quad , \tag{2.7}$$

and

$$\hat{\beta}_2 = \frac{-m}{\sum_{j=1}^{m} \log[1 - G(y_j)^{\alpha}]} \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.4), gives

$$\frac{\partial \log L}{\partial \alpha} = \frac{(n+m)}{\alpha} + \left(\frac{n}{\sum_{i=1}^{n} \log[1 - G(x_i)^{\alpha}]} + 1\right) \sum_{i=1}^{n} \frac{G(x_i)^{\alpha} \log G(x_i)}{1 - G(x_i)^{\alpha}}$$

$$+ \sum_{i=1}^{n} \log G(x_i) + \sum_{j=1}^{m} \log G(y_j) + \left(\frac{m}{\sum_{j=1}^{m} \log[1 - G(y_j)^{\alpha}]} + 1\right) \sum_{j=1}^{m} \frac{G(y_j)^{\alpha} \log G(y_j)}{1 - G(y_j)^{\alpha}} \tag{2.9}$$

On solving the nonlinear equation (2.9) with respect to α and using Eqs (2.7) and (2.8), the ML estimators of β_1 and β_2 will be given by

$$\hat{\beta}_1 = \frac{-n}{\sum_{i=1}^{n} \log[1 - G(x_i)^{\hat{\alpha}}]} \quad , \tag{2.10}$$

and

$$\hat{\beta}_2 = \frac{-m}{\sum_{j=1}^{m} \log[1 - G(y_j)^{\hat{\alpha}}]} \tag{2.11}$$

Once the estimators of β_1 and β_2 are obtained and by the invariance property of the ML estimators, the ML estimator of R becomes

$$\hat{R} = \frac{\hat{\beta}_2}{\hat{\beta}_1 + \hat{\beta}_2} = \frac{m \sum_{i=1}^{n} \log[1 - G(x_i)^{\hat{\alpha}}]}{\left(m \sum_{i=1}^{n} \log[1 - G(x_i)^{\hat{\alpha}}] + n \sum_{j=1}^{m} \log[1 - G(y_j)^{\hat{\alpha}}] \right)} \tag{2.12}$$

The maximum likelihood estimator of R with known and common shape parameter α:

In this section, the problem of estimation of R when α is known is considered, i.e., it is assumed that the independent samples $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_m)$ are drawn from *Kw-G* distribution with parameters (α, β_1), and (α, β_2) respectively with α known. Based on that the ML estimator of R and its distributional properties, the ML estimators of $\hat{\beta}_1$, $\hat{\beta}_2$ and R are given by,

$$\hat{\beta}_1 = \frac{-n}{\sum_{i=1}^{n} \log[1 - G(x_i)^{\alpha}]} \quad , \tag{2.13}$$

$$\hat{\beta}_2 = \frac{-m}{\sum_{j=1}^{m} \log[1 - G(y_j)^{\alpha}]} \quad , \tag{2.14}$$

and

$$\hat{R} = \frac{m\sum_{i=1}^{n}\log[1-G(x_i)^{\alpha}]}{\left(m\sum_{i=1}^{n}\log[1-G(x_i)^{\alpha}] + n\sum_{j=1}^{m}\log[1-G(y_j)^{\alpha}]\right)} . \tag{2.15}$$

Exact and asymptotic Confidence Intervals of R:

Exact Confidence Intervals:

Under the assumption that the shape parameter α is known, it can be noted that the random variable $U_i = -\log[1-G(x_i)^{\alpha}]$ is distributed as exponential random variable with mean $(1/\beta_1)$, and the random variable $V_i = -\log[1-e^{-1/y_j}]$ is distributed as exponential random variable with mean $(1/\beta_2)$. Therefore, $2\beta_1\sum_{i=1}^{n}\log[1-e^{-1/x_i}] \sim \chi^2_{2n}$ and $2\beta_2\sum_{j=1}^{m}\log[1-e^{-1/y_j}] \sim \chi^2_{2m}$ (Gupta and Kundu, 2002). $\hat{R}$ can then be written as $\hat{R} \approx 1/(1+\beta_1/\beta_2\ F)$, where F has a Fisher distribution with $2n$ and $2m$ degrees of freedom respectively and therefore, the pdf of $\hat{R}$ is given by

$$f_{\hat{R}}(u) = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}\left(\frac{n\beta_1}{m\beta_2}\right)^n\left(\frac{1}{u^2}\right)\frac{\left(\frac{1-u}{u}\right)^{n-1}}{\left(1+\frac{n\beta_1}{m\beta_2}\left(\frac{1-u}{u}\right)\right)^{n+m}} \tag{2.16}$$

where $0 < u < 1$, and $\beta_1, \beta_2 \geq 0$. Based on this information a $(1-\tau)$ 100% confidence interval of R can be obtained as

$$\left[\frac{1}{1+\left(\frac{1}{\hat{R}}-1\right)F_{2n,2m,1-\tau/2}}, \frac{1}{1+\left(\frac{1}{\hat{R}}-1\right)F_{2n,2m,\tau/2}}\right], \tag{2.17}$$

where $F_{2n,2m,\tau/2}$ and $F_{2n,2m,1-\tau/2}$ are the lower and upper $\tau/2$ th percentile of a Fisher distribution with $2n$ and $2m$ degrees of freedom respectively.

