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A New Approach for Solving Urysohn Integral Equations by Using Spline-Picard Method

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Abstract: In this paper we present a new numerical method for Urysohn integral equation. The given method is based on using spline piece-wise functions and Picard's method. The convergence and the numerical stability of the mentioned method are mathematically proved. Finally, the mentioned method are applied to some integral equations with known exact solutions.

Key words: Nonlinear integral equation; Spline interpolation; Picard's method; successive approximations.

INTRODUCTION

Integral equations play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of elasticity, engineering, mathematical physics and contact mixed problems (Delves, L.M. and J.L. Mohamed, 1985; Baker, C.T.H., G.F. Miller,; Jerri, A.J.,; Anselone, P., 1964; Argyros, I.K., 1985; Arino, O. and M. Kimmel, 1985). Therefore, many different methods are used to obtain the solution of the nonlinear integral equation. In (Atkinson, K., 1992), Atkinson's *et al*., introduced a class of methods depending on some parameters to obtain the numerical solution of Abel integral equation of the second kind. In (Maleknejad, K., *et al*., 2010), Maleknejad *et al*. used iterated methods to obtain the solution of Urysohn integral equations. The linear multistep methods were applied in (Abdou, M.A., *et al*., 1998), to obtain the numerical solution of a singular nonlinear Volterra integral equation. In (Baker, C.T.H., G.F. Miller,), a Product Nystrom method used as a numerical method to obtain the solution of nonlinear Volterra integral equation, when its kernel takes a logarithmic and Carleman forms. Moreover, some methods can be found in Refs. (Jerri, A.J.,; Jerri, A.J., Kumar, S., 1987;) Ulo Lepik, Enn Tamme, 2007) to discuss and obtain the solution of nonlinear integral equation. Fredholm integral equations arise in physics (solid state physics, plasma physics, quantum mechanics), astrophysics (the radiative transfer being modeled with the well-known Chandrasekhar integral equation), fluid dynamics (the study of water waves on liquids of infinite depth uses the Nekrasov's integral equation), cell kinetic (Arino, O. and M. Kimmel, 1985), chemical kinetic, the theory of gases, mathematical economics, hereditary phenomena in biology. The existence and uniqueness of the solution for Fredholm functional integral equations can be studied using topological methods by completely continuous operators as in (Atkinson, K., 1992; Jerri, A.J.,). For these and other applications (Derili, H., *et al*., 2011). Our attention in this paper is focused on numerical methods based on spline functions and successive approximations for Urysohn functional integral equations (Ahlberg, J.H., E.N. Nilson, J.L. Walsh, 1967; Allouch, C., *et al*., 2011). Consider the nonlinear operator equation

$$x = g + f(x) \tag{1}$$

where f is a completely continuous operator defined on the closure $\overline{D}$ of a Banach space X. An example of such an operator f is the Urysohn integral operator

$$f(x)(t) = \int_{\Omega} f(t, s, x(s))ds, t \in \Omega, x \in D. \tag{2}$$

with Ω, $(\Omega = [a,b])$ a closed bounded region in $\mathbb{R}^m$ for some $m \geq 1$.

In this paper, we investigate the property of the spline functions for Urysohn equations. For this, In section 2, we review necessary conditions for Urysohn equations. In section 3, the properties of splines for Urysohn equations is established. The numerical algorithm discussed with details in section 4. Section 5 devoted to stability and error analysis of the presented method. Some examples are also included in section 6.

Mathematical Results:

Existence, uniqueness of the solution and properties of the sequence of successive approximations,consider the following conditions:

(i) $g \in C[a,b],\ f \in C([a,b] \times \mathbb{R} \times \mathbb{R})$

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(ii) there exist, $\alpha, \beta > 0$ such that

$$| f(s,u,v) - f(s,u',v') | \le \alpha | u - u' | + \beta | v - v' |$$

for all $s \in [a,b], (u,v), (u',v') \in \mathbb{R} \times \mathbb{R},$

(iii) $b - a < \dfrac{1}{(\alpha + \beta)}$

(iv) there exist $\gamma, \rho, > 0$ such that

$$| f(s,u,v) - f(s',u,v) | \le \gamma | s - s' |$$

$$| g(t) - g(t') | \le \rho | t - t' |$$

for all $t, s, t', s' \in [a,b], (u,v) \in \mathbb{R} \times \mathbb{R}.$

let $f_0 : [a,b] \to \mathbb{R},\ f_0(t,s) = f(t,s,x(s)).$ Since f is continuous we infer that f_0 is continuous on the compact set $[a,b]$ and there for exists $M_0 \ge 0$ such that $| f_0(t,s) | \le M_0$ for all $t, s \in [a,b] \times [a,b].$

