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A Model of Forming Relations between a Liaison and *K* Members of the Same Level in a Pyramid Organization Structure with *K* Subordinates

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Abstract: This paper proposes a model of placing a liaison which forms relations to K members of the same level in a pyramid organization structure with K subordinates such that the communication of information between every member in the organization becomes the most efficient. For the model of adding a node of liaison which gets adjacent to K nodes with the same depth N in a complete K-ary tree of height H which can describe the basic type of a pyramid organization, we obtained an optimal depth $N^* = 2$ which maximizes the sum of shortening lengths of shortest paths between every pair of all nodes, irrespective of K and H. This means that the most efficient level of forming relations to a liaison is the second level of below the top, irrespective of the number of subordinates and the number of levels in the organization structure.

Key words: pyramid organization structure; liaison; complete K-ary tree; shortest path

INTRODUCTION

A pyramid organization is a formal organization structure which is a hierarchical structure based on the principle of unity of command that every member except the top in the organization should have a single immediate superior (Koontz, H., 1980; Takahashi, N., 1988). There exist relations only between each superior and his direct subordinates in the pyramid organization. However, it is desirable to have formed additional relations other than that between each superior and his direct subordinates in advance in case they need communication with other departments in the organization.

The pyramid organization structure can be expressed as a rooted tree, if we let nodes and edges in the rooted tree correspond to members and relations between members in the organization respectively. Then the pyramid organization structure is characterized by the number of subordinates of each member, that is, the number of children of each node and the number of levels in the organization, that is, the height of the rooted tree (Robbins, S.P., 2003; Takahara, Y. and M. Mesarovic, 2003). Moreover, the path between a pair of nodes in the rooted tree is equivalent to the route of communication of information between a pair of members in the organization, and adding edges to the rooted tree is equivalent to forming additional relations other than that between each superior and his direct subordinates.

We have proposed some models of adding relations between members in a pyramid organization structure such that the communication of information between every member in the organization becomes the most efficient (Sawada, K. and R. Wilson, 2006; Sawada, K., 2008). For each model we have obtained a set of additional edges to a complete K-ary tree minimizing the sum of lengths of shortest paths between every pair of all nodes.

Liaisons which have roles of coordinating different sections are also placed as a means to become effective in communication of information in an organization (Gittell, J.H., 2000; Lievens, A. and R.K. Moenaert, 2000). However, it has not been theoretically discussed which members of an organization should form relations to the liaisons.

This paper proposes a model of placing a liaison which forms relations to K members of the same level in a pyramid organization structure which is a complete K-ary tree of height H (H = 2, 3, ...). We obtain the level with which the liaison forms relations to K members such that the communication of information between every member in the organization becomes the most efficient. This means that we obtain the optimal depth N^* minimizing the sum of lengths of shortest paths between every pair of all nodes in the complete K-ary tree when an added node of liaison gets adjacent to K nodes with the same depth N (N = 2, 3, ..., H) of the

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complete *K*-ary tree. A complete *K*-ary tree is a rooted tree in which all leaves have the same depth and all internal nodes have *K* (K = 2, 3, ...) children (Cormen, T.H., 2001). Fig. 1 shows an example of a complete *K*-ary tree ($K = 2$, $H = 5$). In Fig. 1 the value of *N* expresses the depth of each node.

If $l_{i,\,j}$ (= $l_{j,\,i}$) denotes the path length, which is the number of edges in the shortest path from a node v_i to a node v_j (i, j = 1, 2, ..., $(K^{H+1}–1) / (K–1)$) in the complete *K*-ary tree of height *H*, then $\sum_{i<j} l_{i,\,j}$ is the total path length. Furthermore, if $l'_{i,\,j}$ denotes the path length from v_i to v_j after getting adjacent in the above model, $l_{i,\,j} - l'_{i,\,j}$ is called the shortening path length between v_i and v_j, and $\sum_{i<j} (l_{i,\,j} - l'_{i,\,j})$ is called the *total shortening path length*. Minimizing the total path length is equivalent to maximizing the total shortening path length.

