Australian Journal of Basic and Applied Sciences, 5(3): 580-586, 2011
ISSN 1991-8178

Mixture of Discrete Weibull Model

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Abstract: There are few studies on the mixture of discrete distributions, where most of the studies introduced a mixture of negative binomial, Poisson and binomial distributions. Recently, a sequence of studies on the mixture discrete distributions were published, but still very few. This paper introduced the new idea of mixture of discrete Weibull models. These models can be obtained by combining two or more discrete Weibull distributions using a mixing parameter. These models contain five or more parameters because each discrete Weibull contains two shape parameters q and β where 0<q<1 and β>0 and a mixing parameter w>0. In this paper, the distribution function, hazard (failure rate) function and second failure rate, called pseudo –hazard function of the mixture of discrete Weibull distribution were found using different values for the shape parameters and the mixing parameters. The distribution functions, hazard (failure rate) functions and pseudo- hazard functions were plotted depending on the data tables. It was found that the hazard (failure rate) functions and the pseudo-hazard functions can be increasing, decreasing or constant depending on the value of the shape parameter β, similar to the continuous Weibull distribution. Different values of the mixing parameter and the shape parameter were considered.

Key words: Mixture Weibull; Discrete Weibull; hazard; pseudo-hazard

INTRODUCTION

Discrete Weibull model can be obtained as a discrete counterpart of either the distribution function or the failure rate function of the standard Weibull model Murthy, *et al.,* (2004). These lead to different models that are the counterparts of the standard two-parameter Weibull distribution. This paper considered mixture of discrete Weibull model which can be obtained from two or more discrete Weibull distributions, determining the distribution function, failure rate function (hazard function) and pseudo-hazard function

Discrete of mixtures are applied by Medgyessi to the counter current method of identifying the constituents of small amounts of organic chemicals, each component of the chemical mixture is distributed independently of the others according to binomial distributions across the cell, the final result is a binomial mixture, see Medgyessi, (1961). A distribution formed by mixing two negative binomials was considered, see Rider, (1961). A mixture distribution formed from a Poisson component and binomial component, see Cohen, (1963). A mixture of binomial distributions when the mixing parameters are known was studied by Blichke, (1964). A mixture distribution of the form shown below, was considered by Cohen, (1966);

$$P(x;w)=\begin{cases}(1-w)+wP_1(0;\lambda) & x=0\\ wP_1(x;\lambda) & x=1,2,...\end{cases}$$

where $P_1(x;\lambda)$ is any discrete distribution defined over the domain x=0,1,2,… with parameters $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$. Cohen considered the special case where P_1 is a negative binomial distribution and in an earlier paper, see Cohen, (1960), where P_1 is a Poisson distribution. A mixture of multinomial distributions arising in a model of observer rating was introduced, see Dawid and Skene, (1979).

2. Materials and Methods:

a. Discrete Weibull Model:

Ali Khan et al. mentioned that the discrete Weibull data arise in reliability problems when the observed

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variable is discrete, see Ali Khan *et al.,* (1989). Probability distributions like exponential, gamma, Weibull, normal and log normal are used to model the lifetime of a component or a structure. These distributions therein are well-known. In many practical situations the failure data are measured as discrete variables, such as the number of runs, cycles or shocks. In this context geometric and negative binomial distributions are known to be discrete alternatives for the exponential and gamma distributions, respectively, as a discrete Weibull distribution alternative to the continuous Weibull distribution.

Discrete Weibull model can be obtained as the discrete counterparts of either the distribution function or the failure rate function of the standard Weibull models, that only assume nonnegative integer values. This model is useful for modeling the number of cycles to failure when components are subjected to cyclical loading. There are different models of discrete Weibull distribution, one of them was suggested by Nakagawa and Osaki, (1975). They gave the distribution function as follows;

$$F(x)=\begin{cases}1-q^{x^{\beta}} & x=0,1,2,... \\ 0 & x<0\end{cases} \tag{1}$$

with q and β denoting the shape parameters, where 0<q<1 and β>0. This form is similar to the two-parameter Weibull distribution, and has application in reliability when the response of interest is a discrete variable. The hazard (failure rate) function is given by;

