Australian Journal of Basic and Applied Sciences, 5(11): 602-606, 2011
ISSN 1991-8178

Convergence of Ishikawa Iterative Sequances for Lipschitzian Strongly Pseudocontractive Operator

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Abstract: The Main result of this paper is to show that the Ishikawa iteration Schemes with mixed errors for $\phi - strongly$ pseudocontractive Lipschitzian operators, with parameter between zero and one, in an arbitrary Banach space is convergent.

Key words: Ishikawa iteration sequences with mixed errors, strongly pseudocon-tractive operator.

INTRODUCTION

Throughout this paper X is Banach space with real field and X^* stands for its dual space, $<.,.>$ will denote for generalized duality pairing between X and X^*.we assume that the mapping $J: X \longrightarrow 2^{X^*}$ defined by

$$J(x) = \{f \in X^* : < x, f >= \|x\| \, \|f\|, \ \|x\| = \|f\|\}, \qquad \forall x \in X,$$

is the normalized duality mapping. By Hahn-Banach theorem $J(x) \neq \emptyset$ for each $x \in X$. Let $T: D \to X$ be an operator, where D is a subset of X as domain of T with range $R(T) \subseteq X$.

(1) T is called Lipschitzian if there is $L \geq 0$ such that for each $x, y \in D$
$\|Tx - Ty\| \leq L \, \|x - y\|$.

(2) T is called strongly pseudocontractive if for each $x, y \in D$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2 \tag{1.1}$$

for some constant $k \in (0,1)$. The T is called $\phi - strongly\, pseudocontractive$ if there exists a strictly increasing function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that the inequality

$$\langle Tx - Ty, j(x-y)\rangle \geq \|x-y\|^2 - \phi(\|x-y\|)\, \|x-y\|$$

holds for all $x, y \in D$.

(3) T is called strongly accretive if for given $x, y \in D$ there is $j(x-y) \in J(x-y)$ and a constant $k \in (0,1)$ such that
$\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2$.
and T is $\phi - strongly\, accretive$ if for any $x, y \in D$ there exists $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that
$\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\, \|x-y\|$.

Definition 1.1: Let $T: X \longrightarrow X$ be an operator, x_0 and u_0 be two given points in X, $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequances in $[0,1]$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ be two sequences in X satisfying the following conditions:
$(i)\ v_n = v_n' + v_n'',\ \|v_n'\| = o(\alpha_n)\ (n \geq 0)$ and $\sum_{n=0}^{\infty}\|v_n''\| < \infty$,
$(ii)\ \lim_{n\to\infty}\|w_n\| = 0$.
Then

(1) The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

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$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, & n \geq 0, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, & n \geq 0, \end{cases}$$

is called the Ishikawa iterative sequence (Ishikawa, 1974; Kim, J.K., 2006).

(1) Let D be a nonempty convex subset of X and $T: D \longrightarrow D$ be an operator. For any given the $x_0, u_0, v_0 \in D$, process defined by

$$\begin{cases} x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n r_n, & n \geq 0, \\ y_n = \alpha'_n x_n + \beta'_n T x_n + \gamma'_n s_n, & n \geq 0, \end{cases}$$

where $\{s_n\}_{n=0}^{\infty}$, $\{r_n\}_{n=0}^{\infty}$ are bounded sequences in D and the real sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{\alpha'_n\}_{n=0}^{\infty}$, $\{\beta'_n\}_{n=0}^{\infty}$, $\{\gamma'_n\}_{n=0}^{\infty} \subset [0,1]$ satisfy the conditions $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, $\forall n \geq 0$, called the Ishikawa iterative sequence with errors (Chidume, C., 2009).

(2) The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + v_n, & n \geq 0, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + w_n, & n \geq 0, \end{cases}$$

is called the Ishikawa iterative sequence with mixed errors (Kim, J. K., 2006; Liu, L.S., 1995).

It is shown in (Osilike, M.O., 1996) that the class of strongly pseudocontractive mapping is a proper subclass of $\phi - strongly$ pseudo contractive mapping. Furthermore, the example in (Chidume, C.E. and Osilike, M.O., 1994) shows that the class of $\phi - strongly\ pseudocontractive$ mapping with the nonempty fixed point set is a proper subclass of $\phi - hemicontractive$ mapping (Zhiqun, X., 2005).

In the present paper we show that the Ishikawa iteration schemes wih mixed errors for $\phi - s$trongly pseudocontractive Lipschitzian operators, in an arbitrary real Banach space is convergent.