Asymptotic confidence interval:

Based on the asymptotic properties under general conditions of the ML estimators $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$. the asymptotic distribution of the ML estimators immediately follows from the Fisher information matrix of α, β_1 and β_2 (Lehmann, 1999). That is, when $n \to \infty, m \to \infty$ and $n/m \to p, 0 < p < 1$,

$$\left(\sqrt{n}\left(\hat{\beta}_1-\beta_1\right), \sqrt{m}\left(\hat{\beta}_2-\beta_2\right), \sqrt{n}\left(\hat{\alpha}-\alpha\right)\right) \xrightarrow{D} N_3(0,\Sigma_3) \tag{2.18}$$

where

$$\Sigma_3 = I^{-1}(\Omega) = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}^{-1}, \tag{2.19}$$

and the matrix $I(\Omega)$ is the Fisher information matrix of the parameter vector $\Omega = \{\beta_1, \beta_2, \alpha\}$, and the *ij*-th element is given by the second derivatives $I_{ij} = \partial^2 \ln L(\Omega)/\partial\omega_i\partial\omega_j,\ \ i, j = 1,2,3$. Therefore, it can be shown that

$$\sqrt{n}\left(\hat{R}-R\right) = \sqrt{n}\left(\frac{\hat{\beta}_2}{\hat{\beta}1+\hat{\beta}_2} - \frac{\beta_2}{\beta_1+\beta_2}\right) \xrightarrow{D} N(0,\psi^2) \tag{2.20}$$

where

$$\psi^2 = E\left(\sqrt{n}\left(\hat{R} - R\right)\right)^2, \tag{2.21}$$

A $(1-\tau)$ 100% approximate confidence interval of R can be constructed based on the asymptotic results found. This asymptotic confidence interval is given by

$$\hat{R} \pm Z_{1-\tau/2}\hat{\psi}, \tag{2.22}$$

where $\hat{\psi}$ is the asymptotic standard deviation of $\hat{R}$.

Bayesian estimation of R:

In this section, the Bayes estimator of R denoted as $\hat{R}_{BS}$ is obtained under the assumption that the shape parameters β_1 and β_2 are independent random variables with prior distributions $\Gamma(a_1, b_1)$ and $\Gamma(a_2, b_2)$ respectively with pdf's

$$\pi(\beta_1) = \frac{b_1^{a_1}}{\Gamma(a_1)}\beta_1^{a_1-1}e^{-b_1\beta_1}; \qquad \beta_1 > 0, \tag{2.23}$$

and

$$\pi(\beta_2) = \frac{b_2^{a_2}}{\Gamma(a_2)}\beta_2^{a_2-1}e^{-b_2\beta_2}; \qquad \beta_2 > 0. \tag{2.24}$$

Based on the above assumptions and from equation (2.2), the joint density of the data, β_1 and β_2 can be obtained as

$$L(data, \beta_1, \beta_2) = L(data; \beta_1, \beta_2)\ \pi(\beta_1)\ \pi(\beta_2), \tag{2.25}$$

Therefore, the joint posterior density of the data, α and β given the data can be obtained as

$$L(\beta_1, \beta_2 / data) = \frac{L(data, \beta_1, \beta_2)\ \pi(\beta_1)\ \pi(\beta_2)}{\int_0^\infty\int_0^\infty L(data, \beta_1, \beta_2)\ \pi(\beta_1)\ \pi(\beta_2)\, d\beta_1\, d\beta_2}, \tag{2.26}$$

The posterior pdf's of, β_1 and β_2 are as follows

$$\beta_1 / data \sim Gamma(n + a_1, b_1 - T_1), \tag{2.27}$$

and

$$\beta_2 / data \sim Gamma(m + a_2, b_2 - T_2), \tag{2.28}$$

respectively, where $T_1 = \sum_{i=1}^{n}\log[1 - G(x_i)^\alpha]$ and $T_2 = \sum_{j=1}^{m}\log[1 - G(y_j)^\alpha]$. Since β_1 and β_2 are assumed independent, the posterior pdf of R becomes

$$\pi(r / x, y) = K\ \frac{r^{m+a_2-1}\ (1-r)^{n+a_1-1}}{[(1-r)(b_1 - T_1) + r\ (b_2 - T_2)]^{n+m+a_1+a_2}}, \quad 0 < r < 1, \tag{2.29}$$

where

$$K = \frac{\Gamma(n + m + a_1 + a_2 - 1)}{\Gamma(n + a_1)\Gamma(m + a_2)}(b_1 - T_1)^{n+a_1}(b_2 - T_2)^{m+a_2}.$$

