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A New Method for Solving Fuzzy Volterra Integro-Differential Equations

[1]T. Allahviranloo, [2]A. Amirteimoori, [3]M. Khezerloo, [3]S. Khezerloo

[1]Department of Mathematics, Ardabil Branch, Islamic Azad University, Ardabil, Iran.
[2]Department of Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran.
[3]Islamic Azad University, Young Researcher Club, Ardabil Branch, Ardabil, Iran.

Abstract: In this paper, the LU-representation of fuzzy number is considered and based on the LU-representation of fuzzy volterra integro-differential equation (FVIDE) is discussed. The existence of solution of FVIDE is brought in details. Then, the solution is found in three cases of FVIDE that is concluded from the kernel. Finally, the method is illustrated by solving two examples.

Key words: Fuzzy integro-differential equations; Fuzzy solution; Fuzzy valued functions; LU-fuzzy representation;

INTRODUCTION

The fuzzy differential and integral equations are important part of the fuzzy analysis theory and they have the important value of theory and application in control theory.

Seikkala (1987) has defined the fuzzy derivative which is the generalization of the Hukuhara derivative in (Puri, M.L., D.A. Ralescu, 1983), the fuzzy integral which is the same as that of Dubois and Prade (1982), and by means of the extension principle of Zadeh, showed that the fuzzy initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0$$

has a unique fuzzy solution when f satisfies the generalized Lipschitz condition which guarantees a unique solution of the deterministic initial value problem. Kaleva (1990) studied the Cauchy problem of fuzzy differential equation, characterized those subsets of fuzzy sets in which the peano theorem is valid. Park *et al*. (1995) have considered the existence of solution of fuzzy integral equation in Banach space and Subrahmaniam and Sudarsanam (1994) have proved the existence of solution of fuzzy functional equations.

Rouhparvar *et al*. (2010) have discussed the existence and uniqueness of solution of the Cauchy reaction-diffusion equation by Adomian decomposition method and Abbasbandy and Allahviranloo (2006) have applied Adomian decomposition method for solving fuzzy systems of the socond kind.

Bede *et al* (2007) have introduced a more general definition of the derivative for fuzzy mappings, enlarging the class of differentiable. Park and Jeong (1999, 2000) studied existence of solution of fuzzy integral equations of the form

$$x(t) = f(t) + \int_0^t f(t, s, x(s))ds, \quad 0 \le t$$

where f and x are fuzzy functions and k is a crisp function on real numbers.
This paper is organized as following:

In Section 2, the basic concept of fuzzy number operation is brought. In Section 3, the main section of the paper, is discussed. The proposed idea are illustrated by some examples in the Section 4. Finally conclusion is drawn in Section 5.

2 Basic Concepts:

There are various definitions for the concept of fuzzy numbers ([3, 4]).

Definition 2.1: An arbitrary fuzzy number u in the parametric form is represented by an ordered pair of

Corresponding Author: T. Allahviranloo, Department of Mathematics, Ardabil Branch, Islamic Azad University, Ardabil, Iran.

functions (u_α^-, u_α^+) which satisfy the following requirements:

- u_α^- is a bounded left-continuous non-decreasing function over [0, 1].
- u_α^+ is a bounded left-continuous non-increasing function over [0, 1].
- $u_\alpha^- \le u_\alpha^+, \quad 0 \le \alpha \le 1$.

A crisp number r is simply represented by $u_\alpha^- = u_\alpha^+ = r, \quad 0 \le \alpha \le 1$. If $u_1^- < u_1^+$, we have a fuzzy interval and if $u_1^- = u_1^+$, we have a fuzzy number. In this paper, we do not distinguish between numbers or intervals and for simplicity we refer to fuzzy numbers as interval. We also use the notation $u_\alpha = [u_\alpha^-, u_\alpha^+]$ to denote the α-cut of arbitrary fuzzy number u. If $u = (u_\alpha^-, u_\alpha^+)$ and $v = (v_\alpha^-, v_\alpha^+)$ are two arbitrary fuzzy numbers, the arithmetic operations are defined as follows:

Definition 2.2 (Addition)

$$u + v = (u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+) \tag{1}$$

and in the terms of α-cuts

$$(u+v)_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+], \quad \alpha \in [0,1] \tag{2}$$

