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Generalized AOR Method for Solving System of Linear Equations

Davod Khojasteh Salkuyeh

Department of Mathematics, University of Mohaghegh Ardabili,
P.O.Box. 179, Ardabil, Iran

Abstract: The accelerated overrelaxation (AOR) iterative method is a stationary iterative method for solving linear system of equations. In this paper, a generalization of the AOR iterative method is presented and its convergence properties are studied. Some numerical experiments are given to show the efficiency of the proposed method. AMS Mathematics Subject Classification: 65F10.

Key words: AOR, generalized AOR, M-matrix, regular splitting, convergence.

INTRODUCTION

Consider the linear system of equations

$$Ax = b, \tag{1}$$

where the matrix $A \in \Re^{n\times n}$ and $x, b \in \Re^{n}$. Let A be a nonsingular matrix with nonzero diagonal entries and $A = D-E-F$, where D is the diagonal of A, $-E$ its strict lower part, and $-F$ its strict upper part. Then the Jacobi and the Gauss-Seidel methods for solving Eq. (1) are defined as

$$x^{(k+1)} = D^{-1}(E+F)x^{(k)} + D^{-1}b,$$
$$x^{(k+1)} = (D-E)^{-1}Fx^{(k)} + (D-E)^{-1}b,$$

respectively. In the accelerated overrelaxation (AOR) iterative method (Hadjidimos, A., 1978), system (1) is written as $\omega Ax = \omega b$ where $\omega \neq 0$ is a parameter. Then the coefficient matrix ωA is decomposed in the form

$$\omega A = (D-\gamma E) - [(1-\omega)D + (\omega-\gamma)E + \omega F],$$

where $\gamma \in \Re$ Next, the system $\omega Ax = \omega b$ is written as

$$x = (D-\gamma E)^{-1}[(1-\omega)D + (\omega-\gamma)E + \omega F]x + \omega(D-\gamma E)^{-1}b,$$

and then the AOR iterative method is defined as

$$x^{(k+1)} = (D-\gamma E)^{-1}[(1-\omega)D + (\omega-\gamma)E + \omega F]x^{(k)} + \omega(D-\gamma E)^{-1}b.$$

It is well-known that for specific values of ω and γ the AOR iterative method reduces to Jacobi, Gauss-Seidel and successive overrelaxation iterative (SOR) methods:

$\gamma = 0,\ \omega = 1:$ Jacobi method,

Corresponding Author: Davod Khojasteh Salkuyeh, Department of Mathematics, University of Mohaghegh Ardabili, P.O.Box. 179, Ardabil, Iran
E-mail: khojaste@uma.ac.ir

$\gamma = \omega = 1:$ Gauss-Seidel method,

$\gamma = \omega:$ SOR method.

Although there are several iterative methods such as GMRES (Saad, Y., M.H. Schultz, 1986) and Bi-CGSTAB (van der Vorst, H. A., 1992) algorithms for solving Eq. (1) which are more effective than these four stationary iterative methods, they have been used as preconditioners for common iterative solvers (see for example (DeLong, M., J.M. Ortega, 1995; DeLong, M., J.M. Ortega, 1996; DeLong, M., J.M. Ortega, 1998)). In (Salkuyeh, D.K., 2007), Salkuyeh proposed the generalized Jacobi (GJ) and Gauss-Seidel (GGS) methods and studied their convergence properties. In this paper, we propose a generalization of the AOR (hereafter denoted by GAOR) method and verify its convergence properties.

This paper is organized as follows. In section 2, we review the GJ and GGS iterative methods and propose the GAOR iterative method. Section 2 also provides the background for the convergence of the proposed method. Convergence properties of the GAOR method are presented in section 3. Some numerical experiments are given in section 4. Concluding remarks are given in section 5.

2. The GJ, GGS and GAOR methods:

Let $A = (a_{ij})$ be an $n \times n$ matrix and $T_m = (t_{ij})$ be a banded matrix of bandwidth $2m+1$ defined as

$$t_{ij} = \begin{cases} a_{ij}, & |i-j| \le m, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the decomposition

$$A = T_m - E_m - F_m, \tag{2}$$

where $-E_m$ and $-F_m$ are the strict lower and upper part of the matrix $A_m - T_m$ respectively. In other words, matrices T_m, E_m, and F_m, are defined as following

