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Solution of m×3 or 3×n Rectangular Interval Games using Graphical Method

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Abstract: A new approach for solving m×3 or 3×n rectangular games based on imprecise number instead of a real number such as interval number takes into account is introduced. To reduce m×3 or 3×n rectangular interval game into simpler 3×3 interval game, it is introduced here the graphical method which works well in interval numbers under consideration. The interval graphical method is a state of the art technique to deal with the games to an inexact environment. The approach is illustrated by numerical examples showing that a m×3 or 3×n interval games can be reduced to 3×3 interval games.

Key words: Crisp game; Interval numbers; Interval games.

INTRODUCTION

The theory of fuzzy sets, proposed by Zadeh (1965), has gained successful applications in various fields. Interval game (IG) theory which is a special case of fuzzy game theory, is an important content in interval fuzzy mathematics. IG theory has been widely applied to manufacturing company, economics, management etc., where three decision makers are present or three possible conflicting objectives should be taken into account in order to reach optimality. In (Narayan, A.L., 2002), it is shown that how crisp game can be fuzzified and how 3×3 sub games involving interval number can be solved by different approaches, mainly interval number ranking process. This approach is suitable when each player has chosen a procedure such that it is always possible to give the maximum or minimum of three interval numbers. But reduction a rectangular m×3 or 3×n IG without saddle point to a 3×3 interval sub-game, is the basic problem in 'Interval Game theory'. In dominance method (Nayak, P.K., 2003), if the convex combination of any three rows (columns) of a pay-off matrix is dominated by the fourth row (column) which indicates that the fourth move of the row (column) player will be an optimal move but It is not certain which one of the first three moves will be an optimal one. To determine it the graphical method can be used. The purpose of this paper is to introduce a graphical method for reducing an m×3 or 3×n IG to 3×3 interval sub games.

Three Person Interval Game:

Interval game theory deals with making decisions under conflict caused by opposing interests. An IG involving 3 players is called a 3 person IG. Here A is maximizing (row) or optimistic player and B is the minimizing (column) or pessimistic player. Here it is considered a three persons non-zero sum IG with single pay-off matrix. It is assumed that A is maximization (optimistic) player (row player) i.e., he/she will try to make maximum profit and B is minimization (pessimistic) player (column player) i.e., he/she will try to minimize the loss.

Pay-off Matrix:

The IG can be considered as a natural extersion of classical game. The pay-off of the game are affected by various sources of fuzziness. The course of the IG is determined by the desire of A to maximize his/her gain and that of restrict his/her loss to a minimum. Interval number ranking process is suitable when each player has chosen a procedure such that it is always possible to give the maximum or minimum of three interval numbers. The table showing how payments should be made at the end of the game is called a pay-off matrix.

If the player A has *m* strategies available to him and the player B has *n* strategies available to him, then the pay-off for various strategies combinations is represented by an m×n pay-off matrix. The pay-off is considered as interval numbers. The pay-off matrix can be written in the matrix form as

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$$\begin{array}{c} \\ A_1 \\ A_2 \\ \cdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{pmatrix} [a_{11},b_{11},c_{11}] & [a_{12},b_{12},c_{12}] & \cdots & [a_{1n},b_{1n},c_{1n}] \\ [a_{21},b_{21},c_{21}] & [a_{22},b_{22},c_{22}] & \cdots & [a_{2n},b_{2n},c_{2n}] \\ \cdots & \cdots & \cdots & \cdots \\ [a_{m1},b_{m1},c_{m1}] & [a_{m2},b_{m2},c_{m2}] & \cdots & [a_{mn},b_{mn},c_{mn}] \end{pmatrix} \end{array}$$

Intervals of real numbers enable on to describe the uncertainty about the actual values of the numerical variable. Here it is assumed that when player A chooses the starategy A_i and the player B selects strategy B_j it results in a pay-off of the closed interval $[a_{ij}, b_{ij}, c_{ij}]$ to the player A with $c_{ij} - a_{ij} - b_{ij} > 0$. Intervals do not admit of comparison among themselves. Here it is provided an order relation for the entries in the pay-off matrix, so that comparison of intervals is possible.