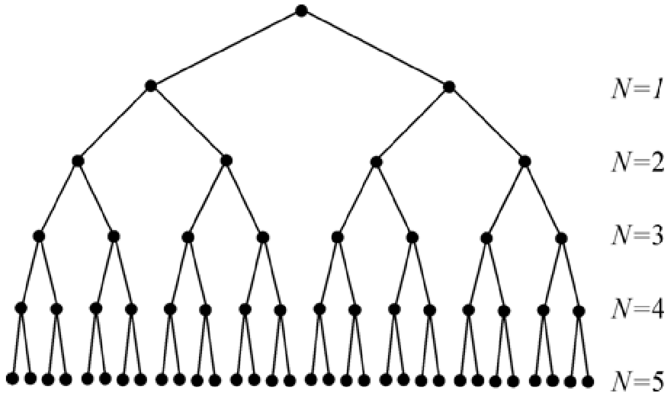

Fig. 1: An example of a complete *K*-ary tree (K = 2, H = 5).

In Section 2 when an added node of liaison gets adjacent to two nodes with the same depth *N* in a complete *K*-ary tree of height *H*, we formulate the total shortening path length and obtain an optimal pair of two nodes with an optimal depth N* by maximizing the total shortening path length. In Section 3 when an added node of liaison gets adjacent to *K* nodes with the same depth *N* in a complete *K*-ary tree of height *H*, we obtain an optimal set of *K* nodes with an optimal depth N* which maximizes the total shortening path length. Since we don't consider efficiency of communication of information between the liaison and the other members, the total shortening path length doesn't include the shortening path length between the node of liaison and nodes in a complete *K*-ary tree.

Forming Relations between a Liaison and Two Members:

This section obtains an optimal pair of two nodes with an optimal depth N* by maximizing the total shortening path length, when a node of liaison is added and gets adjacent to two nodes with the same depth *N* (N = 2, 3, ..., H) in a pyramid organization structure which is a complete *K*-ary tree of height *H* (H = 2, 3, ...).

Formulation of Total Shortening Path Length:

A node of liaison can get adjacent to two nodes with the same depth *N* of a complete *K*-ary tree in $N–1$ ways that lead to non-isomorphic graphs. Let $R_H(N, L)$ denote the total shortening path length by getting adjacent to two nodes, where L (L = 0, 1, 2, ..., N–2) is the depth of the deepest common ancestor of the two nodes to which the node of liaison gets adjacent. For the case of $L = 0$, the total shortening path length is denoted by $S_H(N)$. Since getting adjacent to two nodes shortens path lengths only between pairs of descendants of the deepest common ancestor of the two nodes to which the node of liaison gets adjacent, we obtain

$$R_H(N, L) = S_{H-L}(N - L). \tag{1}$$

We formulate $S_H(N)$ in the following.

Let v_0^X and v_0^Y denote the two nodes to which the node of liaison gets adjacent and assume that $L = 0$. Let v_i^X and v_i^Y denote ancestors of v_0^X and v_0^Y, respectively, with depth $N–i$ for

i = 1, 2, ..., N–2. The sets of descendants of v_0^X and v_0^Y are denoted by V_0^X and V_0^Y respectively. (Note that every node is a descendant of itself (Cormen, T.H., 2001).) Let V_i^X denote the set obtained by removing the descendants of v_{i-1}^X from the set of descendants of v_i^X and let V_i^Y denote the set obtained by removing the descendants of v_{i-1}^Y from the set of descendants of v_i^Y, where i = 1, 2, ..., N–2. Fig. 2 shows the above sets, where $K = 2$, $H = 5$ and $N = 4$. In Fig.

2 a white node and two bold edges signify the node of liaison and two edges for getting adjacent to two nodes with depth $N = 4$ respectively.

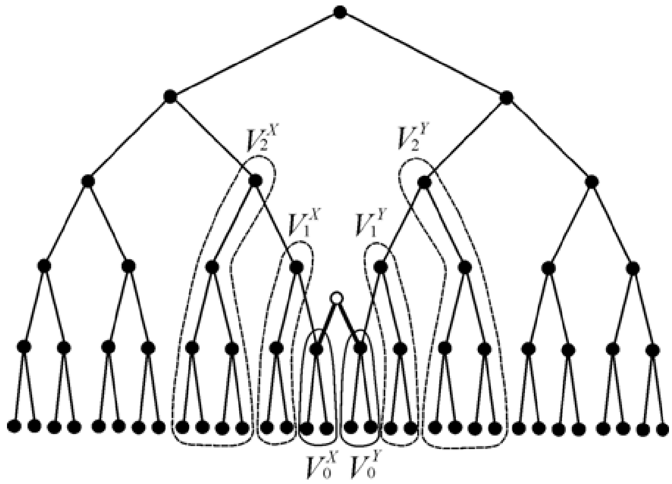

Fig. 2: The sets of nodes for formulation of the total shortening path length.

