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Convergence and Oscillation in a Rational Fourth Order Difference Equation

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Abstract: In this paper the dynamics of the following rational fourth-order difference equation

$$x_n = \alpha + \beta \frac{x_{n-3}}{x_{n-4}}, \quad n = 0,1,\ldots$$

is studied where the initial conditions x_{-4}, x_{-3}, x_{-2}, x_{-1} together with the param-eters α and β are positive real numbers. We discuss some asymptotic behavior of this equation including asymptotic stability, semicycles, and oscillations.

Key words: Difference equation; equilibrium; stability; oscillation; semicycle.

Introduction and Preliminaries:

Consider the following k-th order difference equation

$$x_n = f(x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \quad (1.1)$$

where k≥1, f∈C $[R^k, R]$, and initial conditions $x_{-k}, \ldots, x_{-1}$ are given real numbers. This equation has received increasing attention in the last two decades, for example see (Abu-saris, 2003;Amleh, 1999; Devault, 2003; Devault, 2001; Kuruklis, 1994; Sedaghat, 2003; Sedaghat, 2007; Shojaei, 2009). However, the basic theory of global behavior of its solutions when k>1 is still in its infancy. There are a lot of works about Eq. (1.1) when k = 2 and f is a rational function (for example see the monograph of Kulenovic and Ladass (2002). Also, there are some works about Eq. (1.1) in the following form:

$$x_n = f(x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots$$

where $k > 1$ and initial conditions $x_{-k}, \ldots, x_{-1}$ are given (for example see (Abu-saris, 2003; Amleh, 1999; Devault, 2003; Devault, 2001; Kuruklis, 1994).

On the other hand there are less works about Eq. (1.1) in the following form

$$x_n = f(x_{n-l}, x_{n-k}),$$

where $1 < l < k$. In this respect we examine the global behavior of solutions of the following rational difference equation

$$x_n = f(x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \quad (1.2)$$

where l = 3, k = 4 and initial conditions x_{-4}, x_{-3}, x_{-2}, x_{-1} together with the parameters α and β are all positive real numbers. Our objective in this paper is to study the asymptotic behavior of the solutions of Eq. (1.2) such as convergence, semicycle analysis, and oscillation rather than obtaining an explicit formula for its solutions.

Let I be some interval of positive real numbers. For initial conditions $x_{-k}, \ldots, x_{-1}$ Eq. (1.1) has a unique solution $\{x_n\}^{\infty}_{n=-k}$. The point $\bar{x} \in I$ is called an equilibrium point (or simply an equilibrium) of Eq. (1.1) if $\bar{x}$ $(\bar{x}, \ldots, \bar{x})$.

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The solution $x_n = \bar{x}$ for all $n \geq -k$ is called a trivial solution. The linearized equation associated with Eq. (1.1) about the equilibrium $\bar{x}$ is

$$z_n = \sum_{i=1}^{k} \frac{\partial f}{\partial u_i}(\bar{x}, \dots, \bar{x}) z_{n-i} \,,$$

and its characteristic equation is

$$\lambda^k - \sum_{i=1}^{k} \frac{\partial f}{\partial u_i}(\bar{x}, \dots, \bar{x}) \lambda^{k-i} = 0. \tag{1.3}$$

Definition 1.1:

Let $\bar{x}$ be an equilibrium of Eq. (1.1). Then $\bar{x}$ is said to be

I. Locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for the solution $\{x_n\}^{\infty}_{n=-k}$ of the Eq. (1.1) with $x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_{-1} - \bar{x}| < \delta$ we have $|x_{-1} - \bar{x}| < \epsilon$ for all $n \geq 0$.

II. A global attractor on an interval I if for every solution $\{x_n\}^{\infty}_{n=-k}$ of Eq. (1.1) with $x_{-k}, \dots, x_{-1} \; \varepsilon \; I$ we have $\lim_{n\to\infty} x_n = \bar{x}$.