Definition 2.3 *(Subtraction)*

$$u - v = (u_\alpha^- + v_\alpha^+, u_\alpha^+ - v_\alpha^-) \tag{3}$$

and in the terms of α-cuts

$$(u-v)_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-], \quad \alpha \in [0,1] \tag{4}$$

Definition 2.4 *(Scalar multiplication)* For given $k \in \Re$

$$ku = \begin{cases} (ku_\alpha^-, ku_\alpha^+), & k > 0 \\ (ku_\alpha^+, ku_\alpha^-), & k < 0 \end{cases} \tag{5}$$

and

$$(ku)_\alpha = [\min\{ku_\alpha^-, ku_\alpha^+\}, \max\{ku_\alpha^-, ku_\alpha^+\}] \tag{6}$$

In particular, if k = 1, we have

$$-u = (-u_\alpha^+, -u_\alpha^-)$$

and with α-cuts

$$(-u)_\alpha = [-u_\alpha^+, -u_\alpha^-], \quad \alpha \in [0,1]$$

Definition 2.5 *(Multiplication)*

$$uv = ((uv)_\alpha^-, (uv)_\alpha^+) \tag{7}$$

and

$$(uv)^-_\alpha = \min\{u^-_\alpha v^-_\alpha, u^-_\alpha v^+_\alpha, u^+_\alpha v^-_\alpha, u^+_\alpha v^+_\alpha\}$$
$$(uv)^+_\alpha = \max\{u^-_\alpha v^-_\alpha, u^-_\alpha v^+_\alpha, u^+_\alpha v^-_\alpha, u^+_\alpha v^+_\alpha\}, \quad \alpha \in [0,1] \tag{8}$$

Definition 2.6 *(Division)*

If $0 \notin [v^-_0, v^+_0]$

$$\frac{u}{v} = ((\frac{u}{v})^-_\alpha, (\frac{u}{v})^+_\alpha) \tag{9}$$

and

$$(\frac{u}{v})^-_\alpha = \min\{\frac{u^-_\alpha}{v^-_\alpha}, \frac{u^-_\alpha}{v^+_\alpha}, \frac{u^+_\alpha}{v^-_\alpha}, \frac{u^+_\alpha}{v^+_\alpha}\}$$
$$(\frac{u}{v})^+_\alpha = \max\{\frac{u^-_\alpha}{v^-_\alpha}, \frac{u^-_\alpha}{v^+_\alpha}, \frac{u^+_\alpha}{v^-_\alpha}, \frac{u^+_\alpha}{v^+_\alpha}\}, \quad \alpha \in [0,1] \tag{10}$$

Definition 2.7 (Stefanini, L., 2006). LU-representation of an arbitrary fuzzy number *u* by a vector of 8 component of the interval [0, 1], with $N = 1$ (without internal points) and $\alpha_0 = 0$ and $\alpha_1 = 1$, is as follows:

$$u = (u^-_0, \delta u^-_0, u^+_0, \delta u^+_0, u^-_1, \delta u^-_1, u^+_1, \delta u^+_1) \tag{11}$$

where $u^-_0, \delta u^-_0, u^-_1, \delta u^-_1$ are used for the lower branch u^-_α , and $u^+_0, \delta u^+_0, u^+_1, \delta u^+_1$ are used for the upper branch u^+_α , by application of a monotonic interpolator on the whole interval $\alpha \in [0,1]$.

In particular, the slops corresponding to u^-_i are denoted by δu^-_i , etc. By definition (2.1), it is clear that $\delta u^-_0 \geq 0$, $\delta u^-_1 \geq 0$, $\delta u^+_0 \leq 0$ and $\delta u^+_1 \leq 0$. For an arbitrary trapezoidal fuzzy number *u*, we have $\delta u^-_0 = \delta u^-_1, \quad \delta u^+_0 = \delta u^+_1$

and if $u^-_1 = u^+_1$, then *u* is an triangular fuzzy number.