Since getting adjacent to two nodes doesn't shorten path lengths between pairs of nodes other than between pairs of nodes in V_i^X ($i = 0, 1, 2, ..., N–2$) and nodes in V_j^Y ($j = 0, 1, 2, \ldots, N–2$), the total shortening path length can be formulated by adding up the following three sums of shortening path lengths: (i) the sum of shortening path lengths between every pair of nodes in V_0^X and nodes in V_0^Y, (ii) the sum of shortening path lengths between every pair of nodes in V_0^X and nodes in V_i^Y ($i = 1, 2, ..., N–2$) and between every pair of nodes in V_0^Y and nodes in V_i^X ($i = 1, 2, ..., N–2$) and (iii) the sum of shortening path lengths between every pair of nodes in V_i^X ($i = 1, 2, ..., N–2$) and nodes in V_j^Y ($j = 1, 2, ..., N–2$).

The sum of shortening path lengths between every pair of nodes in V_0^X and nodes in V_0^Y is given by

$$A_H(N) = \left\{ M(H-N) \right\}^2 2(N-1), \tag{2}$$

where $M(h)$ denotes the number of nodes of a complete K-ary tree of height h ($h = 0, 1, 2, ...$). The sum of shortening path lengths between every pair of nodes in V_0^X and nodes in V_i^Y ($i = 1, 2, \ldots, N–2$) and between every pair of nodes in V_0^Y and nodes in V_i^X ($i = 1, 2, ..., N–2$) is given by

$$B_H(N) = 2M(H-N)\sum_{i=1}^{N-2} \left\{ (K-1)M(H-i-2)+1 \right\} 2i, \tag{3}$$

and the sum of shortening path lengths between every pair of nodes in V_i^X ($i = 1, 2, ..., N–2$) and nodes in V_j^Y ($j = 1, 2, ..., N–2$) is given by

$$\begin{aligned} C_H(N) = &\sum_{i=1}^{N-3} \left\{ (K-1)M(H-i-3)+1 \right\} \\ &\times \sum_{j=1}^{i} \left\{ (K-1)M(H-N+j-1)+1 \right\} 2(i-j+1), \end{aligned} \tag{4}$$

where we define

$$\sum_{i=1}^{0} \cdot = 0, \tag{5}$$

$$\sum_{i=1}^{-1} \cdot = 0. \tag{6}$$

From the above equations, the total shortening path length $S_H(N)$ is given by

$$\begin{aligned} S_H(N) &= A_H(N) + B_H(N) + C_H(N) \\ &= \{M(H-N)\}^2\, 2(N-1) \\ &\quad + 2M(H-N)\sum_{i=1}^{N-2}\{(K-1)M(H-i-2)+1\}\,2i \\ &\quad + \sum_{i=1}^{N-3}\{(K-1)M(H-i-3)+1\} \\ &\quad \times \sum_{j=1}^{i}\{(K-1)M(H-N+j-1)+1\}\,2(i-j+1). \end{aligned} \tag{7}$$

An Optimal Depth

From (1) and (7), we have the following theorem.

Theorem 1:

$L^* = 0$ maximizes $R_H(N, L)$ for each N.

Proof:

For $N = 2$, $L^* = 0$ trivially. For $N = 3, 4, ..., H$, let

$$\Delta R_H(N, L) \equiv R_H(N, L+1) - R_H(N, L). \tag{8}$$

We then have

$$\begin{aligned} &\Delta R_H(N, L) \\ &= S_{H-(L+1)}(N-(L+1)) - S_{H-L}(N-L) \\ &= -2\{M(H-N)\}^2 \\ &-2M(H-N)\{(K-1)M(H-N)+1\}\,2(N-L-2) \\ &-2M(H-N) \\ &\times \sum_{i=1}^{N-L-3}(K-1)\{M(H-L-i-2)-M(H-L-i-3)\}\,2i \\ &-\{(K-1)M(H-N)+1\} \\ &\times \sum_{i=1}^{N-L-3}\{(K-1)M(H-N+j-1)+1\}\,2(N-L-j-2) \\ &- \sum_{i=1}^{N-L-4}(K-1)\{M(H-L-i-3)-M(H-L-i-4)\} \\ &\times \sum_{j=1}^{i}\{(K-1)M(H-N+j-1)+1\}\,2(i-j+1), \end{aligned} \tag{9}$$

for $L = 0, 1, 2, ..., N–3$. Since $M(h)$ increases with h, we obtain

$$\Delta R_H(N, L) < 0. \tag{10}$$

Therefore, $R_H(N, L)$ takes its maximum at $L^* = 0$ for each N. The proof is done.