Preliminarise:

Lemma 2.1:

(Liu, Z., 2008) Suppose that $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\omega_n\}_{n=0}^{\infty}$ are nonnegative sequences such that

$$\alpha_{n+1} \leq (1-\omega_n)\alpha_n + \beta_n \omega_n + \gamma_n, \qquad \forall n \geq 0$$

with $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum_{n=0}^{\infty} \omega_n = \infty$, $\sum_{n=0}^{\infty} \gamma_n < \infty$ and $\lim_{n\to\infty} \beta_n = 0$. Then $\lim_{n\to\infty} \alpha_n = 0$.

The Main Result:

Theorem 3.1.

Let X be a real Banach space and *D* a nonempty, closed convex subset of X and $T: D \longrightarrow D$ a uniformly continuous Lipschitzian operator with a Lipschitz constant *L=1* and ϕ-strongly pseudo contractive. Suppose $F(T) \neq \emptyset$ and $x_0 \in D$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iterative sequence with mixed errors defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + v_n, & n \geq 0, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + w_n, & n \geq 0, \end{cases} \tag{3.1}$$

where $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ are two sequences in D and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are two real sequences in $[0,1]$ satisfying the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii)(a) $v_n = v'_n + v''_n$, $\|v'_n\| = o(\alpha_n)$ $(n \geq 0)$ and $\sum_{n=0}^{\infty} \|v''_n\| < \infty$,

(b) $\lim_{n\to\infty} \|w_n\| = 0$.

If

$$\lim_{n\to\infty} \|Tx_{n+1} - Ty_n\| = 0,$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a unique fixed point of T in D.

Proof. From (1.1) follows that $F(T)$ is a singleton, say $F(T) = \{p\}$. Thus

$$x_{n+1} - p = (1-\alpha_n)x_n + \alpha_n T y_n + v_n - p + \alpha_n p - \alpha_n p$$
$$= (1-\alpha_n)(x_n - p) + \alpha_n (T y_n - p) + v_n.$$

which implies that

$$\|x_{n+1}-p\| \le (1-\alpha_n)\|x_n-p\| + \alpha_n\|Ty_n - Tp\| + \|v_n\|. \qquad (3.2)$$

Since $v_n = v_n' + v_n''$ and $\|v_n'\| = o(\alpha_n)$, there exists a nonnegative sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ with $\varepsilon_n \longrightarrow 0$ such that $\|v_n'\| = \varepsilon_n \alpha_n$ and so

$$\|v_n\| \le \varepsilon_n\alpha_n + \left\|v_n''\right\| \qquad (n \ge 0) \qquad (3.3)$$

Since T is Lipschitzian, it follow from (3.2) and (3.3) that

$$\|x_{n+1}-p\| \le (1-\alpha_n)\|x_n-p\| + \alpha_n(\|Ty_n - Tx_n\| + \|Tx_n - Tp\|) + \alpha_n\varepsilon_n + \left\|v_n''\right\|$$
$$\le (1-\alpha_n)\|x_n-p\| + \alpha_n\|Ty_n - Tx_n\| + \alpha_n L\|x_n-p\| + \alpha_n\varepsilon_n + \left\|v_n''\right\|$$
$$= (1-(1-L)\alpha_n)\|x_n-p\| + (1-L)\alpha_n\frac{\|Ty_n-Tx_n\|}{1-L} + (1-L)\alpha_n\frac{\varepsilon_n}{1-L} + \left\|v_n''\right\|$$
$$= (1-(1-L)\alpha_n)\|x_n-p\| + (1-L)\alpha_n[\frac{\|Ty_n-Tx_n\|}{1-L} + \frac{\varepsilon_n}{1-L}] + \left\|v_n''\right\|.$$

Therefore it follows from lemma 2.1 that $\lim_{n\to\infty}\|x_n-p\| = 0$.Hence the Ishikawa iterative sequence $\{x_n\}_{n=0}^{\infty}$ with mixed errors defined by (3.1) converges strongly to a unique fixed point p of T in D.

In (3.1), if $v_n = w_n = 0 \ (n \ge 0)$, then:

Theorem 3.2:

Let X be a real Banach space and D a nonempty, convex subset of X and $T: D \longrightarrow D$ a co- ntinuous Lipschitzian operator with a Lipschitz constant *L=1* and $\phi - strongly\ pseudocontractive$. Suppose $F(T) \neq \emptyset$ and $x_0 \in D$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$ be Ishikawa iterative sequence defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, & n \ge 0, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, & n \ge 0, \end{cases} \qquad (3.4)$$

If

$$\lim_{n\to\infty}\|Tx_{n+1} - Ty_n\| = 0,$$

then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (3.4) converges strongly to a unique fixed point of T in D.