As reported in (Stefanini, L., 2006), using LU-representation (11), the membership function $\mu(x)$ of the LU-fuzzy number *u* is obtained by

$$\mu(x) = \sup\{\alpha \mid x \in [u^-_\alpha, u^+_\alpha]\} \tag{12}$$

In particular, corresponding to the nodes of the α-cuts, we have

$$\mu(u^-_i) = \mu(u^+_i) = \alpha_i, \qquad i = 0,1,\ldots,N$$

and, in the differentiable case

$$\mu'(u^-_i) = \frac{1}{\delta u^-_i}, \quad \mu'(u^+_i) = \frac{1}{\delta u^+_i}, \quad i = 0,1,\ldots,N$$

The membership function $\mu(x)$ can be approximated by the use of monotonic spline interpolation (see (Guerra, M.L., L. Stefanini, 2005)).
In this paper, we will suppose the differentiable case, for which we use the representation

$$u = (u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,1,\ldots,N} \tag{13}$$

with the data

$$u_0^- \le u_1^- \le \cdots \le u_N^- \le u_N^+ \le u_{N-1}^+ \le \cdots \le u_0^+ \tag{14}$$

and the slops

$$\delta u_i^- \ge 0, \qquad \delta u_i^+ \le 0 \tag{15}$$

Stefanini *et al*. (2006) defined corresponding spaces of fuzzy numbers by the LU-representation. In the differentiable case, they denoted by

$$\hat{F}_N = \{u \mid u = (u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,1,\ldots,N}\}$$

the set of LU-fuzzy numbers. $\hat{F}_N$ is a $4(N+1)-$ dimension space.

Let u and v are two LU-fuzzy numbers in form

$$u = (u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,1,\ldots,N}, \quad v = (v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,1,\ldots,N}$$

An Euclidean-like distance on $\hat{F}_N$ was defined by

$$d_N(u,v) = \frac{\|u-v\|_N}{4N}$$

where

$$\|u-v\|_N^2 = \sum_{i=0}^{N}\left[(u_i^- - v_i^-)^2 + (u_i^+ - v_i^+)^2 + (\delta u_i^- - \delta v_i^-)^2 + (\delta u_i^+ - \delta v_i^+)^2\right] \tag{16}$$

It is worthwhile to observe that also the one-sided fuzzy numbers can be easily represented by a form similar to the LU-representation: a left-sided fuzzy number has α-cuts of the form $[u_\alpha^-, +\infty[$ and a right-sided fuzzy number has α-cuts of the form $]-\infty, u_\alpha^+]$. Then, a left-sided fuzzy number can be written as $u = (u_i^-, \delta u_i^-)_{i=0,1,\ldots,N}$ or a right-sided fuzzy number as $u = (u_i^+, \delta u_i^+)_{i=0,1,\ldots,N}$ (see (Stefanini, L., 2006)).

The arithmetic operators associated to the LU-representation can be obtained as follows.

Definition 2.8 (Addition), (Stefanini, L., 2006).

$$u + v = (u_i^- + v_i^-, \delta u_i^- + \delta v_i^-, u_i^+ + v_i^+ 0, \delta u_i^+ + \delta v_i^+)_{i=0,1,\ldots,N} \tag{17}$$

Definition 2.9 (Scalar multiplication), (Stefanini, L., 2006). For given $k \in \Re$

$$ku = \begin{cases} (ku_i^-, k\delta u_i^-, ku_i^+, k\delta u_i^+)_{i=0,1,\ldots,N}, & k > 0 \\ (ku_i^+, k\delta u_i^+, ku_i^-, k\delta u_i^-)_{i=0,1,\ldots,N}, & k < 0 \end{cases} \tag{18}$$

In particular, if $k = -1$ then

$$-u = (-u_i^+, -\delta u_i^+, -u_i^-, -\delta u_i^-)_{i=0,1,\ldots,N}$$

and subtraction is defined by

$$u - v = u + (-v)$$

Note that the scalar multiplication is always reproduced exactly in all the models for all $\alpha \in [0,1]$ but, in general, this is not true for the addition as the sum of rational or mixed functions is not always a rational or a mixed function of the same orders (see (Stefanini, L., 2006)).

Integrals and derivatives of fuzzy-valued functions have discussed by Dubois and Prade (1982), Kaleva (1987, 1990) and Puri and Ralescu (1983); see also (Kim, Y.L., 1997) for some recent results.