$$T_m = \begin{bmatrix} a_{11} & \cdots & a_{1,m+1} & & \\ \vdots & \ddots & & \ddots & \\ a_{m+1} & & \ddots & & a_{n-m,n} \\ & \ddots & & \ddots & \vdots \\ & & a_{n,n-m} & \cdots & a_{nn} \end{bmatrix}, \quad E_m = \begin{bmatrix} & & & & \\ & & & & \\ -a_{m+2,1} & & & & \\ \vdots & \ddots & & & \\ -a_{n,1} & \cdots & -a_{n,n-m-1} & & \end{bmatrix}.$$

$$F_m = \begin{bmatrix} & & -a_{1,m+2} & \cdots & -a_{1,n} \\ & & & \ddots & \vdots \\ & & & & -a_{n-m-1,n} \\ & & & & \\ & & & & \end{bmatrix}$$

Now, similar to the classical AOR method its generalized version is defined as following

$$x^{(k+1)} = (T_m - \gamma E_m)^{-1}[(1-\omega)T_m + (\omega-\gamma)E_m + \omega F_m]x^{(k)} + \omega(T_m - \gamma E_m)^{-1}b. \quad (3)$$

Note that, if $m = 0$, then the GAOR iterative method results in the classical AOR method. As the AOR method, for some specific values of γ and ω the GAOR method reduces to GJ, GGS and the GSOR (generalized SOR) methods. In this section, we focus our attention on the GAOR method and refer the readers to (Salkuyeh, D.K., 2007) for more details about GJ and GGS methods. Evidently, results for the GAOR method also are valid for the GSOR method. Let

$$G_{AOR}(\gamma,\omega) = (D-\gamma E)^{-1}[(1-\omega)D + (\omega-\gamma)E + \omega F],$$
$$G_{AOR}^{(m)}(\gamma,\omega) = (T_m - \gamma E_m)^{-1}[(1-\omega)T_m + (\omega-\gamma)E_m + \omega F_m]$$

be the iteration matrix of the AOR and GAOR methods, respectively.

For convenience, some notations, definitions and results that will be used in the next section are given below. A matrix is called nonnegative, semi-positive and positive if each entry of A is nonnegative, nonnegative but at least a positive entry and positive, respectively. We denote them by $A \geq 0$, $A > 0$ and $A >> 0$. Similarly, for n- dimensional vectors, by identifying them with $n \times 1$ matrices, we can also define $x \geq 0$, $x > 0$ and $A >> 0$. A matrix A is said to be reducible if there exists a permutation matrix P such that

$$P^T AP = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix},$$

where X and Z are both square matrices. Otherwise, A is said to be irreducible. Additionally, we denote the spectral radius of A by $\rho(A)$.

Definition 1:

A matrix $A = (a_{ij})$ is said to be an M-matrix if $a_{ii} > 0$ for $i = 1,\cdots,n$, $a_{ij} \leq 0$, for $i \neq j$, A is nonsingular and $A^{-1} \geq 0$.

Theorem 1:

(Saad, Y., 1995) Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices such that $A \leq B$ and $b_{ij} \leq 0$ for all $i \neq j$.

Then, if A is an M-matrix, so is the matrix B.

Definition 2:

Let $A \in \Re^{n \times n}$. The splitting $A = M - N$ is called:

(a) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$;

(b) regular if $M^{-1} \geq 0$ and $N \geq 0$.

Theorem 2:

(Wang, L., Y. Song, 2009) Let $A \geq 0$ be an irreducible matrix. If $Ax \leq \alpha x$ for some $x > 0$, then $\rho(A) \leq \alpha$.

Theorem 3:

(Wang, L., Y. Song, 2009) If $A \geq 0$ is irreducible, then $\rho(A)$ is a simple eigenvalue and *A* has an eigenvector $x >> 0$ corresponding to $\rho(A)$.

Theorem 4:

(Wang, L., Y. Song, 2009) Let *A* be an M-matrix and $A = M - N$ be a regular or weak regular splitting of *A*. Then, $\rho(M^{-1}N) < 1$.

In the next section, the convergence properties of the GAOR method are studied.

3. Main Results:

Lemma 1:

Let *A* be an M-matrix and $A = T_m - E_m - F_m$ be the splitting defined as (2). Then T_m is an matrix and $\rho(T_m^{-1}E_m) < 1.$

Proof:

Let $S_m = T_m - E_m$. Obviously, we have $A \leq S_m$. Therefore, from Theorem 1, S_m is an M-matrix. Similarly, it is easy see that T_m is also an M-matrix. Hence $T_m^{-1} \geq 0$. On the other hand, $E_m \geq 0$. This shows that $S_m = T_m - E_m$ is a regular splitting. Now, by Theorem 4, $\rho(T_m^{-1}E_m) < 1$.