III. Locally asymptotically stable if it is locally stable and in addition there exists $\gamma > 0$ such that for the solution $\{x_n\}^{\infty}_{n=-k}$ of Eq. (1.1) with we have $x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_{-1} - \bar{x}| < \gamma$ we have $\lim_{n\to\infty} x_n = \bar{x}$

IV. Globally asymptotically stable if it is locally stable and a global attractor.

V. Unstable if it is not stable.

Definition 1.2:

Let $\{x_n\}^{\infty}_{n=-k}$ be a solution of Eq. (1.1).

A. A positive semicycle $\{x_n\}^{\infty}_{n=-k}$ contains a string of consequtive values $x_l, x_{l+1}, \dots, x_m$ all greater than or equal to the equilibrium x with $l \geq -k$ and $m \leq \infty$ such that

either $l > -k$ and $x_{l-1} < \bar{x}$ or $l = -k$,

and

either $m < \infty$ *and* $x_{m+1} < \bar{x}$ *or* $m = \infty$.

B. A positive semicycle $\{x_n\}^{\infty}_{n=-k}$ contains a string of consequtive values $x_l, x_{l+1}, \dots, x_m$ all less than the equilibrium x^{-} with $l \geq -k$ and $m \leq \infty$ such that

either $l > -k$ and $x_{l-1} < \bar{x}$ or $l = -k$,

and

either $m < \infty$ *and* $x_{m+1} < \bar{x}$ *or* $m = \infty$.

C. $\{x_n\}^{\infty}_{n=-k}$ is oscillatory about $\bar{x}$ if for every $N \geq -k$ there exist integers $m,n \geq N$ such that $x_m < \bar{x}$ and $x_n > \bar{x}$.
The following theorems are necessary for our analysis in the next sections.

Theorem 1.1:
(Kocic, 1993), Linearized Stability Theorem] Let $\bar{x}$ be an equilibrium of Eq. (1.1).

I. If all roots of Eq. (1.3) lie inside the open unit disk $|\lambda|<1$, then the equilibrium $\bar{x}$ is locally asymptotically stable.
II. If at least one of the roots of Eq. (1.3) has absolute value greater than one, then the equilibrium

equilibrium $\bar{x}$ x is unstable.

Theorem 1.2:
(Grove, 2005; Kulenovic, 2002) Consider the following fourth-degree polynomial equation

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda a_0 = 0 \tag{1.4}$$

where a_0, a_1, a_2, a_3, a_4 are real numbers. The following conditions are necessary and sufficient conditions for all roots of Eq. (1.4) to lie inside the open unit disk $|\lambda|<1$

1. $|a_1 + a_3| < 1 + a_0 + a_2$

2. $|a_1 - a_3| < 2(1-a_0)$

3. $a_2 - 3_{a0} < 3$

4. $\alpha_0 + \alpha_2 + \alpha_0^2 + \alpha_1^2 + \alpha_0^2\,\alpha_2 + \alpha_0\,\alpha_3^2 < 1 + 2\alpha_0\,\alpha_2 + \alpha_1\,\alpha_3 + \alpha_0\,\alpha_1\,\alpha_3 + \alpha_0^3$

2. Asymptotic Stability:

The change of variables $x_n = \beta_{yn}$ followed by the change $y_n = x_n$ reduces Eq. (1.2) to

$$x_n = A + \frac{x_{n-3}}{x_{n-4}}, \quad n = 0,1,\ldots \tag{2.1}$$

where $A = \frac{\alpha}{\beta} > 0$. Hereafter, we focus our attention on Eq. (2.1) instead of Eq. (1.2). Clearly $\bar{x} = A + 1$.

Simple calculations show that the linearized equation associated with Eq. (2.1) about $\bar{x}$ is

$$z_n = \frac{1}{A+1} z_{n-3} + \frac{1}{A+1} z_{n-4}, \quad n = 0,1,\ldots$$

and the corresponding characteristic equation is

$$\lambda^4 - \frac{1}{A+1}\lambda + \frac{1}{A+1} = 0. \tag{2.2}$$

Lemma 2.1:
Consider the following cubic polynomial

$$x^3 + 2x^2 - x - 1. \tag{2.3}$$

a. Eq. (2.3) has a unique positive root γ and $\gamma < 1$.

b. The unique equilibrium $\bar{x}$ of Eq. (2.1) is locally asymptotically stable if $A > \gamma$ and is unstable if $A \leq \gamma$.