Let $u:[a,b] \to \hat{F}_N$ is a function where $u(t) = (u^-(t), u^+(t))$ for $t \in [a,b]$ is an LU fuzzy number of the form

$$u(t) = (u_i^-(t), \delta u_i^-(t), u_i^+(t), \delta u_i^+(t))_{i=0,1,\ldots,N}$$

the integral of $u(t)$ with respect to $t \in [a,b]$. is given by

$$v_\alpha := \left(\int_a^b u(t)dt\right) = \left[\int_a^b u_\alpha^-(t)dt, \int_a^b u_\alpha^+(t)dt\right], \quad \alpha \in [0,1]$$

Definition 2.10 (Integration), (Stefanini, L., 2006). In the LU-fuzzy representation, the integral of $u(t)$ is defined as

$$v = (v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,1,\ldots,N}$$

where

$$v_i^\pm = \int_a^b u_i^\pm(t)dt, \qquad \delta v_i^\pm = \int_a^b \delta u_i^\pm(t)dt, \qquad i = 0,1,\ldots,N$$

The (Hukuhara) derivative of the fuzzy-valued function $u(t)$ at the point $\hat{t}$ is as follows:

$$(u^{'}(\hat{t}))_\alpha = \left[\frac{d}{dt}u_\alpha^-(t)\bigg|_{t=\hat{t}}, \frac{d}{dt}u_\alpha^+(t)\bigg|_{t=\hat{t}}\right]$$

provided that the following conditions hold:

- each $\dfrac{d}{dt}u_\alpha^-(t)$ is nondecreasing with respect to $t \in [a,b]$.
- each $\dfrac{d}{dt}u_\alpha^+(t)$ is nondecreasing with respect to $t \in [a,b]$.
- $\dfrac{d}{dt}u_\alpha^-(t) \le \dfrac{d}{dt}u_\alpha^+(t), \quad \forall \alpha \in [0,1]$

Definition 2.11 (Derivation), (Stefanini, L., 2006). In the LU-fuzzy representation, the derivative of $u(t)$ at the point $\hat{t}$ is defined as

$$u'(\hat{t}) = (u_i^{'-}(\hat{t}), \delta u_i^{'-}(\hat{t}), u_i^{'+}(\hat{t}), \delta u_i^{'+}(\hat{t}))_{i=0,1,\ldots,N}$$

and the conditions for a valid fuzzy derivative are, for $i = 0,1,\ldots,N$

$$\begin{aligned} u_0^{'-}(\hat{t}) \quad & \le u_1^{'-}(\hat{t}) \le \cdots \le u_N^{'-}(\hat{t}) \\ & \le u_N^{'+}(\hat{t}) \le u_{N-1}^{'+}(\hat{t}) \le \cdots \le u_0^{'+}(\hat{t}), \\ \delta u_i^{'-}(\hat{t}) \quad & \ge 0 \\ \delta u_i^{'+}(\hat{t}) \quad & \le 0 \end{aligned}$$

3 Fuzzy Volterra Integro-differential Equations:

In this section, we consider the fuzzy Volterra integro-differential equation

$$\begin{cases} u'(x) = f(x) + \int_0^x k(x,t)u(t)dt, \\ u(0) = c \in \hat{F}_N \end{cases} \tag{19}$$

where $f:[0,a] \to \hat{F}_N$ and $k:\Delta \to \Re$, where $\Delta = \{(x,t): 0 \le t \le x \le a\}$ and $u:[0,a] \to \hat{F}_N$ are continuous.

Let $c = (c_i^-, \delta c_i^-, c_i^+, \delta c_i^+)_{i=0,1,\ldots,N}$ be the LU-fuzzy initial condition and let *u(x)* and *f(x)* have the following forms, respectively,

$$\begin{aligned} u(x) &= (u_i^-(x), \delta u_i^-(x), u_i^+(x), \delta u_i^+(x))_{i=0,1,\ldots,N}, \\ f(x) &= (f_i^-(x), \delta f_i^-(x), f_i^+(x), \delta f_i^+(x))_{i=0,1,\ldots,N} \end{aligned}$$

then Eq. (19) can replace by

$$\begin{cases} \dfrac{d}{dx}(u_i^-(x), \delta u_i^-(x), u_i^+(x), \delta u_i^+(x)) & = (f_i^-(x), \delta f_i^-(x), f_i^+(x), \delta f_i^+(x)) \\ & + \int_0^x k(x,t)(u_i^-(t), \delta u_i^-(t), u_i^+(t), \delta u_i^+(t))dt \\ (u_i^-(0), \delta u_i^-(0), u_i^+(0), \delta u_i^+(0)) & = (c_i^-, \delta c_i^-, c_i^+, \delta c_i^+) \end{cases} \tag{20}$$