$$h(x)=1-q^{x^{\beta}-(x-1)^{\beta}} \tag{2}$$

The hazard function can be increasing, decreasing or constant depending on the value of β, similar to the continuous distribution. Roy and Gupta proposed an alternate discrete failure rate function r(x), see Roy and Gupta, (1992). It was called second rate of failure and defined as follows;

$$r(x)=\ln\left[\frac{\bar{F}(x-1)}{\bar{F}(x)}\right] \qquad x=1,2,... \tag{3}$$

where $\bar{F}(x)$ is the survivor (reliability) function. From(3.3) we have

$$r(x)=\ln\left[\frac{q^{(x-1)^{\beta}}}{q^{x^{\beta}}}\right]=\ln\left(q\left[(x-1)^{\beta}-x^{\beta}\right]\right) \tag{4}$$

Xie et al. advocate that r(x) be called the discrete failure rate, see Xie, *et al.,* (2002). However it is important to note that r(x) is not a conditional probability where as h(x) is conditional probability and used the term pseudo-hazard function for r(x) so as to differentiate it from the hazard function h(x). The pseudo-hazard function can be increasing or decreasing depending on the values of β, and when β=2, it increases linearly which is similar to the continuous case.

b. Mixture of Discrete Weibull Model:

The distribution function of discrete Weibull mixture model can be obtained by

$$F(x;w,\theta)=\sum_{i=1}^{n} w_i F_i(x;\theta_i) \tag{5}$$

where $F_i(x;\theta_i)$ is a distribution function of the discrete Weibull distribution i, i=1,2,…,n , x = 0,1,2,… with $\theta_i>0$,

$0<w_i<1$, and $\sum_{i=1}^{n} w_i=1$ is the mixing parameter. The hazard function of the discrete Weibull mixture model is given by;

$$h(x;w,\theta)=\sum_{i=1}^{n} w_i h_i(x;\theta_i) \tag{6}$$

where $h_i(x;\theta_i)$ is a hazard function of discrete Weibull distribution i. The pseudo-hazard function of discrete Weibull mixture model is given by

$$r(x;w,\theta) = \sum_{i=1}^{n} w_i r_i(x;\theta_i) \tag{7}$$

where $r_i(x;\theta_i)$ is the pseudo-hazard function of the discrete Weibull distribution i.

If we simplified this case to two discrete components of Weibull, we will have the distribution function of discrete Weibull mixture model as follows;

$$F(x;w,\theta) = wF_1(x;\theta_1) + (1-w)F_2(x;\theta_2) \tag{8}$$

where $F_i(x;\theta_i)$ is the distribution function of the discrete Weibull distribution i, and i=1,2. The hazard function (failure rate) of discrete Weibull mixture model as follows

$$h(x;w,\theta) = wh_1(x;\theta_1) + (1-w)h_2(x;\theta_2) \tag{9}$$

where $h_i(x;\theta_i)$ is the hazard function of discrete Weibull distribution i, and i=1,2.

The pseudo-hazard function of discrete Weibull mixture model is as follows

$$r(x;w,\theta) = wr_1(x;\theta_1) + (1-w)r_2(x;\theta_2) \tag{10}$$

where $r_i(x;\theta_i)$ is the pseudo –hazard function of the discrete Weibull distribution i and i =1,2.

RESULTS AND DISCUSSION

Consider the data set representing the number of shocks before failure for 20 items, see Murthy *et al.,* (2004). We assume that the data can be adequately modeled by a discrete Weibull mixture model with the parameters q and β as follows; (q_1=0.3, β_1=0.5, q_2=0.4, β_2=0.5), (q_1=0.8, β_1=1, q_2=0.5, β_2=1) , (q_1=0.2, β_1=2, q_2=0.7, β_2=1.5). The distribution function, hazard (failure rate) function and pseudo-hazard function of the discrete Weibull mixture model was determined using different values for the mixing parameter w= 0.30, 0.50 and 0.80. Table 1 shows the distribution function of the discrete Weibull mixture model using different values for the mixing parameter. Table 2 shows the hazard (failure rate) function of the discrete Weibull mixture model using different values for the mixing parameter w=0.30, 0.50 and 0.80. Table 3 shows the pseudo-hazard function for the discrete Weibull mixture model using different values for the mixing parameter w=0.30, 0.50 and 0.80.