Proof. (a):

Define $p(x) = x^3 + 2x^2 - x - 1$. Note that p has a unique positive extremum (minimum) at $\frac{-2+\sqrt{7}}{3}$.

So p is decreasing on the interval $(0, \frac{-2+\sqrt{7}}{3})$ and is increasing on the interval $(\frac{-2+\sqrt{7}}{3}, \infty)$. On the other hand $p(0)$ (and therefore $p((-2+\sqrt{7})/3))$ is negative and $p(1) > 0$. Therefore, using intermediate value theorem p has a unique positive root which should be in the interval $(\frac{-2+\sqrt{7}}{3}, 1)$

Now we proceed to (b). Consider the characteristic equation (2.2). Let $a_0 = 1/(A+1)$, $a_1 = a - 1/(A+1)$, $a_2 = a_3 = 0$. Simple calculations show that conditions one and three in Theorem 1.2 hold, condition two is equivalent to $A > 1/2$, and finally condition four is equivalent to $p(A) > 0$ or equivalently $A > \gamma$. Also $p(½) < 0$ and the Intermediate value Theorem imply that $\gamma > 1/2$. Therefore, for $A > \gamma$, all conditions in Theorem 1.2 hold and by Theorem 1.1(i) the equilibrium is locally asymptotically stable. When $A \leq \gamma$, at least one of the conditions in Theorem 1.2 does not hold and therefore by Theorem 1.1(ii) the equilibrium $\bar{x}$ is unstable. The proof is complete.

Theorem 2.1:

For $A > 1$, the unique equilibrium $\bar{x} = A + 1$ of Eq. (2.1) is globally asymptotically stable.

Proof:

Since $A > 1 > \gamma$, by Lemma 2.1 (b) the equilibrium $\bar{x} = A + 1$ of Eq. (2.1) is locally stable. It remains to verify that $\bar{x}$ is a global attractor of all the solutions of Eq. (2.1). Let the sequence $\{x_n\}$ be a solution of Eq. (2.1) and

$L = \liminf_{n\to\infty} x_n \geq A > 0$, $\quad U = \limsup_{n\to\infty} x_n$

Then, by Eq. (2.1) we can write

$U \leq A + \frac{U}{L}$ and $L \geq A + \frac{L}{U}$

Therefore

$AU + L \leq LU \leq AL + U$

and hence $U \leq L$. This together with the fact that $L \leq U$ yields $L = U$. The proof is complete.

Lemma 2.2:

Consider the subsequences $\{x_{3n}\}$, $\{x_{3n+1}\}$, and $\{x_{3n+2}\}$ of the sequence $\{x_n\}^{\infty}_{n=-4}$. It is not possible for two of these subsequences to simultaneously converge to ∞.

Proof:

For the sake of contradiction, suppose that two of subsequences $\{x_{3n}\}$, $\{x_{3n+1}\}$, and $\{x_{3n+2}\}$ diverge to ∞. Assume, without loss of generality, that

$x_{3n} \to \infty$, $x_{3n+1} \to \infty$ as $n \to \infty$ (2.4)

The proof of other cases is similar and will be omitted. Consider the sequences $\left\{\frac{x_{3n+1}}{x_{3n}}\right\}$ and $\left\{\frac{x_{3n}}{x_{3n-1}}\right\}$ By (2.4) and Eq. (2.1) these sequences converge to 1. Thus for every $M > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\frac{x_{3n}}{x_{3n-1}} > M \quad and \quad \frac{x_{3n+1}}{x_{3n}} > M. \tag{2.5}$$

