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Half-sweep Conjugate Gradient Method for Solving First Order Linear Fredholm Integro-differential Equations

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Abstract: The main purpose of this paper is to examine the effectiveness of Half-Sweep Conjugate Gradient (HSCG) method. The trapezoidal and central difference scheme will be used to discretize linear Fredholm integro-differential equations of the first order to formulate system linear equation. The basic formulation and implementation of the proposed method is also presented. Some numerical tests were carried out to show the effectiveness of the proposed method compared to the Full-Sweep Conjugate Gradient (FSGS) iterative method.

Key words: Integro-differential equations, trapezoidal, central difference and Half-Sweep conjugate gradient

INTRODUCTION

In the recent years, the studies of integro-differential equations (IDE) are developed very rapidly and intensively. IDE is an equation that the unknown function appears under the sign of integration and it also contains the derivatives and functional arguments of the unknown function. It can be classified into Fredholm equations and Volterra equations. The upper bound of the region for integral part of Volterra type is variable, while it is a fixed number for that of Fredholm type (1). However, in this paper, we focus on Fredholm integro-differential. Generally, first-order linear Fredholm integro-differential equations can be defined as follows

$$y'(x) = q(x)y(x) + \int_a^b K(x,t)y(t)dt + f(x)\,,\ x,t \in \Gamma = [a,b]\ \ y(a) = y_a \tag{1}$$

where the functions *f (x)*, *q (x)* and the kernel *K (x,t)* are known and y(x) is the solution to be determined.

Habitually, Fredholm IDE of the first order cannot be solved analytically for y(x). In many application areas, it is necessary to use the numerical approach to obtain an approximation solution for the problem (1). To solve a linear integro-differential equation numerically, discretization of integral equation to the solution of system of linear algebraic equations is the basic concept used by researchers to solve integro-differential problems. By taking into consideration numerical techniques, there are many methods can be used to discretize problem (1) such as compact finite difference (Zhao, 2006) Wavelet-Galerkin (Avudainayagam,2000) rationalized Haar functions (Maleknejad, 2004) Lagrange interpolation (Rashed, 2003) Tau (Hosseini, 2003) quadrature-difference (Fedotov, 2008) and Generalised minimal residual (GMRES) method (Aruchunan, 2009).

The idea of the half-sweep iteration method has been introduced by Abdullah (1991) via the Explicit Decoupled Group (EDG) iterative method to solve two-dimensional Poisson equation. Half-sweep iteration is also known as the complexity reduction approach (Hasan, 2007). Following to that, an application of the half-sweep iteration concept with the iterative methods have been extensively studied by many researchers; see (Yousif, 1995; Abdullah, 1996; Othman, 2000; Sulaiman, 2004; Sulaiman, 2008; Abdullah, 2006).

The purpose of this paper is to examine Half-Sweep Conjugate Gradient (HSCG) iterative methods for solving linear algebraic equations produced by the discretization of the first-order linear Fredholm integro-differential equations by using repeated trapezoidal (RT) and central difference (CD) scheme. The integral term is discretized by RT scheme and the differential term is approximated by CD scheme. The standard CG method

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also can be called as Full Sweep Conjugate Gradient (FSCG).

In Section 2 of this paper, the formulation of the RT and CD schemes are elaborated. The latter section of this paper discussed the formulations of the HSCG iterative methods in solving dense linear systems generated from discretization of the Eq. (1). Meanwhile, some numerical outcome are shown in fourth section to emphasize the efficiency of the proposed method and concluding remarks are given in Section 5.

II. Full- and Half-sweep Approximation Equations:

As afore-mentioned, a discretization scheme based on method of quadrature and finite difference were used to construct approximation equations for problem (1) Generally, quadrature method can be defined as follows

$$\int_a^b y(t)\,dt = \sum_{j=0}^{n} A_j y\left(t_j\right) + \varepsilon_n\left(y\right) \tag{2}$$

where t_j (j = 0,1,2,.....*n*) is the abscissas of the partition points of the integration interval [a,b], A_j (j = 0,1,2,....,n) is numerical coefficients that do not depend on the function y (t) and ε_n (y) is the truncation error of Eq. (3). To ease in formulating the full- and half-sweep quadrature approximation equations for problem (1), further discussion will be restricted onto repeated trapezoidal (RT) scheme, which is based on linear interpolation formula with equally spaced data.

$$A_j = \begin{cases} \frac{1}{2}ph, & j = 0, n \\ ph, & j = p, 2p, \cdots, n-p \end{cases} \tag{4}$$

where the constant step size, *h* is defined as

$$h = \frac{b-a}{n} \tag{5}$$

and *n* is the number of subintervals in the interval [a,b]. Meanwhile, the value of *p*, which corresponds to 1 and 2, represents the full and half-sweep cases respectively.