The Bayes estimate of R under squared error loss function cannot be computed analytically and alternatively, using the approximate method of Lindley (1980) and Ahmad et. al., (1997) and as mentioned in Rezaei et. al., (2010), it can be easily seen that the approximate Bayes estimate of R, $\hat{R}_{BS}$ under squared error loss function is

$$\hat{R}_{BS} = \tilde{R}\left[1 + \frac{\tilde{\beta}_1 \tilde{R}^2 (\tilde{\beta}_1 (n + a_1 - 1) - \tilde{\beta}_2 (m + a_2 - 1))}{\tilde{\beta}_2 (n + a_1 - 1)(m + a_2 - 1)}\right], \tag{2.30}$$

where

$$\tilde{R} = \frac{\tilde{\beta}_2}{\tilde{\beta}_1 + \tilde{\beta}_2}, \qquad \tilde{\beta}_1 = \frac{n + a_1 - 1}{b_1 - T_1} \qquad \text{and } \tilde{\beta}_2 = \frac{m + a_2 - 1}{b_2 - T_2}$$

Estimation Of R In The General Case:

Let X and Y be two independent *Kw-G* random variables with parameters (α_1, β_1) and (α_2, β_2) respectively. Therefore, the reliability of the system

$$R = P(Y < X)$$
$$= \int_0^\infty \int_0^x \alpha_1 \beta_1 g(x) G(x)^{\alpha_1 - 1} (1 - G(x)^{\alpha_1})^{\beta_1 - 1} \alpha_2 \beta_2 g(y) G(y)^{\alpha_2 - 1} (1 - G(y)^{\alpha_2})^{\beta_2 - 1} dy\, dx,$$
$$= 1 - \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l} \Gamma(\beta_1 - 1) \Gamma(\beta_2)}{\Gamma(\beta_1 - k - 1) \Gamma(\beta_2 - l) k! l! (\alpha_1 (k+1) + \alpha_2 l)}. \tag{3.1}$$

The Maximum Likelihood Estimator Of R:

Let $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_m)$ be two independent random samples from *Kw-G*(α_1, β_1), and *Kw-G*(α_2, β_2) respectively. The log-likelihood function of $\alpha_1, \alpha_2, \beta_1$ and β_2 for the observed samples is

$$\log L = n \log \alpha_1 + m \log \alpha_2 + n \log \beta_1 + m \log \beta_2 + \sum_{i=1}^{n} \log g(x_i) + \sum_{j=1}^{m} \log g(y_j)$$
$$+ (\alpha_1 - 1) \sum_{i=1}^{n} \log G(x_i) + (\alpha_2 - 1) \sum_{j=1}^{m} \log G(y_j) + (\beta_1 - 1) \sum_{i=1}^{n} \log[1 - G(x_i)^{\alpha_1}]$$
$$+ (\beta_2 - 1) \sum_{j=1}^{m} \log[1 - G(y_j)^{\alpha_2}] \tag{3.2}$$

The estimators $\hat{\alpha}_1, \hat{\alpha}_2$, $\hat{\beta}_1$ and $\hat{\beta}_2$ of the parameters of $\alpha_1, \alpha_2, \beta_1$ and β_2 respectively are obtained by solving the corresponding likelihood equations, and then, the estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ are

$$\hat{\beta}_1 = \frac{-n}{\sum_{i=1}^{n} \log[1 - G(x_i)^{\hat{\alpha}_1}]}, \tag{3.3}$$

and

$$\hat{\beta}_2 = \frac{-m}{\sum_{j=1}^{m} \log[1 - G(y_j)^{\hat{\alpha}_2}]}, \tag{3.4}$$

where $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are the solution of the two nonlinear equation

$$\frac{n}{\hat{\alpha}_1} + \sum_{i=1}^{n} \log G(x_i) + \left(\frac{n}{\sum_{i=1}^{n} \log[1 - G(x_i)^{\hat{\alpha}_1}]} + 1\right) \sum_{i=1}^{n} \frac{G(x_i)^{\hat{\alpha}_1} \log G(x_i)}{1 - G(x_i)^{\hat{\alpha}_1}} = 0 \tag{3.5}$$

$$\frac{m}{\hat{\alpha}_2} + \sum_{j=1}^{m} \log G(y_j) + \left(\frac{m}{\sum_{j=1}^{m} \log[1 - G(y_j)^{\hat{\alpha}_2}]} + 1\right) \sum_{j=1}^{m} \frac{G(y_j)^{\hat{\alpha}_2} \log G(y_j)}{1 - G(y_j)^{\hat{\alpha}_2}} = 0 \tag{3.6}$$