Theorem 1:

Let $A=([a_{ij}, b_{ij}, c_{ij}])$; $i=1,2,\ldots m$; $j=1,2,\ldots n$ be an $m\times n$ pay-off matrix for a 3 person IG, where $[a_{ij}, b_{ij}, c_{ij}]$ are interval numbers. Then the following inequality is satisfied

$$\vee_i \left\{ \wedge_j [a_{ij}, b_{ij}, c_{ij}] \right\} \le \wedge_j \left\{ \vee_i [a_{ij}, b_{ij}, c_{ij}] \right\}$$

Definition1. Saddle Point:

The concept of saddle point is introduced by Neumann (1947). The (k,r)th position of the pay-off matrix will be called a saddle point, if and only if,

$$[a_{kr}, b_{kr}, c_{kr}] = \vee_i \left\{ \wedge_j [a_{ij}, b_{ij}, c_{ij}] \right\} = \wedge_j \left\{ \vee_i [a_{ij}, b_{ij}, c_{ij}] \right\}$$

IG Without Saddle Point:

There are interval games having no saddle point. Consider a simple 3×3 IG with no saddle point with the pay-off matrix

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{pmatrix} [a_{11},b_{11},c_{11}] & [a_{12},b_{12},c_{12}] & [a_{13},b_{13},c_{13}] \\ [a_{21},b_{21},c_{21}] & [a_{22},b_{22},c_{22}] & [a_{23},b_{23},c_{23}] \\ [a_{31},b_{31},c_{31}] & [a_{32},b_{32},c_{32}] & [a_{33},b_{33},c_{33}] \end{pmatrix} \end{array}$$

Where $\vee_i \left\{ \wedge_j [a_{ij}, b_{ij}, c_{ij}] \right\} \ne \wedge_j \left\{ \vee_i [a_{ij}, b_{ij}, c_{ij}] \right\}$. In such IG the principle of saddle point solution breaks down and the players do not have a single best plan as their best strategy. The IG in this case is said to be unstable.

To solve such IG, Neumann (1947) introduced the concept of mixed strategy in classical form. In all such cases to solve games, both the players must determine an optimal mixture of strategies to find a saddle point. An example of 3×3 IG without saddle point is

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{pmatrix} [-1,1,2] & [1,6,8] & [0,4,6] \\ [0,3,5] & [-2,0,1] & [-1,2,3] \\ [-4,-1,1] & [0,4,3] & [1,5,7] \end{pmatrix} \end{array}$$

Because $\vee_i \left\{ \wedge_j [a_{ij}, b_{ij}, c_{ij}] \right\} = [-1,1,2] \ne [0,3,5] = \wedge_j \left\{ \vee_i [a_{ij}, b_{ij}, c_{ij}] \right\}$

Solution of IG Without Saddle Point:

Consider the following IG with the given pay-off matrix for which there is no saddle point

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{pmatrix} [\alpha_{11},\beta_{11},\gamma_{11}] & [\alpha_{12},\beta_{12},\gamma_{12}] & [\alpha_{13},\beta_{13},\gamma_{13}] \\ [\alpha_{21},\beta_{21},\gamma_{21}] & [\alpha_{22},\beta_{22},\gamma_{22}] & [\alpha_{23},\beta_{23},\gamma_{23}] \\ [\alpha_{31},\beta_{31},\gamma_{31}] & [\alpha_{32},\beta_{32},\gamma_{32}] & [\alpha_{33},\beta_{33},\gamma_{33}] \end{pmatrix} \end{array}$$

Where $\underset{i}{\vee}\left\{\underset{j}{\wedge}[\alpha_{ij},\beta_{ij},\gamma_{ij}]\right\} \neq \underset{j}{\wedge}\left\{\underset{i}{\vee}[\alpha_{ij},\beta_{ij},\gamma_{ij}]\right\}$. Let $\lambda_{ij}= c_{ij} - a_{ij} - b_{ij} > 0$ for i=1,2 and j=1,2.

The normalized pay-off matrix is then

$$\begin{pmatrix} [\frac{\alpha_{11}}{\lambda_{11}},\frac{\beta_{11}}{\lambda_{11}},\frac{\gamma_{11}}{\lambda_{11}}] & [\frac{\alpha_{12}}{\lambda_{12}},\frac{\beta_{12}}{\lambda_{12}},\frac{\gamma_{12}}{\lambda_{12}}] & [\frac{\alpha_{13}}{\lambda_{13}},\frac{\beta_{13}}{\lambda_{13}},\frac{\gamma_{13}}{\lambda_{13}}] \\ [\frac{\alpha_{21}}{\lambda_{21}},\frac{\beta_{21}}{\lambda_{21}},\frac{\gamma_{21}}{\lambda_{21}}] & [\frac{\alpha_{22}}{\lambda_{22}},\frac{\beta_{22}}{\lambda_{22}},\frac{\gamma_{22}}{\lambda_{22}}] & [\frac{\alpha_{23}}{\lambda_{23}},\frac{\beta_{23}}{\lambda_{23}},\frac{\gamma_{23}}{\lambda_{23}}] \\ [\frac{\alpha_{31}}{\lambda_{31}},\frac{\beta_{31}}{\lambda_{31}},\frac{\gamma_{31}}{\lambda_{31}}] & [\frac{\alpha_{32}}{\lambda_{32}},\frac{\beta_{32}}{\lambda_{32}},\frac{\gamma_{32}}{\lambda_{32}}] & [\frac{\alpha_{33}}{\lambda_{33}},\frac{\beta_{33}}{\lambda_{33}},\frac{\gamma_{33}}{\lambda_{33}}] \end{pmatrix} = \begin{pmatrix} [a_{11},b_{11},c_{11}] & [a_{12},b_{12},c_{12}] & [a_{13},b_{13},c_{13}] \\ [a_{21},b_{21},c_{21}] & [a_{22},b_{22},c_{22}] & [a_{23},b_{23},c_{23}] \\ [a_{31},b_{31},c_{31}] & [a_{32},b_{32},c_{32}] & [a_{33},b_{33},c_{33}] \end{pmatrix}$$