Description of The Method:

Under the conditions (i)-(iii), applying the classical Picard-Banach' s fixed point technique, the existence and uniqueness of the solution of (2) is obtained. Let $x \in C[a,b]$ be the solution of (2) and the sequence of successive approximations

$$x_0(t) = g(t); \forall t \in [a;b]$$

$$x_m(t) = g(t) + \int_a^b f(t,s,x(s))ds, \forall t \in [a,b], m \in N \tag{3}$$

which uniformly converges to x.The following error estimations hold:

$$\left| x(t) - x_m(t) \right| \frac{(b-a)^m \alpha^m}{1 - \alpha(b-a)} M_0 (b-a), \forall t \in [a,b], m \in N \tag{4}$$

and

$$\left| x(t) - x_m(t) \right| \frac{(b-a)\alpha}{1 - \alpha(b-a)} .max \left| x_m(t) - x_{m-1}(t) \right|, \tag{5}$$

for all $t \in [a,b], m \in N$ For this purpose we use the trapezoidal quadrature rule with recent remainder estimation (which holds for Lipschitzian functions):

$$\int_a^b F(x)dx = \frac{(b-a)}{2n} . \sum_{i=1}^{n} [F(a + t_{i+1}) + F(a + t_i)] + R_n(f) \tag{6}$$

That $h = \dfrac{b-a}{n}, t_i = t_0 + ih$ and,

$$| R_n(f) | \le \frac{(b-a)^2 L}{4n} \tag{7}$$

where $L > 0$ is the Lipschitz constant of F (see[1]).

Consider the uniform partition of $[a,b]$:

$$\Delta : a = t_0 < t_1 < ... < t_{n-1} < t_n = b \tag{8}$$

with $t_i = a + \dfrac{i(b-a)}{n}, i = 0,, n$. Let $h = \dfrac{b-a}{n}$ On these knots, the relation (3) becomes:

$$x_m(t_i) = g(t_i) + \int_a^b f(t,s,x_{m-1}(s))ds, \quad i = 0,, n \tag{9}$$

Applying the quadrature rule $(6) - (7)$ to the relations (9) , we obtain the numerical method:

$$x_0(t_i) = g(t_i), \quad i = 0,,n$$
$$x_m(t_i) = g(t_i) + \int_a^b f(t_i, s, x_{m-1}(s))ds = g(t_i) + \int_a^b F_{m,i}(s)ds$$

$$= g(t_i) + \sum_{j=1}^{n}[f(t_i, t_{j-1}, x_{m-1}(t_{j-1})) + f(t_i, t_j, x_{m-1}(t_j))] + R_{m,i}.i = 0,,n, m \in \mathbb{N} \tag{10}$$

with the reminder estimation

$$|R_{m,i}| \frac{(b-a)^2 L}{4n}.\forall i = 0,,n, m \in \mathbb{N} \tag{11}$$

The Spline Functions:

Let Δ be a partition of an interval $[a,b]$, $\Delta : a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ and $s:[a,b] \to \mathbb{R}$ a cubic spline generated by initial conditions introduced by C. Iancu (see $[34]$) interpolating the values $f_i, i = 0,,n$ with natural boundary conditions $s''(a) = s''(b) = 0$. The restrictions s_i of this cubic spline s to the intervals $[t_{i-1}, t_i], i = \overline{1,n}$ are uniquely determined solving the initial value problems:

$$\begin{cases} s_i''(t) = M_{i-1} + \frac{1}{h_i}.(M_i + M_{i-1})(t - t_{i-1}), \quad x \in [t_{i-1}, t_i] \\ s_i(t_{i-1}) = f_{i-1} \\ s_i'(t_{i-1}) = m_{i-1} \end{cases}$$

getting,

$$s_i(t) = \frac{1}{6h_i}.(M_i - M_{i-1})(t - t_{i-1})^3 + \frac{M_{i-1}}{2}.(t - t_{i-1})^2 + m_{i-1}(t - t_{i-1}) + f_{i-1}. \tag{12}$$