Table 1: Distribution Function of the Discrete Weibull Mixture Model

	$\beta < 1$			$\beta=1$			$\beta > 1$		
	(q_1=0.3, β_1=0.5, q_2=0.4, β_2=0.5)			(q_1=0.8, β_1=1, q_2=0.5, β_2=1)			(q_1=0.2, β_1=2, q_2=0.7, β_2=1.5)		
x	F(x) w=0.30	F(x) w=0.50	F(x) w=0.80	F(x) w=0.30	F(x) w=0.50	F(x) w=0.80	F(x) w=0.30	F(x) w=0.50	F(x) w=0.80
2	0.6300	0.6500	0.6800	0.6330	0.5550	0.4380	0.7594	0.8277	0.9301
3	0.7736	0.7914	0.8180	0.7589	0.6815	0.5654	0.8594	0.8995	0.9598
6	0.9471	0.9545	0.9656	0.9104	0.8611	0.7872	0.9718	0.9798	0.9919
6	0.9471	0.9545	0.9656	0.9104	0.8611	0.7872	0.9718	0.9798	0.9919
7	0.9672	0.9724	0.9801	0.9316	0.8912	0.8307	0.9835	0.9882	0.9953
9	0.9873	0.9897	0.9932	0.9584	0.9319	0.8922	0.9943	0.9959	0.9984
9	0.9873	0.9897	0.9932	0.9584	0.9319	0.8922	0.9943	0.9959	0.9984
10	0.9921	0.9937	0.9960	0.9671	0.9458	0.9139	0.9967	0.9976	0.9991
10	0.9921	0.9937	0.9960	0.9671	0.9458	0.9139	0.9967	0.9976	0.9991
11	0.9951	0.9961	0.9976	0.9739	0.9568	0.9312	0.9981	0.9986	0.9994
12	0.9969	0.9976	0.9986	0.9792	0.9655	0.9450	0.9989	0.9992	0.9997
12	0.9969	0.9976	0.9986	0.9792	0.9655	0.9450	0.9989	0.9992	0.9997
12	0.9969	0.9976	0.9986	0.9792	0.9655	0.9450	0.9989	0.9992	0.9997

Table 1: Continue

13	0.9981	0.9985	0.9992	0.9834	0.9725	0.9560	0.9993	0.9995	0.9998
13	0.9981	0.9985	0.9992	0.9834	0.9725	0.9560	0.9993	0.9995	0.9998
13	0.9981	0.9985	0.9992	0.9834	0.9725	0.9560	0.9993	0.9995	0.9998
15	0.9992	0.9994	0.9997	0.9894	0.9824	0.9718	0.9998	0.9998	0.9999
16	0.9995	0.9996	0.9998	0.9915	0.9859	0.9775	0.9999	0.9999	1.0000
16	0.9995	0.9996	0.9998	0.9915	0.9859	0.9775	0.9999	0.9999	1.0000
18	0.9998	0.9999	0.9999	0.9946	0.9910	0.9856	1.0000	1.0000	1.0000