Theorem 5:

(Wu, M., L. Wang, Y. Song, 2007) If *A* is an M-matrix and $0 \leq \gamma \leq, \omega \leq 1$ with $\omega \neq 0,$ then the AOR iterative method is convergent, i.e., $\rho\left(G_{AOR}(\gamma, \omega)\right) < 1$.

Theorem 6:

If *A* is an M-matrix and $0 \leq \gamma \leq \omega \leq 1$ with $\omega \neq 0,$ then the GAOR method is convergent, i.e., $\rho\left(G_{GAOR}^{(m)}(\gamma, \omega)\right) < 1$.

Proof:

In the GAOR iterative method we have $A_m = M_m - N_m$ where $M_m = T_m - \gamma E_m,$ and $N_m = (1-\omega)T_m + (\omega - \gamma)E_m + \omega F_m.$ Obviously, we have $A \leq M_m$. Therefore, by Theorem 1, M_m is an M-matrix, and as a result $M_m^{-1} \geq 0.$ On the other hand, from Lemma 1 we have $\rho(T_m^{-1}E_m) < 1$. Since $0 \leq \gamma \leq 1$ we have $\rho(\gamma T_m^{-1}E_m) < 1$, and therefore

$$M_m^{-1}N_m = (T_m - \gamma E_m)^{-1}[(1-\omega)T_m + (\omega-\gamma)E_m + \omega F] = (I - \gamma T_m^{-1}E_m)^{-1}[(1-\omega)I + (\omega-\gamma)T_m^{-1}E_m + \omega T_m^{-1}F]$$

$$= \sum_{j=0}^{\infty} (\gamma T_m^{-1}E_m)^{-1}[(1-\omega)I + (\omega-\gamma)T_m^{-1}E_m + \omega T_m^{-1}F] \geq 0.$$

Hence, we conclude that $\omega A = M_m - N_m$ is a weak regular splitting of ωA.

Now, from Theorem 4, we observe that $\rho(M_m^{-1}N_m) < 1$ and this completes the proof.

Example 1:
Consider the M-matrix

$$A = \begin{bmatrix} 4 & -2 & -1 & -2 \\ -1 & 5 & -5 & -1 \\ -2 & -1 & 9 & -1 \\ -1 & -1 & -1 & 5 \end{bmatrix}$$

From Theorem 6, we have $\rho\left(G_{GAOR}^{(m)}(\gamma,\omega)\right) < 1$. For example, we have

$$\rho\left(G_{GAOR}^{(1)}(0.5,0.9)\right) = 0.6776 < 1 \quad , \quad \rho\left(G_{GAOR}^{(1)}(0.4,0.7)\right) = 0.7629 < 1 \ ,$$

$$\rho\left(G_{GAOR}^{(2)}(0.5,0.9)\right) = 0.5053 < 1 \quad , \quad \rho\left(G_{GAOR}^{(2)}(0.4,0.7)\right) = 0.6271 < 1 \ ,$$

These results show that Theorem 6 holds here.

Example 2:
Consider the matrix

$$A = \begin{bmatrix} 7 & -2 & 1 & 2 & 1 \\ 0 & 5 & -1 & 0 & -4 \\ 1 & 1 & 7 & -1 & -1 \\ 2 & -1 & 1 & 6 & 4 \\ 9 & -6 & 6 & 6 & 8 \end{bmatrix},$$

This matrix is not an M-matrix. Here we have

$$\rho\left(G_{GAOR}(0.6,0.8)\right) = 0.8450 \quad , \quad \rho\left(G_{GAOR}^{(1)}(0.6,0.8)\right) = 0.7721 \ ,$$

$$\rho\left(G_{GAOR}^{(2)}(0.6,0.8)\right) = 0.7907.$$

This example shows that, if $p > q$, then in general $\rho\left(G_{GAOR}^{(p)}(\gamma,\omega)\right)$ is not less than $\rho\left(G_{GAOR}^{(q)}(\gamma,\omega)\right)$

The next theorem shows that this is true in the special case.