Case 1. Suppose $k(x,t) \ge 0$ then

$$\begin{aligned} \int_0^x k(x,t) \quad & (u_i^-(t), \delta u_i^-(t), u_i^+(t), \delta u_i^+(t))dt \\ & = \int_0^x (k(x,t)u_i^-(t), k(x,t)\delta u_i^-(t), k(x,t)u_i^+(t), k(x,t)\delta u_i^+(t))dt \\ & = (\int_0^x k(x,t)u_i^-(t)dt, \int_0^x k(x,t)\delta u_i^-(t)dt, \int_0^x k(x,t)u_i^+(t)dt, \int_0^x k(x,t)\delta u_i^+(t)dt) \end{aligned} \tag{21}$$

we can calculate $u(x)$ by solving the following $2(N+1)$ integro-differential equations (IDEs), for $i := 0,1,\ldots,N$

$$\begin{aligned}
&\frac{d}{dx}u_i^-(x) = f_i^-(x) + \int_0^x k(x,t)u_i^-(t)dt \\
&\frac{d}{dx}u_i^+(x) = f_i^+(x) + \int_0^x k(x,t)u_i^+(t)dt \\
&u_i^\pm(0) = c_i^\pm
\end{aligned} \tag{22}$$

To determine the corresponding slopes we add to (22) the following $2(N+1)$ IDEs, for $i = 0,1,\ldots,N$:

$$\begin{aligned}
&\frac{d}{dx}\delta u_i^-(x) = \delta f_i^-(x) + \int_0^x k(x,t)\delta u_i^-(t)dt \\
&\frac{d}{dx}\delta u_i^+(x) = \delta f_i^+(x) + \int_0^x k(x,t)\delta u_i^+(t)dt \\
&\delta u_i^\pm(0) = \delta c_i^\pm
\end{aligned} \tag{23}$$

Case 2. Suppose $k(x,t) < 0$ then

$$\begin{aligned}
\int_0^x k(x,t) \quad & (u_i^-(t), \delta u_i^-(t), u_i^+(t), \delta u_i^+(t))dt \\
&= \int_0^x (k(x,t)u_i^+(t), k(x,t)\delta u_i^+(t), k(x,t)u_i^-(t), k(x,t)\delta u_i^-(t))dt \\
&= (\int_0^x k(x,t)u_i^+(t)dt, \int_0^x k(x,t)\delta u_i^+(t)dt, \int_0^x k(x,t)u_i^-(t)dt, \int_0^x k(x,t)\delta u_i^-(t)dt)
\end{aligned} \tag{24}$$

we can calculate $u(x)$ by solving the following $2(N+1)$ integro-differential equations (IDEs), for $i = 0,1,\ldots,N$:

$$\begin{aligned}
&\frac{d}{dx}u_i^-(x) = f_i^-(x) + \int_0^x k(x,t)u_i^+(t)dt \\
&\frac{d}{dx}u_i^+(x) = f_i^+(x) + \int_0^x k(x,t)u_i^-(t)dt \\
&u_i^\pm(0) = c_i^\pm
\end{aligned} \tag{25}$$

and to determine the corresponding slopes we add to (25) the following $2(N+1)$ IDEs, for $i = 0,1,\ldots,N$:

$$\begin{aligned}
&\frac{d}{dx}\delta u_i^-(x) = \delta f_i^-(x) + \int_0^x k(x,t)\delta u_i^+(t)dt \\
&\frac{d}{dx}\delta u_i^+(x) = \delta f_i^+(x) + \int_0^x k(x,t)\delta u_i^-(t)dt \\
&\delta u_i^\pm(0) = \delta c_i^\pm
\end{aligned} \tag{26}$$