Imposing the conditions $s_i(t_i) = f_i, i = \overline{1,n}$ we get

$$f_i = f_{i-1} + h_i m_{i-1} + \frac{(M_i - M_{i-1})h_i^2}{6} + \frac{h_i^2 M_{i-1}}{2}$$

and we can express the values $m_{i-1}, i = \overline{1,n}$:

$$m_{i-1} = \frac{f_i - f_{i-1}}{h_i} - \frac{h_i(M_i + 2M_{i-1})}{6}, i = \overline{1,n}$$

which can be introduced in (13) obtaining the restrictions of s to the intervals $[t_{i-1}, t_i], i = \overline{1,n}$:

$$s_i(t) = [\frac{(t - t_{i-1})^2}{2} - \frac{(t - t_{i-1})^3}{6h_i} - \frac{h_i(t - t_{i-1})}{3}].M_{i-1}$$

$$+[\frac{(t - t_{i-1})^3}{6h_i} - \frac{h_i(t - t_{i-1})}{6}].M_i + \frac{(t - t_{i-1})}{h_i}.f_i + \frac{(t_i - t)}{h_i}.f_{i-1} \tag{13}$$

where $h_i = t_i - t_{i-1}, i = \overline{1,n}$ and $m_i = s^{'}(t_i)$, $M_i = s^{''}(t_i)$, , $i = \overline{1,n}$. Since $s \in C^2[a,b]$, the requirement $s \in C^1[a,b]$ lead to the conditions $s^{'}s_i(t_i) = s^{'}s_{i+1}(t_i)$, $i = \overline{1,n-1}$ obtaining the following linear system:

$$\frac{h_i}{6}.M_{i-1} + \frac{h_i + h_{i+1}}{3}.M_i + \frac{h_{i+1}}{6}.M_{i+1} = \frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}, i = \overline{1,n-1} \tag{14}$$

to be solved for $M_i, i = \overline{0,n}$. The natural boundary conditions $s^{''}(a) = s^{''}(b) = 0$ can be written as $M_0 = M_n = 0$

An algorithm (inspired by the method presented in (Jerri, A.J.,), pages 14-15) in recurrent form which gives the solutions of the system (15) is the following:

Firstly, for $i = \overline{1,n-1}$ let $a_i = 2, b_i = c_i = \frac{1}{2}$ and

$$d_i = \frac{3(f_{i+1} - f_i)}{h_{i+1}^2} - \frac{3(f_i - f_{i-1})}{h_i^2}$$

Recurrently, it computes

$$\alpha_1 = \frac{c_1}{a_1}, \omega_i = a_i - \alpha_{i-1}.b_i, \quad \alpha_i = \frac{c_i}{\omega_i}, i = \overline{2,n-2}$$

$$\omega_{n-1} = a_{n-1} - \alpha_{n-2}.b_{n-1}$$

and

$$z_1 = \frac{d_1}{2}, z_i = \frac{d_i - b_i.z_{i-1}}{\omega_i}, i = \overline{2,n-1}. \tag{15}$$

Finally, with backward recurrence we obtain the moments:

$$M_{n-1} = z_{n-1}, M_i = z_i - \alpha_i.M_{i+1}, i = \overline{n-2,1}. \tag{16}$$

The error estimation in the uniform approximation of uniformly continuous functions by interpolating natural cubic splines is obtained:

Lemma 1 *Let* $h = max\{h_i : i = \overline{1,n}\} \bar{h} = min\{h_i : i = \overline{1,n}\}$ *and*

$\beta \geq 1 such that \frac{h}{\bar{h}} \leq \beta.$

If $f : [a,b] \to \mathbb{R}$ is uniformly continuous function and $s \in C^2[a,b]$ is cubic spline of interpolation generated by initial conditions, with natural boundary conditions $s^{''}(a) = s^{''}(b) = 0$, such that $s(t_i) = f(t_i) = f_i$, $\forall i = \overline{0,n}$, then the following error estimation holds:

$$max|s(t) - f(t)| \leq \frac{3\beta^2}{4}.\omega(f,\bar{h}) + \omega(f,h). \tag{17}$$

where $\omega(f,h) = sup\{|f(t) - f(t\ )| : t, t\ \in [a,b], |t - t\ | \leq h\}$ is the uniform modulus of continuity.