Choose $M > 1$. Now consider the polynomial $p(M) = M^2 + (A-1)M - A$. It is evident that $p(M) > 0$ or equivalently $\frac{A}{A+M-1} < M$. So by (2.5) we have

$$\frac{x_{3n+1}}{x_{3n}} > \frac{A}{A+M-1} \tag{2.6}$$

Using (2.5) and (2.6), we obtain for all $n \geq N$ that

$$\frac{x_{3n+4}}{x_{3n+3}} = \frac{A + \frac{x_{3n+1}}{x_{3n}}}{A + \frac{x_{3n}}{x_{3n-1}}} < \frac{A}{A+M} + \frac{1}{A+M} \cdot \frac{x_{3n+1}}{x_{3n}} = \frac{A}{A+M} + \frac{1}{A+M} \cdot \frac{x_{3n+1}}{x_{3n}} - \frac{x_{3n+1}}{x_{3n}} + \frac{x_{3n+1}}{x_{3n}}$$

$$= \frac{A}{A+M} + \left(\frac{1}{A+M} - 1\right) \frac{x_{3n+1}}{x_{3n}} + \frac{x_{3n+1}}{x_{3n}} < \frac{A}{A+M} + \frac{1-A-M}{A+M} \cdot \frac{A}{A+M-1} + \frac{x_{3n+1}}{x_{3n}} = \frac{A}{A+M} - \frac{A}{A+M}$$

$$+ \frac{x_{3n+1}}{x_{3n}} = \frac{x_{3n+1}}{x_{3n}}$$

This implies that the sequence $\left\{\frac{x_{3n+1}}{x_{3n}}\right\}$ is decreasing for $n \geq N$. This simply contradicts the fact that this sequence diverges to ∞. The proof is complete.

Theorem 2.2:

Consider the subsequences $\{x_{3n}\}$, $\{x_{3n+1}\}$, and $\{x_{3n+2}\}$ of the sequence $\{x_n\}$. If these subsequences have equal limit superior and limit inferior then $\{x_n\}$ converges to the equilibrium $\bar{x} = A + 1$.

Proof:

Since the subsequences $\{x_{3n}\}$, $\{x_{3n+1}\}$, and $\{x_{3n+2}\}$ have equal limit superior and limit inferior, either they converge to ∞, or converge to a finite positive number. By Theorem 2.1 if $A > 1$ then we are done. So we suppose that $A \leq 1$.

Assume that one of subsequences $\{x_{3n}\}$, $\{x_{3n+1}\}$, and $\{x_{3n+2}\}$ is convergent. If $x_{3n+2} \to \infty$ as $n \to \infty$ then Eq. (2.1) implies that $x_{3n} \to A$ as $n \to \infty$. Also, by Lemma 2.2 $\{x_{3n+1}\}$ should be convergent to a finite positive number β. Therefore Eq. (2.1) yields

$$\beta = A + \frac{\beta}{A} \geq A + \beta > \beta,$$

which simply is a contradiction. Thus, $\{x_{3n+2}\}$ should be convergent. Using similar arguments we find that $\{x_{3n+1}\}$ should be convergent, too. In summary, if one of subsequences $\{x_{3n}\}$, $\{x_{3n+1}\}$, and $\{x_{3n+2}\}$ is convergent then so are the other two. From the previous discussions and Lemma (2.2) we conclude that all of subsequences $\{x_{3n}\}$, $\{x_{3n+1}\}$, and $\{x_{3n+2}\}$ are convergent. Assume that $x_{3n} \to \alpha$, $x_{3n+1} \to \beta$, and $x_{3n+2} \to \gamma$ as $n \to \infty$. Then Eq. (2.1) implies that $\alpha = A + \alpha/\gamma$, $\beta = A+\beta/\alpha$, and $\gamma = A+\gamma/\beta$. By some algebra we obtain that

$$\alpha = \beta = \gamma = A+1.$$

This completes our proof.

3. Oscillatory Behavior:

In this section we show that every nontrivial and positive solution of Eq. (2.1) is oscillatory about the equilibrium. The following theorem describes this fact in details.