Table 2: Hazard (Failure Rate) Function of the Discrete Weibull Mixture Model

	$\beta < 1$			$\beta=1$			$\beta > 1$		
	(q_1=0.3, β_1=0.5, q_2=0.4, β_2=0.5)			(q_1=0.8, β_1=1, q_2=0.5, β_2=1)			(q_1=0.2, β_1=2, q_2=0.7, β_2=1.5)		
x	h(x) w=0.30	h(x) w=0.50	h(x) w=0.80	h(x) w=0.30	h(x) w=0.50	h(x) w=0.80	h(x) w=0.30	h(x) w=0.50	h(x) w=0.80
2	0.3345	0.3488	0.3735	0.3431	0.2843	0.2267	0.4801	0.4815	0.4885
3	0.2669	0.2784	0.2998	0.3089	0.2589	0.2173	0.5703	0.5704	0.5708
6	0.1851	0.1922	0.2084	0.2366	0.2169	0.2044	0.7147	0.7147	0.7147
6	0.1851	0.1922	0.2084	0.2366	0.2169	0.2044	0.7147	0.7147	0.7147
7	0.1708	0.1769	0.1918	0.2240	0.2108	0.2028	0.7443	0.7443	0.7443
9	0.1498	0.1543	0.1670	0.2099	0.2043	0.2011	0.7898	0.7898	0.7898
9	0.1498	0.1543	0.1670	0.2099	0.2043	0.2011	0.7898	0.7898	0.7898
10	0.1418	0.1457	0.1573	0.2062	0.2027	0.2007	0.8077	0.8077	0.8077
10	0.1418	0.1457	0.1573	0.2062	0.2027	0.2007	0.8077	0.8077	0.8077
11	0.1349	0.1383	0.1489	0.2039	0.2017	0.2004	0.8233	0.8233	0.8233
12	0.1290	0.1319	0.1415	0.2025	0.2011	0.2003	0.8370	0.8370	0.8370
12	0.1290	0.1319	0.1415	0.2025	0.2011	0.2003	0.8370	0.8370	0.8370
12	0.1290	0.1319	0.1415	0.2025	0.2011	0.2003	0.8370	0.8370	0.8370
13	0.1237	0.1262	0.1349	0.2015	0.2007	0.2002	0.8491	0.8491	0.8491
13	0.1237	0.1262	0.1349	0.2015	0.2007	0.2002	0.8491	0.8491	0.8491
13	0.1237	0.1262	0.1349	0.2015	0.2007	0.2002	0.8491	0.8491	0.8491
15	0.1149	0.1168	0.1238	0.2006	0.2003	0.2001	0.8696	0.8696	0.8696
16	0.1112	0.1128	0.1190	0.2004	0.2002	0.2000	0.8783	0.8783	0.8783
16	0.1112	0.1128	0.1190	0.2004	0.2002	0.2000	0.8783	0.8783	0.8783
18	0.1047	0.1059	0.1107	0.2001	0.2001	0.2000	0.8933	0.8933	0.8933

Table 3: Pseudo-Hazard Function of the Discrete Weibull Mixture Model

	$\beta<1$			$\beta=1$			$\beta>1$		
	(q_1=0.3, β_1=0.5, q_2=0.4, β_2=0.5)			(q_1=0.8, β_1=1, q_2=0.5, β_2=1)			(q_1=0.2, β_1=2, q_2=0.7, β_2=1.5)		
x	r(x) w=0.30	r(x) w=0.50	r(x) w=0.80	r(x) w=0.30	r(x) w=0.50	r(x) w=0.80	r(x) w=0.30	r(x) w=0.50	r(x) w=0.80
2	0.4085	0.4306	0.4689	0.4473	0.3552	0.2650	0.6605	0.6715	0.7286
3	0.3111	0.3272	0.3573	0.3937	0.3154	0.2502	0.8455	0.8468	0.8537
6	0.2050	0.2138	0.2341	0.2805	0.2496	0.2300	1.2543	1.2543	1.2543
6	0.2050	0.2138	0.2341	0.2805	0.2496	0.2300	1.2543	1.2543	1.2543
7	0.1875	0.1949	0.2134	0.2607	0.2400	0.2275	1.3637	1.3637	1.3637
9	0.1624	0.1678	0.1830	0.2386	0.2299	0.2248	1.5596	1.5596	1.5596
9	0.1624	0.1678	0.1830	0.2386	0.2299	0.2248	1.5596	1.5596	1.5596
10	0.1530	0.1576	0.1714	0.2329	0.2274	0.2242	1.6488	1.6488	1.6488
10	0.1530	0.1576	0.1714	0.2329	0.2274	0.2242	1.6488	1.6488	1.6488
11	0.1450	0.1490	0.1615	0.2293	0.2258	0.2238	1.7335	1.7335	1.7335
12	0.1381	0.1415	0.1528	0.2270	0.2248	0.2236	1.8142	1.8142	1.8142
12	0.1381	0.1415	0.1528	0.2270	0.2248	0.2236	1.8142	1.8142	1.8142
12	0.1381	0.1415	0.1528	0.2270	0.2248	0.2236	1.8142	1.8142	1.8142
13	0.1321	0.1350	0.1451	0.2256	0.2242	0.2234	1.8914	1.8914	1.8914
13	0.1321	0.1350	0.1451	0.2256	0.2242	0.2234	1.8914	1.8914	1.8914
13	0.1321	0.1350	0.1451	0.2256	0.2242	0.2234	1.8914	1.8914	1.8914
15	0.1221	0.1242	0.1323	0.2241	0.2236	0.2232	2.0372	2.0372	2.0372
16	0.1179	0.1197	0.1268	0.2237	0.2234	0.2232	2.1063	2.1063	2.1063
16	0.1179	0.1197	0.1268	0.2237	0.2234	0.2232	2.1063	2.1063	2.1063
18	0.1106	0.1119	0.1175	0.2234	0.2232	0.2232	2.2380	2.2380	2.2380