Proof. The system (15) with $M_0 = M_n = 0$ can be written in the diagonally dominant form:

$$\begin{cases} \frac{h_1+h_2}{3}.M_1+\frac{h_2}{6}.M_2=\frac{f_2-f_1}{h_2}-\frac{f_1-f_0}{h_1} \\ \frac{hi}{6}.M_{i-1}+\frac{h_i+h_{i+1}}{3}.M_i+\frac{h_{i+1}}{6}.M_{i+1}=\frac{f_{i+1}-f_i}{h_{i+1}}-\frac{f_i-f_{i-1}}{h_i}, i=\overline{2,n-2} \\ \frac{h_{n-1}}{6}.M_{n-2}+\frac{h_{n-1}+h_n}{3}.M_{n-1}=\frac{f_n-f_{n-1}}{h_n}-\frac{f_{n-1}-f_{n-2}}{h_{n-1}}. \end{cases}$$

and after division by $\frac{h_i+h_{i+1}}{3}$, $i=\overline{1,n-1}$ in each equation we obtain the following form $G.m=d$ of this system with $G=I+A$:

$$\begin{cases} M_1+\frac{h_2}{2(h_1+h_2)}.M_2=\frac{3(f_2-f_1)}{h_2(h_1+h_2)}-\frac{3(f_1-f_0)}{h_1(h_1+h_2)}=d_1 \\ \frac{hi}{2(h_i+h_{i+1})}.M_{i-1}+M_i+\frac{h_{i+1}}{2(h_i+h_{i+1})}.M_{i+1}=\frac{3(f_{i+1}-f_i)}{h_{i+1}(h_i+h_{i+1})}-\frac{3(f_i-f_{i-1})}{h_i(h_i+h_{i+1})}=d_i, i=\overline{2,n-2} \\ \frac{h_{n-1}}{2(h_{n-1}+h_n)}.M_{n-2}+M_{n-1}=\frac{3(f_n-f_{n-1})}{h_n(h_{n-1}+h_n)}-\frac{3(f_{n-1}-f_{n-2})}{h_{n-1}(h_{n-1})+h_n}=d_{n-1}. \end{cases}$$

Since $\|A\|_\infty=\frac{1}{2}1$ we infer that the matrix $I+A$ is invertible with

$$\|G^{-1}\|_\infty=\|(I+A)^{-1}\|_\infty\le\frac{1}{1-\|A\|_\infty}=2$$

and $m=G^{-1}.d$, where $m=(M_1,,M_{n-1}), d=(d_1,,d_{n-1})$. It is easy to see that

$$\|d\|_\infty=max\{|d_1|,,|d_{n-1}|\}\le\frac{3\omega(f,h)}{\bar{h}^2}$$

and

$$\|m\|_\infty=max\{|M_1|,,|M_{n-1}|\}\le\|G^{-1}\|_\infty.\|d\|_\infty\le\frac{6\omega(f,h)}{\bar{h}^2}.$$

Since

$$\frac{(t-t_{i-1})^2}{2}-\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{3}\le 0$$

and

$$\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{6}\le 0$$

for all $t\in[t_{i-1},t_i]$, $i=\overline{1,n}$ we get

$$|\frac{(t-t_{i-1})^2}{2}-\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{3}|+|\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{6}|$$

$$=\frac{h_i(t-t_{i-1})}{2}-\frac{(t-t_{i-1})^2}{2}=\frac{(t-t_{i-1})(t_i-t)}{2}.$$

Then,

$$|s(t)-f(t)|\le|M_{i-1}|.|\frac{(t-t_{i-1})^2}{2}-\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{3}|$$

$$+|M_i|.|\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{6}|+|\frac{t-t_{i-1}}{h_i}.f_i+\frac{t_i-t}{h_i}.f_{i-1}-f(t)|$$

$$\leq \|m\|_\infty(|\frac{(t-t_{i-1})^2}{2}-\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{3}|+|\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{6}|)$$

$$+\frac{t-t_{i-1}}{h_i}.\omega(f,h_i)+\frac{t_i-t}{h_i}.\omega(f,h_i)\leq\frac{6\omega(f,h)}{h^2}.\frac{(t-t_{i-1})(t_i-t)}{2}+\omega(f,h)$$