Theorem 7:
Let *A* be an irreducible M-matrix, $0 \le \gamma \le \omega \le 1$, with $\omega \ne 0,$ and $m \ge p$. If $G_{GAOR}^{(p)}(\gamma,\omega)$ is an irreducible matrix, then

$$\rho\left(G_{GAOR}^{(m)}(\gamma,\omega)\right) \le \rho\left(G_{GAOR}^{(p)}(\gamma,\omega)\right) \le 1.$$

Proof:
For the sake of simplicity let $S_p = G_{GAOR}^{(p)}(\gamma,\omega)$ and $S_m = G_{GAOR}^{(m)}(\gamma,\omega)$. Similar to the proof of Theorem 6, we have $S_p \ge 0$. Since S_p is an irreducible matrix, from Theorem 3, $\lambda = \rho(S_p)$ is an eigenvalue of S_p corresponding to the eigenvector $x >> 0,$ i.e.,

$$S_p x = \lambda x. \tag{4}$$

From Theorem 6 we have $0 \le \lambda = \rho(S_p) < 1.$ Eq. (4) is equivalent to

$$[(1-\omega)T_p + (\omega-\gamma)E_p + \omega F_p]x = \lambda(T_p - \gamma E_p)x,$$

or

$$(\omega - \gamma + \lambda\gamma)E_p x + \omega F_p x = (\lambda + \omega - 1)T_p x \tag{5}$$

Now, we have

$$S_m x - \lambda x = (T_m - \gamma E_m)^{-1}[(1-\omega)T_m x + (\omega-\gamma)E_m x + \omega F_m x - \lambda T_m x - \lambda\gamma E_m x]$$
$$= (T_m - \gamma E_m)^{-1}[(1-\omega-\gamma)T_m x + (\omega-\gamma+\lambda\gamma)E_m x + \omega F_m x] \tag{6}$$

Evidently, $T_p - T_m \ge 0$. We split the matrix $T_p - T_m$ as $T_p - T_m = L_m + U_m$, where L_m and U_m are strictly lower and strictly upper triangular matrices, respectively. Note that $L_m, U_m \ge 0$. On the other hand, we have

$$E_p = E_m + L_m, \qquad F_p = F_m + U_m .$$

Therefore, form (5) and (6) we obtain

$$S_m x - \lambda x = (T_m - \gamma E_m)^{-1}[-(1-\omega-\lambda)(L_m + U_m) - (\omega-\gamma+\lambda\gamma)L_m - \omega U_m]x$$
$$= (T_m - \gamma E_m)^{-1}[-(1-\omega-\gamma+\omega-\gamma+\lambda\gamma)L_m - (1-\omega-\lambda+\omega)U_m]x$$
$$= (T_m - \gamma E_m)^{-1}[-((1-\lambda)-\gamma(1-\lambda))L_m - (1-\lambda)U_m]x$$
$$= (\lambda-1)(T_m - \gamma E_m)^{-1}[(1-\gamma)L_m + U_m]x.$$

From the proof of Theorem 6 we have $(T_m - \gamma E_m)^{-1} \ge 0.$ On the other hand, we have $0 \le \lambda < 1, 0 \le \gamma \le 1,$ and $L_m, U_m \ge 0$. Therefore, $S_m x - \lambda x \le 0.$ Now, from Theorem 2 we have $\rho(S_m) \le \lambda = \rho(S_p),$ and this completes the proof.

Remark 1:
Let *A* be an irreducible M-matrix, $0 \le \gamma \le \omega \le 1$ with $\omega \ne 0,$ and $m \ge 1$. Then

$$\rho\left(G_{GAOR}^{(m)}(\gamma,\omega)\right) \le \rho\left(G_{AOR}(\gamma,\omega)\right) \le 1.$$

Proof:
From Theorem 7 it is enough to show that the matrix $G_{AOR}(\gamma,\omega)$ is irreducible. This has been proved in Theorem 2.5 in (Wang, L., Y.Song, 2009).

Example 3:
Consider the matrix of the Example 1. This matrix is an M-matrix and we have

$$\rho\left(G_{GAOR}^{(2)}(0.5,0.9)\right) = 0.5053 < \rho\left(G_{GAOR}^{(1)}(0.5,0.9)\right) = 0.6776 < \rho\left(G_{AOR}(0.5,0.9)\right) = 0.8272 < 1,$$

$$\rho\left(G_{GAOR}^{(2)}(0.4,0.7)\right)=0.6271<\rho\left(G_{GAOR}^{(1)}(0.4,0.7)\right)=0.7629<\rho\left(G_{AOR}(0.4,0.7)\right)=0.8721<1,$$

These results show that Theorem 7 and Remark 1 hold here.

In the next section, we give some numerical experiments to show the effectiveness of the proposed method.