Thus, the ML estimator of R is

$$R = 1 - \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l} \Gamma(\hat{\beta}_1 - 1) \Gamma(\hat{\beta}_2)}{\Gamma(\hat{\beta}_1 - k - 1) \Gamma(\hat{\beta}_2 - l) k! l! (\hat{\alpha}_1 (k+1) + \hat{\alpha}_2 l)}. \tag{3.7}$$

Asymptotic Confidence Intervals of R:

Similar to the results obtained in section (2.3.2), and based on the asymptotic properties under general conditions of the ML estimators $\hat{\alpha}_1, \hat{\alpha}_2$, $\hat{\beta}_1$ and $\hat{\beta}_2$, the asymptotic distribution of the ML estimators immediately follows from the Fisher information matrix of α_1, α_2, β_1 and β_2 (Lehmann, 1999),. That is, when $n \to \infty,\ m \to \infty$ and $n/m \to p,\ 0 < p < 1$,

$$\left(\sqrt{n}\left(\hat{\beta}_1 - \beta_1\right),\ \sqrt{n}\left(\hat{\alpha}_1 - \alpha_1\right), \sqrt{m}\left(\hat{\beta}_2 - \beta_2\right), \sqrt{m}\left(\hat{\alpha}_2 - \alpha_2\right)\right) \xrightarrow{D} N_4(0, \Sigma_4) \tag{3.8}$$

where

$$\Sigma_3 = I^{-1}(\Omega) = \begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{31} & I_{32} & I_{33} & I_{44} \end{pmatrix}^{-1}, \tag{3.9}$$

where, $I_{ij} = \partial^2 \ln L / \partial\omega_i \partial\omega_j$, $\Omega = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$, and $i, j = 1,2,3,4$. Accordingly, it is possible to present confidence intervals of R similar to those mentioned in Section (2.3.2).

Bayesian Estimation Of R:

In this section, the Bayes estimator of R, $\hat{R}_{BS}$ is obtained under the assumption that α_1, α_2, β_1 and β_2 are independent gamma random variables with prior distributions $\alpha_1 \sim Gamma(a_1, b_1)$, $\alpha_2 \sim Gamma(a_2, b_2)$, $\beta_1 \sim Gamma(a_3, b_3)$ and $\beta_2 \sim Gamma(a_4, b_4)$. Based on that and from the likelihood function of the observed data in equation (3.2), the joint density of the data, α_1, α_2, β_1 and β_2 can be obtained as

$$L(data, \alpha_1, \alpha_2, \beta_1, \beta_2) = L(data; \alpha_1, \alpha_2, \beta_1, \beta_2)\ \pi(\alpha_1)\ \pi(\alpha_2)\pi(\beta_1)\ \pi(\beta_2), \tag{3.10}$$

Therefore, the joint posterior density of the data, α and β given the data can be obtained as

$$L(\alpha, \beta / data) = \frac{L(data, \alpha_1, \alpha_2, \beta_1, \beta_2)\ \pi(\alpha_1)\ \pi(\alpha_2)\pi(\beta_1)\ \pi(\beta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(data, \alpha_1, \alpha_2, \beta_1, \beta_2)\ \pi(\alpha_1)\ \pi(\alpha_2)\pi(\beta_1)\ \pi(\beta_2)}, \tag{3.11}$$

The posterior pdf's of, α_1, α_2, β_1 and β_2 are then

$$\pi_{\alpha_1}(\alpha_1 / data) \sim \alpha_1^{n+a_1} e^{-(b_1 - T_1)\alpha_1}\ e^{\beta_1 T_3}, \tag{3.12}$$

$$\pi_{\alpha_2}(\alpha_2 / data) \sim \alpha_2^{m+a_2} e^{-(b_2 - T_2)\alpha_2}\ e^{\beta_2 T_4}, \tag{3.13}$$

$$\beta_1 / data \sim \Gamma\left(n + a_3, b_3 - T_1\right), \tag{3.14}$$

and

$$\beta_2 / data \sim \Gamma\left(m + a_4, b_4 - T_2\right), \tag{3.15}$$

respectively, where $T_1 = \sum_{i=1}^{n} \log G(x_i)$, $T_2 = \sum_{j=1}^{m} \log G(y_j)$, $T_3 = \sum_{i=1}^{n} \log[1 - G(x_i)^{\alpha_1}]$ and $T_4 = \sum_{j=1}^{m} \log[1 - G(y_j)^{\alpha_2}]$.

The posterior pdfs of α_1 and α_2 are unknown but, by studying them, a proposed distribution for them might be the one parameter gamma distribution. Therefore, the Metropolis–Hastings method with one parameter gamma distribution proposal distribution (Rezaei et. al., 2010) is used to simulate an approximated Bayesian estimator of *R*.