Conclusion:

Mixture of discrete Weibull distribution can be obtained by combining two or more discrete Weibull distributions using mixing parameters. This mixture has at least five or more parameters. This model can be used when we have two or more subpopulations that can be combined to make a mixture.

Fig. 1 until 9 represent the functions of the discrete Weibull mixture model. Fig. 1, 2 and 3 show the distribution function with different values of the parameters and the mixing parameter. Fig. 4 illustrates the idea of hazard function which is decreasing as β_1=0.5 and β_2=0.5, i.e. both are less than 1. Fig. 5 shows that the hazard function is almost constant for all values of x because β_1=1 and β_2=1. Fig. 6 shows that the hazard function is increasing as β_1=2 and β_2=1.5, i.e. both are greater than 1. Fig. 7 shows that the pseudo-hazard function is decreasing as β_1=0.5 and β_2=0.5, i.e. both are less than 1. Fig. 8 shows that the pseudo-hazard function is almost constant for all values of x because β_1=1 and β_2=1. Fig. 9 shows that the pseudo-hazard function is increasing as β_1=2 and β_2=1.5, i.e. both are greater than 1.

It can be seen that the pseudo-hazard function, $r(x)$ has the same trend as the hazard function. If one is increasing (decreasing) the other is also increasing (decreasing). These results agree well with the concept of the hazard and pseudo-hazard functions which state that when shape parameter $\beta > 1$ these functions increase, and when $\beta < 1$ they decrease.

Fig. 1: *F(x)* with the parameters (q_1=0.3, β_1=0.5, q_2=0.4, β_2=0.5)

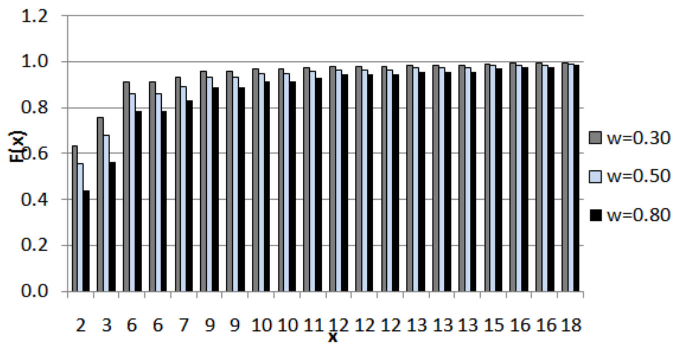

Fig. 2: *F(x)* with the parameters (q_1=0.8, β_1=1, q_2=0.5, β_2=1)

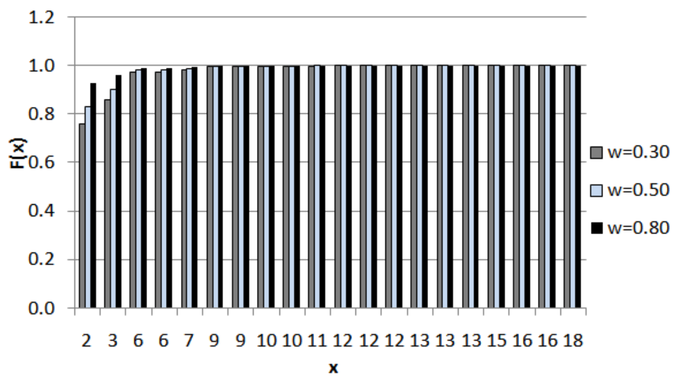

Fig. 3: *F(x)* with the parameters (q_1=0.2, β_1=2, q_2=0.7, β_2=1.5)