4. Numerical Experiments:

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer Pentium 4 - 256 MHz. In all the experiments, vector $b = A(1,1,\dots,1)^T$ was taken to be the right-hand side of the linear system and a null vector as an initial guess. The stopping criterion used was always $\|b-Ax_k\|_k / \|b\|_2 < 10^{-10}$, where x_k is the computed solution at step *k* of each method. We present two examples to compare the numerical results of the GAOR method with that of the AOR method.

Example 4:

In the this example, we consider the $n \times n$ banded matrix

$$A=\begin{bmatrix} 12.5 & -3 & -2 & -1 & & & & & \\ -3 & 12.5 & -3 & -2 & -1 & & & & \\ -2 & -3 & 12.5 & -3 & -2 & -1 & & & \\ -1 & -2 & -3 & 12.5 & -3 & -2 & -1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -1 & -2 & -3 & 12.5 & -3 & -2 & -1 \\ & & & -1 & -2 & -3 & 12.5 & -3 & -2 \\ & & & & -1 & -2 & -3 & 12.5 & -3 \\ & & & & & -1 & -2 & -3 & 12.5 \end{bmatrix}.$$

This matrix is strictly diagonally dominant with positive diagonal and nonpositive off-diagonal entries. Therefore, the matrix *A* is an M-matrix (Axelsson, O., 1996). We let $\gamma = 0.4$ and $\omega = 0.8$ Numerical results of the AOR and the GAOR iterative methods for m = 1,2, for different values of *n* (n = 25000, 50000, 75000, 100000) are given in Table 1. In each iteration of the GAOR method vector of the form $y=(T_m-\gamma E_m)^{-1}z$ should be computed. Hence, to compute *y* we may solve $(T_m-\gamma E_m)y=z$ for *y*. In the implementation of the GAOR method we used the LU factorization of $T_m-\gamma E_m$ to solve this system. Here, we mention that the LU factorization of $T_m-\gamma E_m$ is computed before starting the iterations of the GAOR method. In Table 1 the number of iterations of the method and the CPU time (in parenthesis) for convergence are given (timings are in seconds). The time for the GAOR method is the sum of the time for computing the LU factorization of $T_m-\gamma E_m$ and the time for the convergence. As we observe the GAOR method is more effective than the AOR method.

Example 5:

(Wang, L., Y. Song, 2009) We consider the two dimensional convection-diffusion equation

$$-(u_{xx}+u_{yy})+2e^{x+y}(xu_x+yu_y)=f(x,y), \qquad \text{in } \Omega=(0,1)\times(0,1),$$

with the homogeneous Dirichlet boundary conditions. Discretization of this equation on a $(p+1)\times(p+1)$ grid, by using the second order centered differences for the second and first order differentials gives a linear system of equations of order $n = p^2$ with n unknowns. As the previous example we consider the right hand side the system as $b = A(1,1,\dots,1)^T$. All of the assumptions are as Example 1. Assuming *p = 70, 80, 90, 100,* we obtain four systems of linear equations of dimension *n = 4900, 6400, 8100, 10000*. In Table 2 the numerical results obtained by the AOR and GAOR with γ = 0.5 and ω = 0.9 are reported. In the GAOR method we assumed m = 1. As we observe, for this example, the GAOR method reduces the number of the iterations of the AOR method for the convergence by a factor of two. Table 2 also shows the the CPU times of the GAOR method is slightly less that that of the AOR method.

Table 1: Numerical results of the AOR and GAOR for Example 4.

Method	*n* = 250000	*n* = 50000	*n* = 75000	*n* =100000
AOR	570 (8.48)	570 (17.06)	570 (27.53)	570 (36.78)
GAOR *m = 1*	294 (6.64)	294 (13.53)	294 (22.05)	294 (29.28)
m = 2	109 (2.56)	109 (5.13)	109 (8.34)	109 (11.08)

Table 2: Numerical results of the AOR and GAOR for Example 5.

Method	*n* = 4900	*n = 6400*	*n* = 8100	*n* = 10000
AOR	647 (2.02)	838 (3.48)	1052 (5.45)	1291 (8.52)
GAOR	330 (1.81)	425 (3.20)	532 (5.17)	651 (8.05)

Conclusion:

In this paper, we proposed a generalization of the AOR method say GAOR method and studied its convergence properties for M-matrices. We presented some numerical experiments to show the effectiveness of the proposed method. Numerical results show that the GAOR method is more effective than the AOR method.

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