In this paper, central difference approximation as follow:

$$y'(x_i) = \frac{y_{i+p} - y_{i-p}}{2ph} \tag{6}$$

where $y'(x_i)$ is approximated by the gradient of the line passing $\left(x_i, y'(x_i)\right)$

Based on Fig. 1, the full and half-sweep iterative methods will compute approximate values onto node points of type

- only until the convergence criterion is reached. Then, other approximate solutions at remaining points (points of the different type,) are computed directly as discussed in (Hasan, 2007; Yousif, 1995; Abdullah, 1996; Othman, 2000; Sulaiman, 2004; Sulaiman, 2007; Sulaiman, 2008). By applying Eq. (2) and Eq. (6) into Eq. (1) and neglecting the error, a system of linear equations can be formed for approximation values of. The following linear system generated using RT and CD schemes can be easily shown in matrix form as follows

$$My = f \tag{4}$$

Conjugate Gradient Method (CG):

As mentioned in the previous section, the CG method will be used to solve a system of linear equations. CG method is proposed by Hestenes and Stiefel (Hestenes, 1952) and was originally developed as a direct method designed to solve positive definite linear system. Following is algorithm of general form of CG method.

Algorithm: CG Method:

Computer $r_0 := f - My_0,\ p_0 := r_0$.

For $j = 0,1,\ldots,$ until convergence Do:Computer For until convergence Do:

$a_j := (r_j, r_j)/(Mp_j, p_j)$

$y_{j+1} := y_j + a_j p_j$

$r_{j+1} := r_j - a_j M p_j$

$\beta j := (r_{j+1}, r_{j+1})/(r_j, r_{j+1})$

$P_{j+1} := r_{j+1} + \beta_j p_j$

End do
The vectors p_j are multiples of the p_j 's of algorithm.

***Discussion*:**

Numerical comparison parameters are considered such as number of iterations, execution time and aximum absolute error. As comparisons, the Gauss-Seidel (GS) method acts as the control of comparison of numerical results. Throughout the simulations, the convergence test considered the tolerance error of the $\varepsilon = 10^{-16}$. In order to compare the performances of the methods described in the previous section, several experiments were carried out on the following problems.

Example 1 (Darania, 2007):

$$y'(x) = xe^x + e^x - x + \int_0^1 xy(t)\,dt \tag{7}$$

$$y(0) = 0$$

with exact solution given as.

Results of numerical experiments, which were obtained from implementations of the FSGS, FSGM and HSGM methods for Example 1, have been recorded in Table 1. Figs.2 and 3 show number of iterations and execution time versus mesh size respectively for Example 1.Results of example 1 have been recorded in Tables 1.

Example 2 [19]

$$u'(x) = \frac{1}{(\ln)^2}\int_0^1 \frac{x}{t+1}u(t)dt + u(x) - \frac{1}{2}x + \frac{1}{x+1} - \ln(1+x)$$

$$u(0) = 1$$

with exact solution is given by $u(x) = \ln(x+1)$

V. Conclusion:

Through the results obtained for Example 1(in Table 1) shows that number of iterations of FSCG and HSCG methods have decreased approximately 23.81%-28.57% and 27.27%-28.57% respectively compared to GS method.

In terms of execution time, both the FSCG and HSCG methods are much faster than the GS method about 12.13% - 40.74% and 49.27%-78.57% respectively. Number of iterations for FSCG and HSCG iterative methods for Example 2 as shown in Table 2 decreased approximately 37.20% - 39.03% and 37.20% - 41.87% compared with GS method. Through the observation in Table 2 and Fig. 5, show that execution time for FSCG and HSCG methods decreased about 22.17% - 31.30% and 41.17% - 83.51% respectively compared to the GS method.

Generally, the numerical results have shown that the HSCG method is more superior in term of number of iterations and the execution time than FSCG and GS method.

