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A New Approach for Computing Eigenvalues

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Abstract: In this paper, a new approach for computing eigenvalues of a square matrix based on purely elementary similarity operations is presented. The elementary operations are utilized in such a way as to transform the matrix first into an upper Hessenberg form and then to a Real Schur form. An algorithm for computing the eigenvalues of a 2×2 matrix by elementary similarity operations is presented. It is then possible to compute the real or complex eigenvalues by implementing this algorithm on each block of the Real Schur form. A numerical example is provided to illustrate the efficiency of the algorithm.

Key words: Numerical linear algebra, Eigenvalue problem, Similarity operations, Numerical methods.

INTRODUCTION

A very important topic in numerical linear algebra is obviously the eigenvalue problem; numerical methods of computing eigenvalues and eigenvectors. The computation of eigenvalues has a paramount importance, since if they are known; the eigenvectors can then be easily computed Datta, B.N., (1994).

In the algebraic eigenvalues/eigenvector problem for $A \in \Re^{n\times n}$, one seeks nonzero solution x∈ C which satisfy

$$Ax = \lambda x . \tag{1}$$

The classic reference on the numerical aspects of this problem is Wilkinson, (1965) with Parlett, (1980) providing an equally thorough and up-to-date treatment of the case of symmetric *A*. It is really only rather recently that some of the computational issues associated with solving (1) – in the presence of rounding error- have been resolved or understood. Even now some problems such as the invariant subspace problem continue to be active research areas.

The most common algorithm now used to solve (1) for general *A* is the *QR* algorithm of Francis, (1961). A shifting procedure (see Datta [3,p.441] for a brief explanation) is used to enhance convergence and the usual implementation is called double-Francis-*QR* algorithm.

One of the oldest methods of computing eigenvalues of symmetric matrices is related to Jacobi in 1846. This method was reinvestigated by Van Neumann in 1946. In 1954 Givens presented the bisection method for finding eigenvalues of a real symmetric matrix. Then an algorithm for finding the dominant eigenvalues of a given matrix called power method was introduced. In 1961 Francis, presented an iterative method called *QR* for computing all eigenvalues of a given matrix. In 1981 Cuppen, presented a method for computing eigenvalues of tri-diagonal symmetric matrices using parallel computation. Golub and Van Loan. (1989), Wilkinson, (1965), Barlow and Demmel, (1990) and many others have worked on eigenvalues and have published interesting papers. Also for symmetric matrices, other methods such as Sturm bisection sequence, Lanczos and Rayleigh Ritz methods have been developed (Datta, 1994; Gloub and Van Loan, 1989; Parlett, 1980).

In this letter, an algorithm based on merely elementary similarity operations is presented and it is shown that it works perfectly, especially for symmetric positive definite matrices.

2. Main Results:

Theorem 1:

If *A* is a 2×2 matrix, it can be transformed to Frobenius form by a fixed number of elementary similarity operations.

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Proof:

Consider a 2×2 matrix A in general form:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (2)$$

The following elementary operations will transform A into Frobenius companion from.

Row (2) ← Row (2)/a_{21}, ($a_{21} \neq 0$) (3)

Column (2) ← a_{21} × Column (2) (4)

Thus producing

$$A \leftarrow \begin{bmatrix} a_{11} & a_{12}^{(1)} \\ 1 & a_{22} \end{bmatrix} \quad (5)$$

Now performing

Column (2) ← Column (2) -a_{22} Column (1) (6)

Followed by

Row (1) ← Row (1) + a_{22} Row (2) (7)

Produces

$$A \leftarrow \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(2)} \\ 1 & 0 \end{bmatrix}. \quad (8)$$

Theorem 2:

If A is a 2×2 matrix and in Frobenius form, then it can always be transformed into a lower triangular form by elementary similarity operations from which its eigenvalues can be extracted.