Case 3. Suppose $k(x,t)$ has not constant sign on $[0,x]$, then

$$\begin{cases}
\frac{d}{dx}(u_i^-(x), \delta u_i^-(x), u_i^+(x), \delta u_i^+(x)) & = (f_i^-(x), \delta f_i^-(x), f_i^+(x), \delta f_i^+(x)) \\
& \quad + \int_0^x K(x,t,u(t))dt \\
(u_i^-(0), \delta u_i^-(0), u_i^+(0), \delta u_i^+(0)) & = (c_i^-, \delta c_i^-, c_i^+, \delta c_i^+)
\end{cases} \tag{27}$$

where

$$K(x,t,u(t))=\begin{cases}(k(x,t)u_i^-(t),k(x,t)\delta u_i^-(t),k(x,t)u_i^+(t),k(x,t)\delta u_i^+(t)), & k(x,t)\geq 0\\ (k(x,t)u_i^+(t),k(x,t)\delta u_i^+(t),k(x,t)u_i^-(t),k(x,t)\delta u_i^-(t)), & k(x,t)<0\end{cases} \tag{28}$$

So, if $k(x,t)\geq 0$ on $[0,a]$ and $k(x,t)<0$ on $[a,x]$, $a\leq x$ then

$$\begin{aligned}&\frac{d}{dx}u_i^-(x)= f_i^-(x)+\int_0^a k(x,t)u_i^-(t)dt+\int_a^x k(x,t)u_i^+(t)dt\\&\frac{d}{dx}u_i^+(x)= f_i^+(x)+\int_0^a k(x,t)u_i^+(t)dt+\int_a^x k(x,t)u_i^-(t)dt\\&u_i^\pm(0)=c_i^\pm\end{aligned} \tag{29}$$

and

$$\begin{aligned}&\frac{d}{dx}\delta u_i^-(x)=\delta f_i^-(x)+\int_0^a k(x,t)\delta u_i^-(t)dt+\int_a^x k(x,t)\delta u_i^+(t)dt\\&\frac{d}{dx}\delta u_i^+(x)=\delta f_i^+(x)+\int_0^a k(x,t)\delta u_i^+(t)dt+\int_a^x k(x,t)\delta u_i^-(t)dt\\&\delta u_i^\pm(0)=\delta c_i^\pm\end{aligned} \tag{30}$$

Theorem 3.1 The solution of (20), $u(x)$, is a fuzzy number for all $x\geq 0$.
Proof: In Eq. (19), we suppose

$$F(x,u(x))= f(x)+G(x,u(x))$$

where

$$G(x,u(x))=\int_0^x k(x,t)u(t)dt$$

then we can rewrite Eq. (19) as follows:

$$\begin{cases} u'(x)=F(x,u(x)),\\ u(0)=c\in \hat{F}_N\end{cases}$$

where *F* is continuous mapping. Using Theorem (3.1) in (Seikkala, 1987), it is clear that Eq. (19) has unique fuzzy solution *u(x)* then Eq. (20) has unique fuzzy solution *u(x)*, i.e.

$$u(x)\in \hat{F}_N, \qquad x\geq 0.$$

4 Numerical Examples:
Example 4.1: Consider FVIDE

$$\begin{cases}\dfrac{d}{dx}\quad (u_0^-(x),\delta u_0^-(x),u_1^-(x),\delta u_1^-(x),u_0^+(x),\delta u_0^+(x),u_1^+(x),\delta u_1^+(x))_{i=0,1}\\ \quad =(e^{-x}+xe^{x})(-1,1,0,1,1,-1,0,-1)_{i=0,1}\\ \quad +\int_0^x e^{x+t}(u_0^-(x),\delta u_0^-(x),u_1^-(x),\delta u_1^-(x),u_0^+(x),\delta u_0^+(x),u_1^+(x),\delta u_1^+(x))_{i=0,1}dt\\ \quad (u_0^-(0),\delta u_0^-(0),u_1^-(0),\delta u_1^-(0),u_0^+(0),\delta u_0^+(0),u_1^+(0),\delta u_1^+(0))=(-1,1,0,1,1,-1,0,-1)_{i=0,1}\end{cases}$$