Where $\underset{i}{\vee}\left\{\underset{j}{\wedge}[a_{ij},b_{ij},c_{ij}]\right\} \neq \underset{j}{\wedge}\left\{\underset{i}{\vee}[a_{ij},b_{ij},c_{ij}]\right\}$. It is to be determined the probabilities with which each strategy of A and B will be played to get an optimal solution.

Let x_i and y_j be the probabilities with which A chooses his i[th] strategy and B chooses his j[th] strategy respectively. Then for this problem, the mixed strategies of A and B are respectively, $X=(x_1,x_2,x_3)$ and $Y=(y_1,y_2,y_3)$ such that $x_1+x_2+x_3=1$; $x_1,x_2,x_3 \geq 0$ and $y_1+y_2+y_3 =1$; $y_1,y_2,y_3 \geq 0$.

The expected gains for A when B chooses B_1,B_2 and B_3 are respectively

$$[a_{11},b_{11},c_{11}]x_1 + [a_{21},b_{21},c_{21}]x_2 + [a_{31},b_{31},c_{31}]x_3\,,$$
$$[a_{12},b_{12},c_{12}]x_1 + [a_{22},b_{22},c_{22}]x_2 + [a_{32},b_{32},c_{32}]x_3\,,$$
$$and\ [a_{13},b_{13},c_{13}]x_1 + [a_{23},b_{23},c_{23}]x_2 + [a_{33},b_{33},c_{33}]x_3\,.$$

The values of X has to be selected in such a way that the gain for A remains the same whatever be the strategy chosen by B, i.e.,

$$[a_{11},b_{11},c_{11}]x_1 + [a_{21},b_{21},c_{21}]x_2 + [a_{31},b_{31},c_{31}]x_3 =$$
$$[a_{12},b_{12},c_{12}]x_1 + [a_{22},b_{22},c_{22}]x_2 + [a_{32},b_{32},c_{32}]x_3 =$$
$$[a_{13},b_{13},c_{13}]x_1 + [a_{23},b_{23},c_{23}]x_2 + [a_{33},b_{33},c_{33}]x_3\,.$$

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3$$
$$= a_{13}x_1 + a_{23}x_2 + a_{33}x_3 = k(say)$$
$$\Rightarrow x_1 = \frac{kA_1}{A};\ x_2 = \frac{kA_2}{A};\ x_3 = \frac{kA_3}{A}$$

$$Where\ A_1 = \begin{vmatrix} 1 & a_{21} & a_{31} \\ 1 & a_{22} & a_{32} \\ 1 & a_{23} & a_{33} \end{vmatrix};\ A_2 = \begin{vmatrix} a_{11} & 1 & a_{31} \\ a_{12} & 1 & a_{32} \\ a_{13} & 1 & a_{33} \end{vmatrix}$$

$$A_3 = \begin{vmatrix} a_{11} & a_{21} & 1 \\ a_{12} & a_{22} & 1 \\ a_{13} & a_{23} & 1 \end{vmatrix}; \; A = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

$$and \;\; k = \frac{A}{A_1 + A_2 + A_3}$$

Similarly for

$$b_{11}x_1 + b_{21}x_2 + b_{31}x_3 = b_{12}x_1 + b_{22}x_2 + b_{32}x_3$$
$$= b_{13}x_1 + b_{23}x_2 + b_{33}x_3 = l(say)$$

and

$$c_{11}x_1 + c_{21}x_2 + c_{31}x_3 = c_{12}x_1 + c_{22}x_2 + c_{32}x_3$$
$$= c_{13}x_1 + c_{23}x_2 + c_{33}x_3 = m(say)$$

Thus, the solution for x_1, x_2 and x_3 .
The relations satisfied by x_1 is

$$x_1 = \frac{kA_1}{A};\; x_1 = \frac{lB_1}{B};\; x_1 = \frac{mC_1}{C};$$

As a whole, it follows that a solution exists only when the a_{ij}'s,b_{ij}'s and c_{ij}'s satisfy $\frac{kA_1}{A} = \frac{lB_1}{B} = \frac{mC_1}{C}$ and according to the relation $c_{ij}=a_{ij}+b_{ij}+1$ for i,j=1,2,3 this equality holds, when all the intervals are of same length i.e., $\gamma_{ij} = \alpha_{ij} + \beta_{ij} + \mu$.