Special Cases:

In this section will illustrate the above results for some cases for the baseline cumulative distribution function *G*(*x*). The first case is when *G*(*x*) is the uniform distribution where this will define the original kumaraswamy distribution (Kw distribution). The other case is the Kumaraswamy-Weibull distribution (*kw*-W), where *G*(*x*) is the will known Weibull distribution. The pdf's and cdf's of the two classes are considered in table 1, while the corresponding ML and Bayes estimators of β_1 and β_2 in the case with common shape parameter α and of α_1, α_2, β_1 and β_2 in the general case are presented in table 2 and table 3 respectively.

Table 1: Probability density functions for the kumaraswamy-G distributions where G is distributed as uniform and Weibull

Distribution of G	Probability distribution function	probability density function
Uniform	$F(x)=1-(1-x^{\alpha})^{\beta}$	$f(x)=\alpha\beta\, x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, \qquad 0<x<1$ *
Exponential	$F(x)=1-(1-(1-e^{-\lambda x^{\gamma}})^{\alpha})^{\beta}$	$f(x)=\alpha\beta\gamma\lambda\, x^{\gamma-1}e^{-\lambda x^{\gamma}}(1-e^{-\lambda x^{\gamma}})^{\alpha-1}(1-(1-e^{-\lambda x^{\gamma}})^{\alpha})^{\beta-1}$

* The Kumaraswamy distribution (Kumaraswamy, 1980)

Simulation Results:

In this section a Monte Carlo simulation is performed to study the behavior of the different methods for different sample sizes and for different parameter values. the simulation is when the shape parameter α is common and known and is assumed to be equal to one. The performance of the ML and the Bayes estimates with respect to the squared error loss function are compared in terms of biases and mean squares errors (MSE). Bayes estimates, are computed based on two types of priors, (i) non-informative priors, where prior information about *R* is assumed to be very less (questionable) and in this case, $a_1=a_2=b_1=b_2=0.0001$, (Congdon, 2001, Kundu and Gupta, 2005). (ii) Informative priors, where it is assumed that there are some prior information about the parameters and $a_1, a_2, b_1, b_2>0$, for example, $a_1=a_2=3$, $b_1=b_2=2$. The average estimates of the ML and Bayes estimates of *R* based on 1000 replications are presented in table 4 through table 7 for the two special cases studied, All simulations are based on the following sample sizes; *n*, *m* = 10, 25, and 50 and parameter values as $\beta_2=0.5, 1.5, 2.5, 4.5$, and $\beta_1=1.5$, respectively.

It can be noted that even for small sample sizes, the performance of the ML estimator of *R* is quite satisfactory in terms of biases and MSE which verifies the consistency property of the ML estimators of *R*. It is also observed that when *n* and *m* increase, the MSE and biases decrease for both ML and Bayesian estimation methods. For the Weibull distribution, when the shape parameter $R>0.5$, there is an underestimation with negative biases.. Also, it is noted that for fixed sample sizes and for fixed value for β_1 and as the parameter β_2 increases, MSE and biases decrease for both ML and Bayesian estimation methods.

Table 2: ML and Bayes estimators of β_1, β_2 when the shape parameter α is known and G is distributed as uniform and Weibull

Distribution	ML estimators	Approximate Bayes estimators
The Kumaraswamy distribution	$\hat{\beta}_1=-n\left(\sum_{i=1}^{n}\log[1-x_i^{\alpha}]\right)^{-1}, \hat{\beta}_2=-m\left(\sum_{j=1}^{m}\log[1-y_j^{\alpha}]\right)^{-1}$ $\hat{R}=\dfrac{m\sum_{i=1}^{n}\log[1-x_i^{\alpha}]}{\left(m\sum_{i=1}^{n}\log[1-x_i^{\alpha}]+n\sum_{j=1}^{m}\log[1-y_j^{\alpha}]\right)}$	$\hat{R}_{BS}=\tilde{R}\left[1+\dfrac{\tilde{\beta}_1\tilde{R}^2(\tilde{\beta}_1(n+a_1-1)-\tilde{\beta}_2(m+a_2-1))}{\tilde{\beta}_2(n+a_1-1)(m+a_2-1)}\right]$ $\tilde{R}=\dfrac{\tilde{\beta}_2}{\tilde{\beta}_1+\tilde{\beta}_2}, \tilde{\beta}_1=\dfrac{n+a_1-1}{b_1-T_1}, \tilde{\beta}_2=\dfrac{m+a_2-1}{b_2-T_2}$ $T_1=\sum_{i=1}^{n}\log[1-x_i^{\alpha}], T_2=\sum_{j=1}^{m}\log[1-y_j^{\alpha}]$
The Kumaraswamy-Weibull distribution	$\hat{\beta}_1=-n\left(\sum_{i=1}^{n}\log[1-(1-e^{-\lambda x_i^{\gamma}})^{\alpha}]\right)^{-1}$, $\hat{\beta}_2=-m\left(\sum_{j=1}^{m}\log[1-(1-e^{-\lambda y_j^{\gamma}})^{\alpha}]\right)^{-1}$ $\hat{R}=\dfrac{m\sum_{i=1}^{n}\log[1-(1-e^{-\lambda x_i^{\gamma}})^{\alpha}]}{\left(m\sum_{i=1}^{n}\log[1-(1-e^{-\lambda x_i^{\gamma}})^{\alpha}]+n\sum_{j=1}^{m}\log[1-(1-e^{-\lambda y_j^{\gamma}})^{\alpha}]\right)}$	$\hat{R}_{BS}=\tilde{R}\left[1+\dfrac{\tilde{\beta}_1\tilde{R}^2(\tilde{\beta}_1(n+a_1-1)-\tilde{\beta}_2(m+a_2-1))}{\tilde{\beta}_2(n+a_1-1)(m+a_2-1)}\right]$ $\tilde{R}=\dfrac{\tilde{\beta}_2}{\tilde{\beta}_1+\tilde{\beta}_2}, \tilde{\beta}_1=\dfrac{n+a_1-1}{b_1-T_1}, \tilde{\beta}_2=\dfrac{m+a_2-1}{b_2-T_2}$ $T_1=\sum_{i=1}^{n}\log[1-(1-e^{-\lambda x_i^{\gamma}})^{\alpha}]$, $T_2=\sum_{j=1}^{m}\log[1-(1-e^{-\lambda y_j^{\gamma}})^{\alpha}]$