Theorem 3.1:

Any nontrivial and positive solution of Eq. (2.1) oscillates about the quilibrium $\bar{x}$ = A + 1. Positive and negative semicycles have at most 7 elements. Moreover, examples of positive and negative semicycles with 7 elements exist.

Proof:

Assume that $\{x_n\}^{\infty}_{n=-4}$ is a nontrivial and positive solution of Eq. (2.1). We will prove oscillation by showing that there is an upper bound of 7 for the maximum number of elements in both a positive and a negative semicycle.

At first we consider a positive semicycle. Assume that x_{N-7} $(N \in \mathbb{N}, N \geq 3)$ starts a starts a positive semicycle. So $x_{N-7} \geq A+1$. If $x_{N-6} < x_{N-7}$ then Eq. (2.1) implies that $x_{N-3} < A+1$ and therefore x_{N-3} starts a negative semicycle. Thus we assume that $xN\text{-}6 \geq x_{N-7}$. In a similar manner if $x_{N-5} < x_{N-6}$ or $x_{N-4} < x_{N-5}$ then $x_{N-2} < A+1$ or $x_{N-1} < A+1$ and therefore x_{N-2} or x_{N-1} starts a negative semicycle. So we assume that $x_{N-5} \geq x_{N-6}$ and $x_{N-4} \geq x_{N-5}$. Thus we have

$$1+A \leq x_{N-7} \leq x_{N-6} \leq x_{N-5} \leq x_{N-4}. \tag{3.1}$$

Therefore

$$x_N = A + \frac{x_{N-3}}{x_{N-4}} \leq A + \frac{x_{N-3}}{x_{N-6}} = A + \frac{A + \frac{x_{N-6}}{x_{N-7}}}{x_{N-6}} = A + \frac{A}{x_{N-6}} + \frac{1}{x_{N-7}} \leq A + \frac{A}{A+1} + \frac{1}{A+1} = A+1$$

Now note that since the solution $\{x_n\}$ is nontrivial, at least one of the inequalities in (3.1) should be strict. So one of the inequalities in (3.2) should be strict. Hence, $x_N < A+1$ which means that x_N starts a negative semicycle. In summary, if x_{N-7} starts a positive semicycle then there can be at most 6 additional consecutive elements greater than or equal to $\bar{x}$ A + 1.

Next, we consider negative semicycles. Assume that $x_{N-7} < A+1$ starts a negative emicycle. If $x_{N-6} \geq x_{N-7}$ then Eq. (2.1) implies that $x_{N-3} \geq 1+A$ and therefore x_{N-3} starts a positive semicycle. In a similar fashion if $x_{N-5} \geq x_{N-6}$ or $x_{N-4} \geq x_{N-5}$ then $x_{N-2} \geq 1+A$ or $x_{N-7} \geq 1+A$ an therefore x_{N-2} or x_{N-1} starts a positive semicycle. Thus we assume that $x_{N-5} < x_{N-6}$ and $x_{N-4} < x_{N-5}$. So we have $x_{N-4} < x_{N-5} < x_{N-6} < x_{N-7} < 1+A$. Thus

$$x_N = A + \frac{x_{N-3}}{x_{N-4}} > A + \frac{x_{N-3}}{x_{N-6}} = A + \frac{A + \frac{x_{N-6}}{x_{N-7}}}{x_{N-6}} = A + \frac{A}{x_{N-6}} + \frac{1}{x_{N-7}} > A + \frac{A}{A+1} + \frac{1}{A+1} = A + 1,$$

which means that x_N starts a positive semicycle. In summary, if x_{N-7} starts a negative semicycle then there can be at most 6 additional consecutive elements less than $\bar{x}$ = A+1.

To exhibit a positive semicycle with 7 elements let A +1 < $x_{-4} < x_{-3} < x_{-2} < x_{-1}$. Then simply we have x_0, x_1, $x_2 > A+1$. To exhibit a negative semicycle with 7 elements let $x_{-1} < x_{-2} < x_{-3} < x_{-4} < A+1$. Then simply we have x_0, x_1, $x_2 < A+1$. The proof is complete.

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