Thus $x_1 = \frac{kA_1}{A};\; x_2 = \frac{kA_2}{A} \; and \; x_3 = \frac{kA_3}{A}.$

Similarly $y_1 = \frac{kA_1}{A};\; y_2 = \frac{kA_2}{A} \; and \; y_3 = \frac{kA_3}{A}$ which are crisp numbers and the value of the game can be easily computed as $V = \left[\frac{A_1 + A_2 + A_3}{A}, \frac{B_1 + B_2 + B_3}{B}, \frac{C_1 + C_2 + C_3}{C} \right]$.

Example:

Consider, for example, the IG whose pay-off matrix is given below and which has no saddle point.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & \quad\quad B_2 & \quad\quad B_3 \end{array} \\ \begin{pmatrix} (1,3,4) & (2,0,3) & (3,-1,-2) \\ (2,0,3) & (3,-1,-2) & (1,2,0) \\ (3,-1,-2) & (1,2,0) & (2,3,1) \end{pmatrix} \end{array}$$

Hence the required probabilities are $x_1 = 1/3=x_2=x_3$ and $y_1=y_2=y_3=1/3$, A= -18 and A_1=-9=A_2=A_3 and also k=2. The value of the game [1/2,1,5]

Graphical Method:

If the graphical method is applied for a particular problem, then the same reasoning can be used to solve any IG with mixed strategies that has only three non-terminated pure strategies for one of the players. This method is useful for the IG where the pay-off matrix is of the size 3×n or m×3 i.e., the IG with mixed strategies that has only three pure strategies for one of the players in the IG.Optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. It is clear that if one player has only three strategies, the other will also use the same number of strategies. Hense, graphical method is useful to find out which of the two strategies can be used.

General Rule to Draw a Graph:

Consider the following two 3×n and m×3 IG without saddle point:

$$(a)\quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{pmatrix} [a_{11},b_{11},c_{11}] & [a_{12},b_{12},c_{12}] & \cdots & [a_{1n},b_{1n},c_{1n}] \\ [a_{21},b_{21},c_{21}] & [a_{22},b_{22},c_{22}] & \cdots & [a_{2n},b_{2n},c_{2n}] \\ [a_{31},b_{31},c_{31}] & [a_{32},b_{32},c_{32}] & \cdots & [a_{3n},b_{3n},c_{3n}] \end{pmatrix} \end{array}$$

$$(b)\quad \begin{array}{c} \\ A_1 \\ A_2 \\ \cdots \\ A_m \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{pmatrix} [a_{11},b_{11},c_{11}] & [a_{12},b_{12},c_{12}] & [a_{13},b_{13},c_{13}] \\ [a_{21},b_{21},c_{21}] & [a_{22},b_{22},c_{22}] & [a_{23},b_{23},c_{23}] \\ \cdots & \cdots & \cdots \\ [a_{m1},b_{m1},c_{m1}] & [a_{m2},b_{m2},c_{m2}] & [a_{m3},b_{m3},c_{m3}] \end{pmatrix} \end{array}$$

(i) Draw three parallel vertical lines one unit distance apart and mark a scale on each.
(ii) For the case (a), the three strategies A1 A2 A3 of A are represented by these three straight lines.
(iii) For the case (b) the three strategies B1 B2 B3 of B are represented by these three straight lines.
(iv) For the case (a), since the player A has 3 strategies, let the mixed strategy for the player A is given by

$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ x_1 & x_2 & x_3 \end{pmatrix}$ where $x_1+x_2+x_3=1$; $x_1,x_2,x_3 \geq 0$.

B's pure move A's expected pay-off E(x)

$$\begin{aligned} B_1\quad & [a_{11},b_{11},c_{11}]x_1 + [a_{21},b_{21},c_{21}]x_2 + [a_{31},b_{31},c_{31}](1-x_1-x_2) \\ & = [a_{11}-b_{31}, b_{11}-c_{31}, c_{11}-a_{31}]x_1 + \\ & \quad [a_{21}-b_{31}, b_{21}-c_{31}, c_{21}-a_{31}]x_2 + [a_{31},b_{31},c_{31}] \end{aligned}$$

$$\begin{aligned} B_2\quad & [a_{12},b_{12},c_{12}]x_1 + [a_{22},b_{22},c_{22}]x_2 + [a_{32},b_{32},c_{32}](1-x_1-x_2) \\ & = [a_{12}-b_{32}, b_{12}-c_{32}, c_{12}-a_{32}]x_1 + \\ & \quad [a_{22}-b_{32}, b_{22}-c_{32}, c_{22}-a_{32}]x_2 + [a_{32},b_{32},c_{32}] \end{aligned}$$