Proof:

For simplicity suppose A is in the Frobenius form

$$A = \begin{bmatrix} -2b & -c \\ 1 & 0 \end{bmatrix} \quad (9)$$

The characteristic polynomial of this matrix is:

$$P(\lambda) = \lambda^2 + 2b\lambda + c \quad (10)$$

And its eigenvalues are:

$$\lambda_{1,2} = -b \pm \delta, \qquad where \quad \delta = \sqrt{b^2 - c} \quad (11)$$

Now we perform:

Row (1) ← Row (1) + b Row (2) (12)

followed by

Column (2) ← Column (2) - b Column (1) (13)

yielding :

$$A \leftarrow \begin{bmatrix} -b & b^2 - c \\ 1 & -b \end{bmatrix} = \begin{bmatrix} -b & \delta^2 \\ 1 & -b \end{bmatrix} \tag{14}$$

Now continuing

Row (1) ← Row (1) + δ Row (2) (15)

followed by:

Column (2) ← Column (2) - δ Column (1) (16)

results in:

$$A \leftarrow \begin{bmatrix} -b+\delta & \delta^2 - b\delta + b\delta - \delta^2 \\ 1 & -b-\delta \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \tag{17}$$

Obviously if $\delta<0$ then λ_1, λ_2 are complex conjugate and this operations takes care of complex computations.

For a given $n\times n$ matrix A, it is well known that starting similarity row operations from the bottom left-most corner followed by corresponding column operations, will transform it to an upper Hessenberg form Kincaid, D.R. and E.W.Cheney, (2002). It appears that further similarity operations ruin the Hessenberg form. This is true, but amazingly if the operations are repeated further, for most cases the matrix is transformed to Real Schur Form (RSF). Especially if the given matrix is symmetric and positive definite, it always works. Obviously when the matrix is reduced to RSF, then the eigenvalues are either on the main diagonal if they are real, or appear on 2×2 blocks which indicate the eigenvalues are complex. Further elementary operations as described in theorems 1 and 2 yields the complex conjugate eigenvalues. When the eigenvalues are known, then it is an easy matter to obtain the eigenvectors anyway Datta, (1994).

Following example illustrate the capability of this method.

3. Illustrative Examples:

Example 1:

Consider a simple case where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 4 & 10 & -1 \end{bmatrix}$$

A few steps of the algorithm for finding the eigenvalues of this matrix are annotated here. First we perform the similarity row operation

Row (3) ← Row (3) – 2 Row (2)

followed by the corresponding column operation:

Column (2) ← Column (2) +2 Column (3)

to produce the Hessenberg form:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

Now we intend to make the off diagonal elements zero; column pivoting reduces the round off error, so we first change row two with row one. That is:

Row (2) ← Row (1)

followed by

Column(1) ← Column(2)

producing:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Once again we reduce this to Hessenberg form by performing:

Row (3) ← Row (3) –Row (2)

followed by :

Column (2) ← Column (2) + Column (3)

resulting in:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

This is now in Real Schur Form. One of the eigenvalues is located at $A(3,3)$ and the other two can be found by performing similarity operations as in theorems 1 and 2. This will finally produce the block upper triangular matrix:

$$A = \begin{bmatrix} 3.7321 & 0 & -1 \\ 1 & 0.2679 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues are located on the main diagonal of the first block and are in fact 3.7321 and 0.2679 which are as obtained by MATLAB software. It must be pointed out that even if column pivoting is not performed and the similarity operations are repeated in order to transform A into a lower triangular form, the method works and after 10 iterations we will have (correct to four decimal places):

$$A = \begin{bmatrix} 3.7321 & -3.5638 & -42.4242 \\ 0.0000 & 1.0000 & 9.3444 \\ 0.0000 & 0.0000 & 0.2679 \end{bmatrix}$$

Clearly the eigenvalues are located on the main diagonal. Bigger full matrices were tested with the algorithm and satisfactory results were obtained.

Conclusion:

The purpose of this paper was to show how the eigenvalues of a square matrix can be computed merely by using row and column elementary similarity operations. It is shown that all the eigenvalues are produced at the same time. If the row operations are performed in a unit matrix, evidently the corresponding eigenvectors are produced as well. Further research is under way in order to modify this method as to embrace all kinds of matrices and also to compute the computational effort required.

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