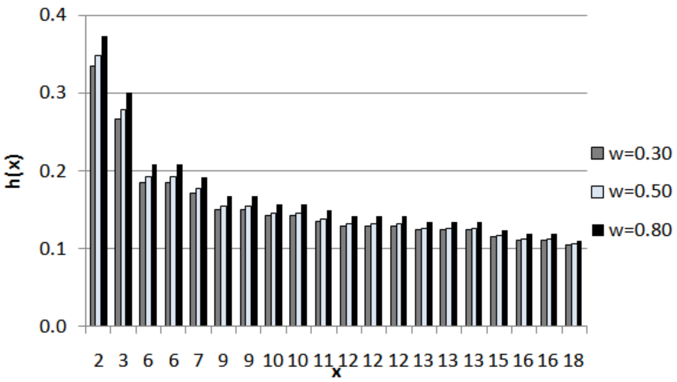

Fig. 4: *h(x)* with the parameters (q_1=0.3, β_1=0.5, q_2=0.4, β_2=0.5)

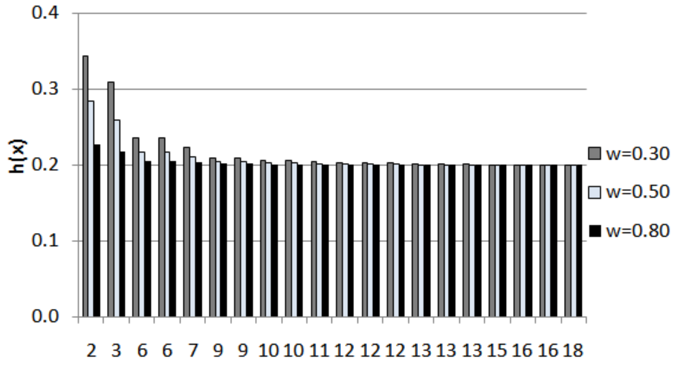

Fig. 5: *h(x)* with the parameters (q_1=0.8, β_1=1, q_2=0.5, β_2=1)

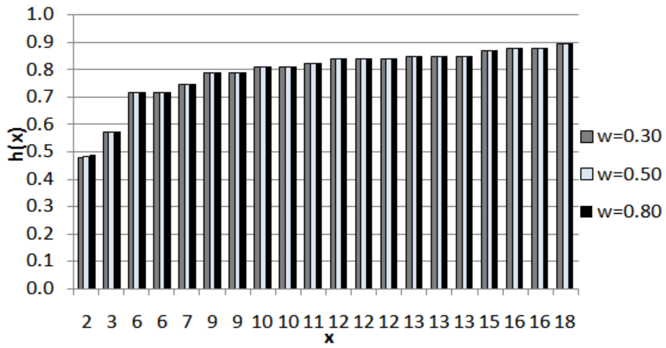

Fig. 6: *h(x)* with the parameters (q_1=0.2, β_1=2, q_2=0.7, β_2=1.5)

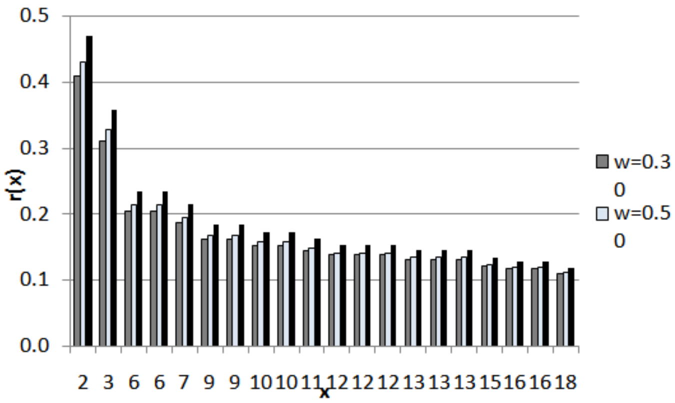

Fig. 7: *r(x)* with the parameters (q_1=0.3, β_1=0.5, q_2=0.4, β_2=0.5)