$$\begin{aligned} B_n\quad & [a_{1n},b_{1n},c_{1n}]x_1 + [a_{2n},b_{2n},c_{2n}]x_2 + [a_{3n},b_{3n},c_{3n}](1-x_1-x_2) \\ & = [a_{1n}-b_{3n}, b_{1n}-c_{3n}, c_{1n}-a_{3n}]x_1 + \\ & \quad [a_{2n}-b_{3n}, b_{2n}-c_{3n}, c_{2n}-a_{3n}]x_2 + [a_{3n},b_{3n},c_{3n}] \end{aligned}$$

This shows that the player A's expected pay-off varies bilinearly with x_1 and x_2. According to the min-max criterion for the mixed strategy games, the player A should select the value of x_1 and x_2 so as to minimize his minimum expected pay-offs. Thus the player B would like to choose that pure moves B_j against S_A for which $E_j(x)$ is a minimum for j=1,2,..n. Let us denote this minimum expected pay-off for A by $v=\min\{E_j(x);j=1,2,..n\}$. The objective of player A is to select x_1 , x_2 and hence x_3 in such a way that v is as large as possible. This may be done by plotting the regions R_1 ,R_2 ,..R_n plotted by three parallel straight lines as

$$\{y=(a_{11}-b_{31})x_1+(a_{21}-b_{31})x_2+a_{31},\ y=(b_{11}-c_{31})x_1+(b_{21}-c_{31})x_2+b_{31},\ y=(c_{11}-a_{31})x_1+(c_{21}-a_{31})x_2+c_{31}\}$$

$$\{y=(a_{12}-b_{32})x_1+(a_{22}-b_{32})x_2+a_{32},\ y=(b_{12}-c_{32})x_1+(b_{22}-c_{32})x_2+b_{32},\ y=(c_{12}-a_{32})x_1+(c_{22}-a_{32})x_2+c_{32}\}$$

$$\{y=(a_{1n}-b_{3n})x_1+(a_{2n}-b_{3n})x_2+a_{3n},\ y=(b_{1n}-c_{3n})x_1+(b_{2n}-c_{3n})x_2+b_{3n},\ y=(c_{1n}-a_{3n})x_1+(c_{2n}-a_{3n})x_2+c_{3n}\}$$

drawn to represent the gains of A corresponding to B_1 ,B_2 ,...B_n respectively B on the line representing A_1 with on line representing A_2 and A_3.

(v) Now the three strategies of player B corresponding to those regions which pass through the maximum region can be determined. It helps in reducing the size of the IG to 3×3.

(vi) Similarly, for the case (b), since the player B has three strategies, let the mixed strategy for the player A is given by $S_B=\begin{pmatrix} B_1 & B_2 & B_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$ where $y_1+y_2+y_3=1$; $y_1,y_2,y_3\ \geq 0$. Thus for each of the pure strategies available to the player A, the expected pay-off for the player B, would be as follows

A's pure moves B's expected pay-off E(y)

$$A_1\quad [a_{11},b_{11},c_{11}]y_1+[a_{12},b_{12},c_{12}]y_2+[a_{13},b_{13},c_{13}](1-y_1-y_2)$$
$$=[a_{11}-b_{13},b_{11}-c_{13},c_{11}-a_{13}]y_1+[a_{12}-b_{13},b_{12}-c_{13},c_{12}-a_{13}]y_2+[a_{13},b_{13},c_{13}]$$

$$A_2\quad [a_{21},b_{21},c_{21}]y_1+[a_{22},b_{22},c_{22}]y_2+[a_{23},b_{23},c_{23}](1-y_1-y_2)$$
$$=[a_{21}-b_{23},b_{21}-c_{23},c_{21}-a_{23}]y_1+[a_{22}-b_{23},b_{22}-c_{23},c_{22}-a_{23}]y_2+[a_{23},b_{23},c_{23}]$$

$$\cdots$$

$$A_m\quad [a_{m1},b_{m1},c_{m1}]y_1+[a_{m2},b_{m2},c_{m2}]y_2+[a_{m3},b_{m3},c_{m3}](1-y_1-y_2)$$
$$=[a_{m1}-b_{m3},b_{m1}-c_{m3},c_{m1}-a_{m3}]y_1+[a_{m2}-b_{m3},b_{m2}-c_{m3},c_{m2}-a_{m3}]y_2+[a_{m3},b_{m3},c_{m3}]$$