Table 3: ML and Bayes estimators of β_1, β_2 in the general case when G is distributed as uniform and Weibull

Distribution	ML estimators
The Kumaraswamy distribution	$\hat{\beta}_1=-n\left(\sum_{i=1}^{n}\log[1-x_i^{\hat{\alpha}_1}]\right)^{-1}$, $\hat{\beta}_2=-m\left(\sum_{j=1}^{m}\log[1-y_j^{\hat{\alpha}_2}]\right)^{-1}$ $\dfrac{n}{\hat{\alpha}_1}+\sum_{i=1}^{n}\log x_i+\left(\dfrac{n}{\sum_{i=1}^{n}\log[1-x_i^{\hat{\alpha}_1}]}+1\right)\sum_{i=1}^{n}\dfrac{x_i^{\hat{\alpha}_1}\log x_i}{1-x_i^{\hat{\alpha}_1}}=0$, $\dfrac{m}{\hat{\alpha}_2}+\sum_{j=1}^{m}\log y_j+\left(\dfrac{m}{\sum_{j=1}^{m}\log[1-y_j^{\hat{\alpha}_2}]}+1\right)\sum_{j=1}^{m}\dfrac{y_j^{\hat{\alpha}_2}\log y_j}{1-y_j^{\hat{\alpha}_2}}=0$

	$\hat{R}=\dfrac{m\sum_{i=1}^{n}\log[1-x_i^{\hat{\alpha}_1}]}{\left(m\sum_{i=1}^{n}\log[1-x_i^{\hat{\alpha}_1}]+n\sum_{j=1}^{m}\log[1-y_j^{\hat{\alpha}_2}]\right)}$
The Kumaraswamy-Weibull distribution	$\hat{\beta}_1=-n\left(\sum_{i=1}^{n}\log[1-(1-e^{-\lambda x_i^{\gamma}})^{\hat{\alpha}_1}]\right)^{-1}, \hat{\beta}_2=-m\left(\sum_{j=1}^{m}\log[1-(1-e^{-\lambda y_j^{\gamma}})^{\hat{\alpha}_2}]\right)^{-1}$ $\frac{n}{\hat{\alpha}_1}+\sum_{i=1}^{n}\log(1-e^{-\lambda x_i^{\gamma}})+(\frac{n}{\sum_{i=1}^{n}\log[1-(1-e^{-\lambda x_i^{\gamma}})^{\hat{\alpha}_1}]}+1)\sum_{i=1}^{n}\frac{(1-e^{-\lambda x_i^{\gamma}})^{\hat{\alpha}_1}\log(1-e^{-\lambda x_i^{\gamma}})}{1-(1-e^{-\lambda x_i^{\gamma}})^{\hat{\alpha}_1}}=0$ $\frac{m}{\hat{\alpha}_2}+\sum_{j=1}^{m}\log(1-e^{-\lambda y_j^{\gamma}})+(\frac{m}{\sum_{j=1}^{m}\log[1-(1-e^{-\lambda y_j^{\gamma}})^{\hat{\alpha}_2}]}+1)\sum_{j=1}^{m}\frac{(1-e^{-\lambda y_j^{\gamma}})^{\hat{\alpha}_2}\log(1-e^{-\lambda y_j^{\gamma}})}{1-(1-e^{-\lambda y_j^{\gamma}})^{\hat{\alpha}_2}}=0$ $\hat{R}=\dfrac{m\sum_{i=1}^{n}\log[1-(1-e^{-\lambda x_i^{\gamma}})^{\hat{\alpha}_1}]}{\left(m\sum_{i=1}^{n}\log[1-(1-e^{-\lambda x_i^{\gamma}})^{\hat{\alpha}_1}]+n\sum_{j=1}^{m}\log[1-(1-e^{-\lambda y_j^{\gamma}})^{\hat{\alpha}_2}]\right)}$