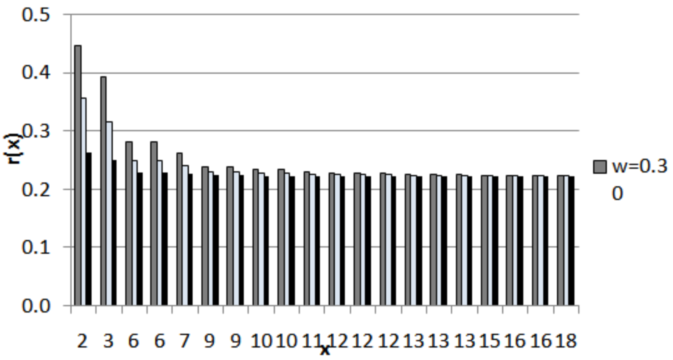

Fig. 8: *r(x)* with the parameters (q_1=0.8, β_1=1, q_2=0.5, β_2=1)

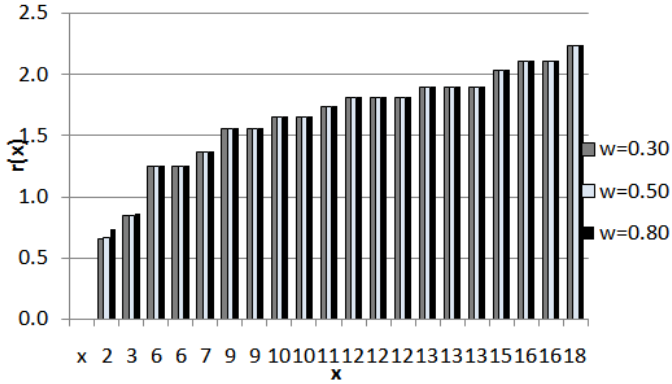

Fig. 9: *r(x)* with the parameters (q_1=0.2, β_1=2, q_2=0.7, β_2=1)

REFERENCES

Ali, M.S., A. Khan, A.A. Khalique, A.A. Abouammoh, 1989. On Estimating Parameters in Discrete Weibull Distribution. IEEE Transactions on Reliability, 38(3): 348-350.

Blichke, W.R., 1964. Estimating the Parameters of Mixtures of Binomial Distributions. Journal of the American Statistical Association., 59: 510-528.

Cohen, A.C., 1960. An Extension of Truncated Poisson Distribution. Biometrics, 16: 446-450.

Cohen, A.C., 1963. Estimation in Mixtures of Discrete Distributions. Proc. Int. Symp. on Classical and Contagious Discrete Distributions. Montreal, Pergamon Press, pp: 373-378.

Cohen, A.C., 1966. A Note on Certain Discrete Mixed Distributions. Biometrics, 22: 566-572.

Dawid, A.P., A.M. Skene, 1979. Maximum Likelihood Estimation of Observer Error-Rates Using the EM Algorithm. Applied Statistics, 28: 20-28.

Murthy, D.N.P., M. Xie, R. Jiang, 2004. Weibull Models. Wiley & Sons, Inc. Hoboken, New Jersey.

Medgyessi, P., 1961. Decomposition of Super Positions of Distribution Functions. Publishing House of the Hungarian Academy of Sciences, Budapest.

Nakagawa, T., S. Osaki, 1975. The Discrete Weibull Distribution. IEEE Transaction on Reliability, 24: 300-301.

Rider, P.R., 1961. The Method of Moments Applied to a Mixture of Two Exponential Distributions. Annals of Math. Stats., 32: 143-147.

Roy, D., R.P. Gupta, 1992. Classifications of Discrete Lives. Microelectronics and Reliability, 32: 1459-1473.

Xie, M., O. Gaudoin, C. Bracquemond, 2002. Redefining Failure Rate Functions for Discrete Distributions. International Journal of Reliability, Quality and Safety Engineering, 7: 17-25.