This shows that the player B's expected pay of varies bi-linearly with y_1 and y_2. According to the max-min criterion for the mixed strategy games, the player B should select the value of y_1 and y_2 so as to minimize his maximum expected pay-offs. This may be done by plotting the regions L_1 ,L_2 ,..L_m plotted by the three parallel straight lines as

$$\{y=(a_{11}-b_{13})y_1+(a_{12}-b_{13})y_2+a_{13},\ y=(b_{11}-c_{13})y_1+(b_{12}-c_{13})y_2+b_{13},\ y=(c_{11}-a_{13})y_1+(c_{12}-a_{13})y_2+c_{13}\}$$

$$\{y=(a_{21}-b_{23})y_1+(a_{22}-b_{23})y_2+a_{23},\ y=(b_{21}-c_{23})y_1+(b_{22}-c_{23})y_2+b_{23},\ y=(c_{21}-a_{23})y_1+(c_{22}-a_{23})y_2+c_{23}\}$$

$$\{y=(a_{m1}-b_{m3})y_1+(a_{m2}-b_{m3})y_2+a_{m3},\ y=(b_{m1}-c_{m3})y_1+(b_{m2}-c_{m3})y_2+b_{m3},\ y=(c_{m1}-a_{m3})y_1+(c_{m2}-a_{m3})y_2+c_{m3}\}$$

drawn to represent the gains of B corresponding to A_1, A_2,..A_m respectively of A on the line representing B_1 with on the line representing B_2 and B_3. This method is illustrated by different types of numerical examples.

Theorem 2:

A region which will gives the bounds of maximize the minimum expected gain of B will give an optimal value of the probability x_1,x_2 ,x_3 and the region will refer three courses of action taken by A for the purpose. Proof: According to the general theory of games, in a first step the player A would try to obtain $\vee_i\left\{\wedge_j[a_{ij},b_{ij},c_{ij}]\right\}$ while for the player B the objective is $\wedge_j\left\{\vee_i[a_{ij},b_{ij},c_{ij}]\right\}$. If the real matrix $\left([a_{ij},b_{ij},c_{ij}]\right)_{m\times n}$ has no saddle points mixed strategies must be used. The mixed strategy problem is solved by applying the min-max criterion in which the maximizing player chooses the probabilities x_i which minimize the smallest expected gains in the columns and the minimizing player chooses his probabilities y_j which minimize the least expected gains of rows. Obviously the players obtain their optimal strategies from $\vee_i\left\{\wedge_j[a_{ij},b_{ij},c_{ij}]x_iy_j\right\}$ and $\wedge_j\left\{\vee_i[a_{ij},b_{ij},c_{ij}]x_iy_j\right\}$. This problem becomes classical game and x_i,y_j is chosen in such a way that $\vee_i\left\{\wedge_j[a_{ij},b_{ij},c_{ij}]x_iy_j\right\} = \wedge_j\left\{\vee_i[a_{ij},b_{ij},c_{ij}]x_iy_j\right\}$. Let V the value of the game given by the pay-off matrix $\left([a_{ij},b_{ij},c_{ij}]\right)_{3\times n}$ as

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} [a_{11},b_{11},c_{11}] & [a_{12},b_{12},c_{12}] & \cdots & [a_{1n},b_{1n},c_{1n}] \\ [a_{21},b_{21},c_{21}] & [a_{22},b_{22},c_{22}] & \cdots & [a_{2n},b_{2n},c_{2n}] \\ [a_{31},b_{31},c_{31}] & [a_{32},b_{32},c_{32}] & \cdots & [a_{3n},b_{3n},c_{3n}] \end{pmatrix} \end{array}$$

without any saddle point. Let the mixed strategies used by A be X=(x_1 ,x_2 ,x_3) and B be Y=(y_1 ,y_2 ,…,y_n), in which $\sum_i x_i = 1\, and\; x_i \ge 0; \sum_j y_j = 1\, and\; y_j \ge 0.$ Then the expected gain of the player A given that B plays his pure strategy B_j is given by

$$E_j(X) = [a_{1j},b_{1j},c_{1j}]x_1 + [a_{2j},b_{2j},c_{2j}]x_2 + [a_{3j},b_{3j},c_{3j}](1-x_1-x_2),\; j=1,2,...n$$

Now all x_1 ,x_2, x_3 must lie in the open interval (0,1), because if either x_1 or x_2 or x_3 = 1 the game is of pure strategy. Hence $E_j(X)$ is a bilinear function of x_1 and x_2. Considering $E_j(X)$ as a bilinear function of x_1 and x_2 and from the limiting values (0,1) of x_1 & x_2;

$$E_j(X) = \begin{cases} [a_{3j},b_{3j},c_{3j}], & x_1 = x_2 = 0 \\ [a_{2j},b_{2j},c_{2j}], & x_1 = 0; x_2 = 1 \\ [a_{1j},b_{1j},c_{1j}], & x_1 = 1; x_2 = 0 \end{cases}$$