Then,

$$\begin{cases} \dfrac{d}{dx}u_0^-(x) = -1(e^{-x}+xe^x)+e^x\int_0^x e^t u_0^-(t)dt, & u_0^-(0)=-1 \\ \dfrac{d}{dx}u_0^+(x) = (e^{-x}+xe^x)+e^x\int_0^x e^t u_0^+(t)dt, & u_0^+(0)=1 \\ \dfrac{d}{dx}u_1^-(x) = e^x\int_0^x e^t u_1^-(t)dt, & u_1^-(0)=0 \\ \dfrac{d}{dx}u_1^+(x) = e^x\int_0^x e^t u_1^+(t)dt, & u_1^+(0)=0 \end{cases}$$

and

$$\begin{cases} \dfrac{d}{dx}\delta u_0^-(x) = (e^{-x}+xe^x)+e^x\int_0^x e^t \delta u_0^-(t)dt, & \delta u_0^-(0)=1 \\ \dfrac{d}{dx}\delta u_0^+(x) = -1(e^{-x}+xe^x)+e^x\int_0^x e^t \delta u_0^+(t)dt, & \delta u_0^-(0)=-1 \\ \dfrac{d}{dx}\delta u_1^-(x) = (e^{-x}+xe^x)+e^x\int_0^x e^t \delta u_1^-(t)dt, & \delta u_1^-(0)=1 \\ \dfrac{d}{dx}\delta u_1^+(x) = -1(e^{-x}+xe^x)+e^x\int_0^x e^t \delta u_1^+(t)dt, & \delta u_1^-(0)=-1 \end{cases}$$

By solving above systems, we obtain

$u(x) = e^{-x}(-1,1,0,1,1,-1,0,-1)_{i=0,1}$

Example 4.2 *Consider FVIDE*

$$\begin{cases} \dfrac{d}{dx}\quad (u_0^-(x),\delta u_0^-(x),u_1^-(x),\delta u_1^-(x),u_0^+(x),\delta u_0^+(x),u_1^+(x),\delta u_1^+(x))_{i=0,1} \\ \quad = x(0,2,2,2,4,-2,2,-2)_{i=0,1}+\dfrac{1}{3}x^4(0,1,1,1,2,-1,1,-1)_{i=0,1} \\ \quad +\int_0^x(-x)(u_0^-(x),\delta u_0^-(x),u_1^-(x),\delta u_1^-(x),u_0^+(x),\delta u_0^+(x),u_1^+(x),\delta u_1^+(x))_{i=0,1}dt \\ \quad (u_0^-(0),\delta u_0^-(0),u_1^-(0),\delta u_1^-(0),u_0^+(0),\delta u_0^+(0),u_1^+(0),\delta u_1^+(0)) = (0,1,1,1,2,-1,1,-1)_{i=0,1} \end{cases}$$

Then,

$$\begin{cases} \dfrac{d}{dx}u_0^-(x) = -x\int_0^x u_0^-(t)dt, & u_0^-(0)=0 \\ \dfrac{d}{dx}u_0^+(x) = 4x+\dfrac{2}{3}x^4-x\int_0^x u_0^+(t)dt, & u_0^+(0)=2 \\ \dfrac{d}{dx}u_1^-(x) = 2x+\dfrac{1}{3}x^4-x\int_0^x u_1^-(t)dt, & u_1^-(0)=1 \\ \dfrac{d}{dx}u_1^+(x) = 2x+\dfrac{1}{3}x^4-x\int_0^x u_1^+(t)dt, & u_1^+(0)=1 \end{cases}$$

and

$$\begin{cases} \dfrac{d}{dx}\delta u_0^-(x) = 2x + \dfrac{1}{3}x^4 - x\displaystyle\int_0^x \delta u_0^-(t)dt, & \delta u_0^-(0) = 1 \\ \dfrac{d}{dx}\delta u_0^+(x) = -2x - \dfrac{1}{3}x^4 - x\displaystyle\int_0^x \delta u_0^+(t)dt, & \delta u_0^-(0) = -1 \\ \dfrac{d}{dx}\delta u_1^-(x) = 2x + \dfrac{1}{3}x^4 - x\displaystyle\int_0^x \delta u_1^-(t)dt, & \delta u_1^-(0) = 1 \\ \dfrac{d}{dx}\delta u_1^+(x) = -2x - \dfrac{1}{3}x^4 - x\displaystyle\int_0^x e^t \delta u_1^+(t)dt, & \delta u_1^-(0) = -1 \end{cases}$$

By solving above systems, we obtain

$$u(x) = x^2(0,1,1,1,2,-1,1,-1)_{i=0,1}$$

5 Conclusion:

In this work, the LU-representation of FVIDE was discussed and also the existence of solutions is proved as a theorem. The structure of LU-representation of FVIDE is comparison with the others for example α-cut and parametric forms is simple.

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