Table 4: The maximum likelihood estimator of R, biases and the mean squared error , when $\beta_1=1.5$ and the shape parameter α is known ($\alpha=1$) and the baseline distribution function *G* is uniform

(*n*,*m*)	β_2	*R*	$\hat{R}$	Bias	*MSE*
(10,10)	0.25	0.2000	0.2112	0.0112	0.0151
	0.50	0.3333	0.3455	0.0122	0.0124
	1.00	0.5000	0.5054	0.0054	0.012
	2.00	0.6667	0.6698	0.0031	0.0089
	3.00	0.7500	0.7544	0.0044	0.0075
(10,25)	0.25	0.2000	0.2017	0.0017	0.0033
	0.50	0.3333	0.3387	0.0054	0.0029
	1.00	0.5000	0.5042	0.0042	0.0053
	2.00	0.6667	0.6685	0.0018	0.0031
	3.00	0.7500	0.7532	0.0032	0.0026
(25,25)	0.25	0.2000	0.2010	0.001	0.0025
	0.50	0.3333	0.3362	0.0029	0.0021
	1.00	0.5000	0.5035	0.0035	0.0035
	2.00	0.6667	0.6679	0.0012	0.0021
	3.00	0.7500	0.7525	0.0025	0.0008
(25,50)	0.25	0.2000	0.2009	0.0009	0.0017
	0.50	0.3333	0.3351	0.0018	0.0014
	1.00	0.5000	0.5018	0.0018	0.0013
	2.00	0.6667	0.6675	0.0008	0.0008
	3.00	0.7500	0.7519	0.0019	0.0006
(50,50)	0.25	0.2000	0.2004	0.0004	0.0012
	0.50	0.3333	0.3343	0.0010	0.0011
	1.00	0.5000	0.5009	0.0009	0.0009
	2.00	0.6667	0.6672	0.0005	0.0005
	3.00	0.7500	0.7508	0.0008	0.0004

Table 5: Bayesian estimation of *R*, in terms of biases and the mean squared error, when $\beta_1=1.5$ and the shape parameter α is known ($\alpha=1$) and the baseline distribution function *G* is uniform

(*n*,*m*)	β_2	*R*	Non-informative priors $a_1=a_2=b_1=b_2=0.0001$			Informative priors $a_1=a_2=3,\ b_1=b_2=2$	
			Bias	*(MSE)*		*Bias*	*(MSE)*
(10,10)	0.25	0.2000	0.0193	0.0200		0.0247	0.0351
	0.50	0.3333	0.0153	0.0195		0.0219	0.0279
	1.00	0.5000	0.0101	0.0152		0.0201	0.0246
	2.00	0.6667	0.0046	0.0109		0.0182	0.0228
	3.00	0.7500	0.0056	0.0101		0.0138	0.0207
(10,25)	0.25	0.2000	0.0075	0.0097		0.0186	0.0258
	0.50	0.3333	0.0083	0.0091		0.0159	0.0206
	1.00	0.5000	0.0051	0.0072		0.0137	0.0199
	2.00	0.6667	0.0030	0.0055		0.01	0.0173
	3.00	0.7500	0.0023	0.0041		0.0085	0.0138
(25,25)	0.25	0.2000	0.0059	0.0067		0.0153	0.0221
	0.50	0.3333	0.0078	0.0049		0.0119	0.0200
	1.00	0.5000	0.0041	0.0041		0.0104	0.0184
	2.00	0.6667	0.0028	0.0026		0.0084	0.0131

	3.00	0.7500	0.0026	0.0021		0.0069	0.0140
(25,50)	0.25	0.2000	0.0027	0.0047		0.011	0.0197
	0.50	0.3333	0.0054	0.0038		0.0099	0.0153
	1.00	0.5000	0.0028	0.0026		0.0057	0.0109
	2.00	0.6667	0.0014	0.0021		0.0036	0.0096
	3.00	0.7500	0.0010	0.0015		0.0019	0.0082
(50,50)	0.25	0.2000	0.0029	0.0033		0.0091	0.0151
	0.50	0.3333	0.0021	0.0021		0.0073	0.0114
	1.00	0.5000	0.0016	0.0016		0.0041	0.0093
	2.00	0.6667	0.0010	0.0011		0.0024	0.0047
	3.00	0.7500	0.0009	0.0007		0.0015	0.0055

Table 6: The maximum likelihood estimator of R, biases and the mean squared error , when $\beta_1 = 1.5$, the shape parameter α is known ($\alpha = 1$) and the baseline distribution function *G* is Weibull with parameters $\lambda = 1,\ \gamma = 3$