Hence $E_j(X)$ represents a region joining the points $(0,0,a_{3j})$, $(0,0,b_{3j})$, $(0,0,c_{3j})$, $(0,1,a_{2j})$, $(0,1,b_{2j})$, $(0,1,c_{2j}), and\, (1,0,a_{1j}), (1,0,b_{1j}), (1,0,c_{1j})$. Now A expects a least possible gain V so that $E_j(X) \ge V$, for all j. The main objective of A is to select x_1,x_2,x_3 in such a way that $x_1+x_2+x_3=1$ and all lie in the open interval (0,1). Consider the following 3×3 pay-off matrix, where B_r ,B_s and B_t are three critical moves of B,

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad B_r \quad & \quad B_s \quad & \quad B_t \quad \end{array} \\ \begin{pmatrix} [a_{1r},b_{1r},c_{1r}] & [a_{1s},b_{1s},c_{1s}] & [a_{1t},b_{1t},c_{1t}] \\ [a_{2r},b_{2r},c_{2r}] & [a_{2s},b_{2s},c_{2s}] & [a_{2t},b_{2t},c_{2t}] \\ [a_{3r},b_{3r},c_{3r}] & [a_{3s},b_{3s},c_{3s}] & [a_{3t},b_{3t},c_{3t}] \end{pmatrix} \end{array}$$

Then the expectation of A is

$$\begin{aligned} &= [a_{1r},b_{1r},c_{1r}]x_1y_1 + [a_{1s},b_{1s},c_{1s}]x_1y_2 + [a_{1t},b_{1t},c_{1t}]x_1y_3 + \\ &\quad [a_{2r},b_{2r},c_{2r}]x_2y_1 + [a_{2s},b_{2s},c_{2s}]x_2y_2 + [a_{2t},b_{2t},c_{2t}]x_2y_3 + \\ &\quad [a_{3r},b_{3r},c_{3r}]x_3y_1 + [a_{3s},b_{3s},c_{3s}]x_3y_2 + [a_{3t},b_{3t},c_{3t}]x_3y_3 \end{aligned}$$

Player A would always try to mix his moves with such probabilities so as to maximize his expected gain. This shows that if A chooses $x_1 = \frac{kA_1}{A}$ $\quad where\, A_1 = \begin{vmatrix} 1 & a_{2r} & a_{3r} \\ 1 & a_{2s} & a_{3s} \\ 1 & a_{2t} & a_{3t} \end{vmatrix}$ $\quad and\;\; A = \begin{vmatrix} a_{1r} & a_{2r} & a_{3r} \\ a_{1s} & a_{2s} & a_{3s} \\ a_{1t} & a_{2t} & a_{3t} \end{vmatrix}$ and $k = \frac{A}{A_1 + A_2 + A_3}$ then he can ensure that his expectation will be at least

$$\left[\frac{A_1 + A_2 + A_3}{A}, \frac{B_1 + B_2 + B_3}{B}, \frac{C_1 + C_2 + C_3}{C} \right].$$

This choice of x_1 will thus be optimal to the player A.

Thus the lower bound of the regions will give the minimum expected gain of A as a function of x_1 and x_2. Hence with the help of graphical method, the position is to find two particular moves which will gives the bounds of maximum pay-off among the choosing minimum expected pay-off on the lower bounds of B. Thus the mixed strategy problem is solved by applying the min-max criterion in which the maximizing player chooses the probabilities x_i which minimize the smallest expected gains in columns. Thus, a region to be found which will give the bounds of maximum pay-off among the minimum expected pay-off on the lower region of B. For this any two pair of lines having opposite signs for their slopes will define an alternative optimal solution region and the optimal value of probability x_1 ,x_2 and x_3.

Theorem 3:

A region which will give the bounds of minimize the maximum expected loss of A, and the lowest bound of all regions will give the minimum expected pay-off (min-max value) and the optimal value of the probability y_1 , y_2 , y_3.

Proof: The proof of this theorem is similar to theorem 2.