Theorem 3.3:

Let X be a real Banach space, D a nonempty, convex subset of X and $T: D \longrightarrow D$ a cont- inuous Lipschitzian operator with a Lipschitz constant $L < 1$ and $\phi - strongly\ accrative$ operator. Define $G: D \longrightarrow D$ by $Gx = f - Tx$ for all $x \in X$ and for given $f \in D$. Let $x_0 \in D$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iterative sequence with mixed errors defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n G y_n + v_n, & n \ge 0, \\ y_n = (1-\beta_n)x_n + \beta_n G x_n + w_n, & n \ge 0, \end{cases}$$

where $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ are two sequences in D and $\{\alpha_n\}_{n=0}^{\infty}$,$\{\beta_n\}_{n=0}^{\infty}$ are two real sequences in $[0,1]$ satisfying the following conditions:

$(i)\ \sum_{n=0}^{\infty}\alpha_n = \infty,$

(ii) (a) $v_n = v_n' + v_n'',\ \|v_n'\| = o(\alpha_n)\ \ (n \ge 0)$ and $\sum_{n=0}^{\infty}\left\|v_n''\right\| < \infty,$

(b) $\lim_{n\to\infty}\|w_n\| = 0.$

If

$$\lim_{n\to\infty}\|Tx_{n+1} - Ty_n\| = 0,$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a unique fixed point of G in D.

Proof: Since $T: D \longrightarrow D$ is a $\phi - strongly\, accrative$ operator, for any $x, y \in D$ there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi: [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that
$\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\,\|x-y\|,$
so that
$\langle Gx - Gy, j(x-y)\rangle = \langle f - Tx - f + Ty, j(x-y)\rangle$
$= -\langle Tx - Ty, j(x-y)\rangle$
$\leq -\phi(\|x-y\|)\|x-y\|$
$\leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|.$

Therefore G is $\phi - strongly\, pseudocontractive$. On the other hand, G is continuous Lipschitzian ope- rator with a Lipschitz constant $L < 1$ and
$\lim_{n\to\infty} \|Gx_{n+1} - Gy_n\| = \lim_{n\to\infty} \|Tx_{n+1} - Ty_n\| = 0.$

Therefore conclusion of theorem (3.3) can be obtained from Theorem 2.1 immediately. This completes the proof.

Theorem 3.4:

Let X be a real Banach space, D a nonempty, convex subset of X and the operator $T: D \to D$ be a continuous Lipschitzian operator with a Lipschitz constant $L < 1$ and ϕ −strongly pseudocontractive. Define $G: D \to D$ by $Gx = f - Tx$ for all $x \in X$ and for given $f \in D$. Let $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{\beta_n'\}_{n=0}^{\infty}$ and $\{\gamma_n'\}_{n=0}^{\infty}$ be four real sequences in $[0,1]$ that

(i) $\beta_n + \gamma_n \leq 1,\ \ \beta_n' + \gamma_n' \leq 1,$

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty,$

(iii) $\sum_{n=0}^{\infty} \gamma_n < \infty,\ \ \gamma_n' \longrightarrow 0.$

Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iterative sequence with errors defined by

$$\begin{cases} x_{n+1} = (1 - \beta_n - \gamma_n)x_n + \beta_n G y_n + \gamma_n r_n, & n \geq 0 \\ y_n = (1 - \beta_n' - \gamma_n')x_n + \beta_n' G x_n + \gamma_n' s_n, & n \geq 0 \end{cases} \tag{3.5}$$

If $\{x_n\}_{n=0}^{\infty}$ be bounded and

$\lim_{n\to\infty} \|Tx_{n+1} - Ty_n\| = 0,$

then G has a unique fixed point $p \in D$ and the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to p.

Proof. From (3.5) follows:

$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_n G y_n + \gamma_n (r_n - x_n), & n \geq 0 \\ y_n = (1 - \beta_n')x_n + \beta_n' G x_n + \gamma_n' (s_n - x_n), & n \geq 0 \end{cases}$$

Let $v_n = \gamma_n (r_n - x_n)$ and $w_n = \gamma_n' (s_n - x_n)$, $\forall n \geq 0$. Then

$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_n G y_n + \gamma_n v_n, & n \geq 0 \\ y_n = (1 - \beta_n')x_n + \beta_n' G x_n + \gamma_n' w_n, & n \geq 0 \end{cases}$$

by the sequences $\{x_n\}_{n=0}^{\infty}$, $\{r_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ are bounded in D and by condition (iii),
$\sum_{n=0}^{\infty} \|v_n\| < \infty$ and $\lim_{n\to\infty} \|w_n\| = 0,$

Which imply that $\{x_n\}_{n=0}^{\infty}$ defined by (3.5) is a special case on the Ishikawa iteration sequence with mixed errors defined by (3.1). Therefore all the conditions in theorem 3.1 are satiafied. This completes the proof.

ACKNOWLEDGMENT

The first author was supported by the research council of Alzahra University.

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