for any $t\in[t_{i-1},t_i], i=\overline{1,n}.$ So,

$$|s(t)-f(t)|\leq\frac{6\omega(f,h)}{h^2}.\frac{h^2}{8}+\omega(f,h)\leq\frac{3\beta^2}{4}.\omega(f,h)+\omega(f,h)$$

for any $t\in[t_{i-1},t_i], i=\overline{1,n}.$

We see that for uniform partition Δ, it follows that $\overline{h}=h=\frac{b-a}{n}$ and $\beta=1$, obtaining

$$max|s(t)-f(t)|\leq\frac{7\omega(f,h)}{4}. \tag{18}$$

The Numerical Algorithm:

The relations (11) lead to the following algorithm:

$$x_0(t_i)=g(t_i).\quad i=\overline{0,n} \tag{19}$$

$$x_1(t_i)=g(t_i)+\frac{(b-a)}{2n}\sum_{j=1}^{n}[f(t_i,t_{j-1},g(t_{j-1}))+f(t_i,t_j,g(t_j))]+R_{1,i}$$

$$=\overline{x_1(t_i)}+R_{1,i}, i=\overline{0,n} \tag{20}$$

$$x_2(t_i)=g(t_i)+\frac{(b-a)}{2n}\sum_{j=1}^{n}[f(t_i,t_{j-1},\overline{x_1(t_{j-1})}+R_{1,j-1})$$

$$+f(t_i,t_j,\overline{x_1(t_j)}+R_{1,j})]+R_{2,j}=g(t_i)+\frac{(b-a)}{2n}.$$

.

$$\sum_{j=1}^{n}[f(t_i,t_{j-1},s_1(t_{j-1})+f(t_i,t_j,s_1(t_j)+R_{1,j})]+\overline{R_{2,i}}=\overline{x_2(t_i)}+\overline{R_{2,i}},\quad i=\overline{0,n} \tag{21}$$

where $s_1:[a,b]\to\mathbb{R}$, is cubic spline of interpolation with natural boundary conditions inspired from relation (7) which interpolates the values $\overline{x_1(t_i)}$, $i=\overline{0,n}$ and has the restrictions to the intervals $[t_{i-1},t_i]$, $i=\overline{1,n}$:

$$s_1^{(i)}(t)=[\frac{(t-t_{i-1})^2}{2}-\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{3}].M_1^{(i-1)}$$

$$+[\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{6}].M_1^{(i)}+\frac{t-t_{i-1}}{h_i}.\overline{x_1(t_i)}+\frac{t_i-t}{h_i}.\overline{x_1(t_{i-1})}$$

where $M_1^{(0)}=M_1^{(n)}=0$ and $M_1^{(i)}$ $i=\overline{1,n-1}$ are given in recurrent way by:

$$a_i=2, b_i=c_i=\frac{1}{2}, d_i=\frac{3}{h^2}.[\overline{x_1(t_{i+1})}+\overline{2x_1(t_i)}-\overline{x_1(t_{i-1})}], i=\overline{1,n-1}$$

and

$$\alpha_1=\frac{c_1}{a_1}, \omega_i=a_i-\alpha_{i-1}.b_i, \quad \alpha_i=\frac{c_i}{\omega_i}, i=\overline{2,n-2}$$

$$\omega_{n-1}=a_{n-1}-\alpha_{n-2}.b_{n-1}$$

$$z_1=\frac{d_1}{2}, z_i=\frac{d_i-b_i.z_{i-1}}{\omega_i}, i=\overline{2,n-1}.$$

With backward recurrence,

$$M_1^{(n-1)}=z_{n-1}, M_1^{(i)}=z_i-\alpha_i.M_1^{(i+1)}, \bar{i}=n-2,1$$

By induction for $m\geq 3$ we obtain:

$$x_m(t_i)=g(t_i)+\frac{(b-a)}{2n}\sum_{j=1}^{n}[f(t_i,t_{j-1},\overline{x_{m-1}(t_{j-1})}+R_{m-1,j-1})$$