We next discuss the optimal depth $N = N^*$ which maximizes $S_H(N) = R_H(N, 0)$. Since the number of nodes of a complete K-ary tree of height h is

$$M(h) = \frac{K^{h+1} - 1}{K - 1}, \tag{11}$$

$S_H(N)$ of (7) becomes

$$S_H(N) = \frac{1}{(K-1)^3}\left\{ 2(N-1)(K-1)K^{2H-N} + 4K^{H-N+1} \right. \\ \left. -4K^H + 2(N-1)(K-1)\right\}. \tag{12}$$

The optimal depth N^* can be obtained and is given in Theorem 2.

Theorem 2:
The optimal depth is N* = 2.

Proof:
If $H = 2$, then $N^* = 2$ trivially. If $H = 3, 4, ...$, then $N^* = 2$, since

$$\begin{aligned} &\Delta S_H(N) \\ &\equiv S_H(N+1) - S_H(N) \\ &= \frac{1}{(K-1)^2}\left\{(-2NK + 2K + 2N)K^{2H-N-1} - 4K^{H-N} + 2\right\} \\ &< 0 \end{aligned} \tag{13}$$

for $N = 2, 3, ..., N–1$. The proof is done.

Forming Relations between a Liaison and K Members:
This section obtains an optimal set of K nodes with an optimal depth N^* by maximizing the total shortening path length, when a node of liaison is added and gets adjacent to K nodes with the same depth N ($N = 2, 3, ..., H$) in a pyramid organization structure which is a complete K-ary tree of height H ($H = 2, 3, ...$).

Let Q denote a set of K nodes $\{n_1, n_2, ..., n_K\}$ with the same depth N and let t_Q denote the total shortening path length when the node of liaison gets adjacent to every node in the set Q, then we have the following theorem.

Theorem 3:
t_Q takes its maximum when the depth of the deepest common ancestor of every pair of K nodes is zero for each N.

Proof:
Let $s_{i,j}$ denote the total shortening path length when the node of liaison gets adjacent to two nodes n_i and n_j ($i < j$) in the set Q, then we have

$$t_Q \le \sum_{i<j} s_{i,j}. \tag{14}$$

Since $s_{i,j}$ takes its maximum when the depth of the deepest common ancestor of the two nodes n_i and n_j is zero from Theorem 1, $\sum_{i<j} s_{i,j}$ takes its maximum when the depth of the deepest common ancestor of every pair of the K nodes is zero. Furthermore since

$$t_Q = \sum_{i<j} s_{i,j} \tag{15}$$

when the depth of the deepest common ancestor of every pair of the K nodes is zero, this theorem is obtained.

Let $T_{H,K}(N)$ denote the total shortening path length when the node of liaison gets adjacent to K nodes with the same depth N of which the deepest common ancestor is zero, so that we have

$$T_{H,K}(N) = \frac{K(K-1)}{2} S_H(N), \tag{16}$$

where $S_H(N)$ is the total shortening path length when the node of liaison gets adjacent to two nodes with the same depth N of which the deepest common ancestor is zero. From (16) and Theorem 2, we have the following theorem.

Theorem 4:

$N^* = 2$ *maximizes* $T_{H,K}(N)$.

Conclusions:

This study considered the placement of a liaison which forms relations to K members with the same level in a pyramid organization structure with K subordinates such that the communication of information between every member in the organization becomes the most efficient. For the model of adding a node of liaison which gets adjacent to K nodes with the same depth N in a complete K-ary tree of height H which can describe the basic type of a pyramid organization, we obtained an optimal depth N^* which maximizes the total shortening path length.

Theorem 4 shows that the optimal depth is $N^* = 2$, irrespective of K and H. This result means that the most efficient level of forming relations to a liaison is the second level of below the top, irrespective of the number of subordinates and the number of levels in the organization structure.

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