Particular Cases and Numerical Examples:

Example:

Consider the 3×6 IG problem whose pay-off matrix is given below. The game has no saddle point

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{array} \\ \begin{pmatrix} (0,2,1) & (2,4,0) & (-1,1,2) & (1,3,4) & (2,0,3) & (3,-1,-2) \\ (2,4,0) & (-1,1,2) & (1,3,4) & (2,0,3) & (3,-1,-2) & (1,2,0) \\ (-1,1,2) & (1,3,4) & (2,0,3) & (3,-1,-2) & (1,2,0) & (2,3,1) \end{pmatrix} \end{array}$$

Let the player A play the mixed strategy $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ x_1 & x_2 & x_3 \end{pmatrix}$ against player B, where $x_1+x_2+x_3=1$; $x_1,x_2,x_3 \geq 0$. The A's expected pay-off's against B's pure moves are given by

B's pure move A's expected pay-off E(x)

B_1	$(-x_1 + x_2 - 1, 0x_1 + 2x_2 + 1, 2x_1 + x_2 + 2)$
B_2	$(-x_1 - 4x_2 + 1, 0x_1 - 3x_2 + 3, -x_1 + x_2 + 4)$
B_3	$(-x_1 + x_2 + 2, -2x_1 + 0x_2 + 0, 0x_1 + 2x_2 + 3)$
B_4	$(2x_1 + 3x_2 + 3, 5x_1 + 2x_2 - 1, x_1 + 0x_2 - 2)$
B_5	$(0x_1 + x_2 + 1, 0x_1 - x_2 + 2, 2x_1 - 3x_2 + 0)$
B_6	$(0x_1 - 2x_2 + 2, -2x_1 + x_2 + 3, -4x_1 - 2x_2 + 1)$

Three parallel axes are drawn , the strategies A_1,A_2 and A_3 of A are represented by these three straight lines. These expected pay-off equations are plotted as functions of x_1 and x_2 as shown the fig.1. Now since the player A wishes to maximize his minimum expected pay-off, it is considered that the highest region of intersection on the lower region of A's expected pay-off equations. From the figure so as to obtain from above drawing a region, PR of this bounds refers to the maximum of minimum gains. At this region B has used his three courses of action and they are B_4 , B_5 and B_6 as it is the region of intersection of B_4 , B_5 and B_6. The solution to the original 3×6 game, therefore, reduces to that of the simpler game with the 3×3 pay-off matrix.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_4 & B_5 & B_6 \end{array} \\ \begin{pmatrix} (1,3,4) & (2,0,3) & (3,-1,-2) \\ (2,0,3) & (3,-1,-2) & (1,2,0) \\ (3,-1,-2) & (1,2,0) & (2,3,1) \end{pmatrix} \end{array}$$

The abscissa of the points P and R are 3 and the ordinates of P and R -3/2,9/2. Therefore the optimal strategies are for A(3/2,3/2,3/2) and the value of the game V is [1/2,1,5]. The probabilities for B_4 , B_5 and B_6 are obtained from graph as (0,0,0,1/2,1/4,1/4) and in this case also the value of the game V is [1/2,1,5]. Here B_1 , B_2 and B_3 are dominated strategies as is obvious from the graph as also from the pay-off matrix.

The method is equally applicable to n×3 IG. But in this case the graph stands for B's loss against the probability y with which B plays B_1. In this case to minimize the maximum loss of B, observed from the above figure. The lowest region of this bound will refer to the three courses of the action taken by A for the purpose.

Conclusion:

This paper presents an application of graphical method for finding the solution of three person non-zero sum IG for fixed strategies. Numerical examples are presented to illustrate the methodology. Our proposed method is simple and more accurate to deal with the general three person of IG problems. Hence the proposed method is suitable for solving the proactical IG problems in real applications. Application of IG takes place in civil engineering, lunching advertisement camping for competing products, planning war strategies for opposing armies and other areas.

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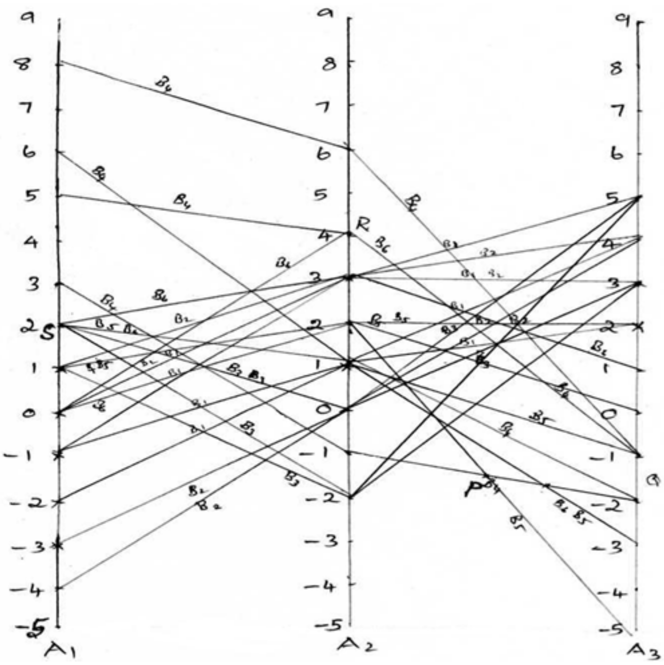

Fig. 1: Graphical solution of 3×6 IG.

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