$$+f(t_i,t_j,\overline{x_m-1(t_j)}+R_{m-1,j})]+R_{m,j}=g(t_i)$$

$$+\frac{(b-a)}{2n}$$

$$\sum_{j=1}^{n}[f(t_i,t_{j-1},s_{m-1}(t_{j-1}))+f(t_i,t_j,s_{m-1}(t_j))]+\overline{R_{m,i}}=\overline{x_m(t_i)}+\overline{R_{m,i}} \quad .i=\overline{0,n} \tag{22}$$

where $s_{m-1}:[0,a]\to\mathbb{R}$ is the natural cubic spline of interpolation as in (7), interpolating the values $x_{m-1}(t_i); i=\overline{0,n}$ and having the restrictions to the intervals $[t_{i-1},t_i]$, $i=\overline{1,n}$:

$$s_{m-1}^{(i)}(t)=[\frac{(t-t_{i-1})^2}{2}-\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{3}].M_{m-1}^{(i-1)} \tag{23}$$

$$+[\frac{(t-t_{i-1})^3}{6h_i}-\frac{h_i(t-t_{i-1})}{6}].M_{m-1}^{(i)}+\frac{t-t_{i-1}}{h_i}.\overline{x_{m-1}(t_i)}+\frac{t_i-t}{h_i}.\overline{x_{m-1}(t_{i-1})}$$

where $M_{m-1}^{(0)}=M_{m-1}^{(n)}=0$ and $M_{m-1}^{(i)}, i=\overline{1,n-1}$ are recurrently given by:

$$a_i=2, b_i=c_i=\frac{1}{2}, d_i=\frac{3}{h^2}.[\overline{x_{m-1}(t_{i+1})}+\overline{2x_{m-1}(t_i)}-\overline{x_{m-1}(t_{i-1})}], i=\overline{1,n-1}$$

and

$$\alpha_1=\frac{c_1}{a_1}, \omega_i=a_i-\alpha_{i-1}.b_i, \quad \alpha_i=\frac{c_i}{\omega_i}, i=\overline{2,n-2}$$

$$\omega_{n-1} = a_{n-1} - \alpha_{n-2}.b_{n-1}$$
$$z_1 = \frac{d_1}{2}, z_i = \frac{d_i - b_i.z_{i-1}}{\omega_i}, i = \overline{2, n-1}.$$

Using the backward recurrence it follows that,

$$M_{m-1}^{(n-1)} = z_{n-1}, M_{m-1}^{(i)} = z_i - \alpha_i.M_{m-1}^{(i+1)}, i = \overline{n-2,1}$$

Numerical Examples:

In this section, numerical examples of the nonlinear Urysohn integral equations (1) are considered to show the accuracy of presented method. In this study, all examples are solved using the method stated in Section 4. All calculations are performed using Maple 14. **Example 1.** As the first example consider the following integral equation

$$x(t) = g(t) + \int_0^1 [(s+t).(x(s))^2]ds, \quad t \in [0,1]$$

where the function g(t) is given by:

$$g(t) = t^2 - \frac{8}{15}t - \frac{7}{6}$$

has exact solution $x^*(t) = t^2 - 1$

Applying the presented method for example 1 we get iterations and the results in Table 1. The order of error is $O(10^{-2})$. The first column represents the knots. The second columns contains the values of the exact solution on these knots. In the third column are approximations on the knots at the last iteration. The fourth column contains the errors.

Example 2. For the second equation, Let

$$x(t) = g(t) + \int_0^1 \frac{ds}{t+s+x(s)}, \quad t \in [0,1]$$

where g is chosen that

$$x^*(t) = \frac{1}{t+1}, c > 0$$

is a solution of above equation (Atkinson, K., 1992). The results are given in Table 2.

Table 1: The results for example(1)

t_i	$x^*(t_i)$	$\overline{x_{21}(t_i)}$	e_i
0	-1	-0.952138	0.048
0.25	-0.9375	-0.938531	0.00103
0.5	-0.75	-0.799923	0.049923
0.75	-0.4375	-0.536316	0.098816
1	0	-0.147709	0.147709

Table 2: The results for example(2)

t_i	$x^*(t_i)$	$\overline{x_{21}(t_i)}$	e_i
0	-1	-1.02463	0.02463
0.25	-0.9375	-0.973987	0.036487
0.5	-0.75	-0.799923	0.049923
0.75	-0.4375	-0.497707	0.060207
1	0	-0.0720667	0.0720667

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