(*n*,*m*)	β_2	*R*	$\hat{R}$	*Bias*	*MSE*
(10,10)	0.25	0.2000	0.2101	0.0101	0.0153
	0.50	0.3333	0.3411	0.0078	0.0049
	1.00	0.5000	0.5005	0.0069	0.0038
	2.00	0.6667	0.6599	-0.0068	0.0062
	3.00	0.7500	0.7410	-0.009	0.0075
(10,25)	0.25	0.2000	0.2044	0.0044	0.0094
	0.50	0.3333	0.3357	0.0024	0.0032
	1.00	0.5000	0.4977	0.0054	0.0027
	2.00	0.6667	0.6603	-0.0064	0.004
	3.00	0.7500	0.7427	-0.0073	0.0017
(25,25)	0.25	0.2000	0.2063	0.0063	0.0051
	0.50	0.3333	0.3403	0.0073	0.0019
	1.00	0.5000	0.5001	0.0032	0.0015
	2.00	0.6667	0.6600	-0.0067	0.0024
	3.00	0.7500	0.7427	-0.0052	0.0017
(25,50)	0.25	0.2000	0.2033	0.0033	0.0027
	0.50	0.3333	0.3345	0.0015	0.0012
	1.00	0.5000	0.5035	0.0025	0.001
	2.00	0.6667	0.6664	-0.0049	0.0015
	3.00	0.7500	0.7456	-0.0042	0.001
(50,50)	0.25	0.2000	0.2049	0.0029	0.0011
	0.50	0.3333	0.3350	0.001	0.001
	1.00	0.5000	0.4996	0.0013	0.0008
	2.00	0.6667	0.6647	-0.0028	0.001
	3.00	0.7500	0.7476	-0.0021	0.0006

Table 7: Bayesian estimation of R biases and the mean squared error, when $\beta_1 = 1.5$, the shape parameter α is known ($\alpha = 1$) and the baseline distribution function *G* is Weibull with parameters $\lambda = 1,\ \gamma = 3$

(*n*,*m*)	β_2	*R*	Non-informative priors $a_1 = a_2 = b_1 = b_2 = 0.0001$			Informative priors $a_1 = a_2 = 3,\ b_1 = b_2 = 2$	
			Bias	*(MSE)*		*Bias*	*(MSE)*
(10,10)	0.25	0.2000	0.0153	0.0241		0.0351	0.0282
	0.50	0.3333	0.0129	0.0199		0.0279	0.0217
	1.00	0.5000	0.0097	0.0163		0.0246	0.0189
	2.00	0.6667	0.0043	0.011		0.0228	0.0153
	3.00	0.7500	0.0039	0.0101		0.0207	0.0128
(10,25)	0.25	0.2000	0.0069	0.0149		0.0258	0.0201
	0.50	0.3333	0.0052	0.0137		0.0206	0.0176
	1.00	0.5000	0.0042	0.0072		0.0199	0.0143
	2.00	0.6667	0.0025	0.0062		0.0173	0.0114
	3.00	0.7500	0.0021	0.0059		0.0138	0.0115
(25,25)	0.25	0.2000	0.0051	0.0107		0.0221	0.0181
	0.50	0.3333	0.0048	0.0098		0.02	0.0137
	1.00	0.5000	0.0036	0.0086		0.0184	0.0102
	2.00	0.6667	0.0019	0.0048		0.0131	0.0095
	3.00	0.7500	0.0016	0.0042		0.014	0.0064
(25,50)	0.25	0.2000	0.0024	0.0083		0.0197	0.0129
	0.50	0.3333	0.0037	0.0052		0.0153	0.0113
	1.00	0.5000	0.0021	0.0039		0.0109	0.0098
	2.00	0.6667	0.0013	0.0022		0.0096	0.0063
	3.00	0.7500	0.0009	0.0018		0.0082	0.0046
(50,50)	0.25	0.2000	0.0019	0.0046		0.0151	0.0099
	0.50	0.3333	0.0016	0.0043		0.0114	0.0091

	1.00	0.5000	0.0011	0.0026		0.0093	0.0052
	2.00	0.6667	0.0009	0.0015		0.0047	0.0038
	3.00	0.7500	0.0005	0.0007		0.0055	0.001

Discussion:

In this paper, the problem of estimating P[$Y<X$] for the kumaraswamy generalized class of distributions has been addressed. The exact and asymptotic distributions of the maximum likelihood estimator have been derived and used to construct exact and asymptotic confidence intervals.. It is observed that the Bayes estimators behave quite similarly to the ML estimators for both cases when the baseline distribution is uniform and Weibull. It is also observed that for fixed (m,n), as the parameter λ increases the MSE of the estimates decrease, and when (m,n), increase then MSEs of all the estimators decrease rapidly. The performance of the Bayes estimators also is quite satisfactory. The MSEs of the ML estimators are smaller than the MSEs of the Bayes estimators. Finally, the average lengths of all intervals decrease as (m,n) increases.

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