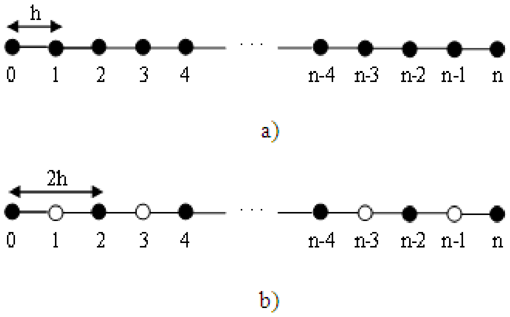

Fig. 1: a and b show distribution of uniform node points for the full- and half-sweep cases respectively.

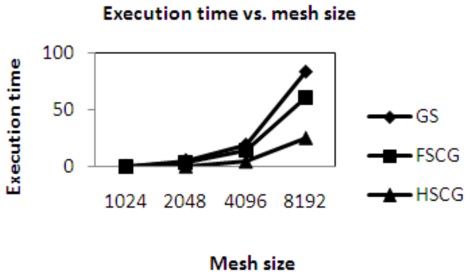

Fig. 2: Number of Iterations Versus Mesh Size of the GS, FSCG and HSCG Methods for Example 1.

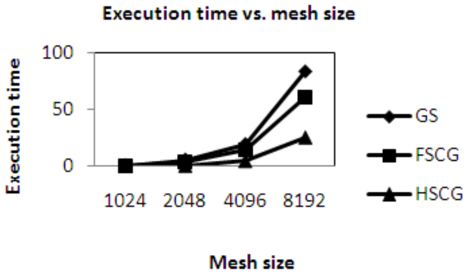

Fig. 3: The Execution Time (Seconds) Versus Mesh Size of the GS, FSCG and HSCG Methods for Example 1

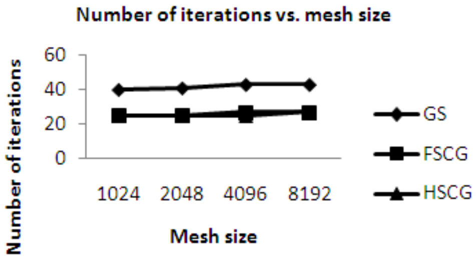

Fig. 2: Number of Iterations Versus Mesh Size of the GS, FSCG and HSCG Methods for Example 2.

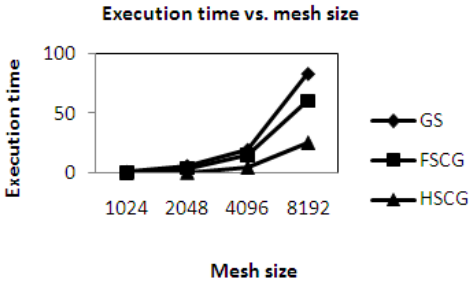

Fig. 3: The Execution Time (Seconds) Versus Mesh Size of the GS, FSCG and HSCG Methods for Example 2.

Table 1: Comparison of a Number of Iterations, Execution Time and Maximum Absolute Error for the Iterative Methods of Example 1

Methods	Number of iteration			
	Mesh Size			
	1024	2048	4096	8192
GS	21	21	22	22
FSCG	15	16	16	16
HSCG	15	15	16	16
Methods	Execution time (seconds)			
	Mesh Size			
	1024	2048	4096	8192
GS	0.27	1.12	4.70	17.23
FSCG	0.16	0.97	3.71	15.14
HSCG	0.07	0.24	1.69	8.74
Methods	Maximum absolute error			
	Mesh Size			
	1024	2048	4096	8192
GS	2.67E-03	1.33E-03	6.67E-04	3.34E-04
FSCG	2.47E-03	1.23E-03	6.51E-04	3.20E-04
HSCG	5.01E-03	2.47E-03	1.23E-03	6.51E-04

Table 2: Comparison of a Number of Iterations, Execution Time and Maximum Absolute Error for the Iterative Methods of Example 2

Methods	Number of iteration			
	Mesh Size			
	1024	2048	4096	8192
GS	40	41	43	43
FSCG	25	25	27	27
HSCG	25	25	25	27
Methods	Execution time (seconds)			
	Mesh Size			
	1024	2048	4096	8192
GS	1.02	5.88	19.57	83.65
FSCG	0.74	4.04	15.23	60.76
HSCG	0.60	0.97	5.11	25.41
Methods	Maximum absolute error			
	Mesh Size			
	1024	2048	4096	8192
GS	4.56E-04	2.85E-04	9.08E-05	8.56E-05
FSCG	3.91E-04	2.06E-04	8.85E-05	7.76E-05
HSCG	5.65E-04	3.19E-04	2.06E-04	8.85E-05

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