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A Comparative Study of Standard and Exact Finite Difference Schemes for Numerical Solution of Ordinary Differential Equations Emanating from the Radioactive Decay of Substances

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Abstract: This research work investigates the numerical solution of ordinary differential equations emanating from the radioactive decay of substances using the standard and exact finite difference schemes. It is assumed that the decay can be modelled into a simple first order differential equation of the form, $y^1 = f(x,y) \quad y(x_0) = y_0$. Numerical results were generated with the two schemes and comparisms were made between the numerical results and the theoretical results.

Key words: *Initial value problems, Standard schemes, Exact finite Schemes and Approximations.*

INTRODUCTION

Nowadays, many problems in chemical, biological, engineering and physical sciences can be modelled in the form of ordinary differential equations. One of such cases is the radioactive decay of substances. Effective methods are usually required to solve such models. We shall consider equation of the form;

$$y^1 = f(x,y), \quad y(x_0) = y_0, \quad x \in [a,b] \tag{1}$$

which occurs most often in the fields of chemical and biological sciences.

We shall develop two different schemes for the solution of equation of the form (1) while the second step is to test the performance of the two schemes in solving a specific equation of the type (1) which arise from the decay of radioactive substances. The first scheme shall be called the standard numerical scheme; the second method will be referred to as the exact finite difference scheme.

Before we consider the two schemes, let us see the formular below which is from the Law of Growth and Decay.

Let *f(t)* represent the amount of a quality at a time *(t)*. We assume that the rate of change *f(t)* is proportional to *f(t)*, then, we have the following differential equation,

$$\frac{d}{dt} f(t) = \lambda f(t) \tag{2}$$

where λ is a constant.
If *f(t)* decreases, then, λ<0 and equation (2) is called the law of natural decay. If *f(t)* increases, then λ>0 and equation (2) is called the law of natural growth.

Now, given equation (2), we shall derive a formular for *f(t)* expressed in terms of *λ,t,* and *f*(0) (the original amount of the quantity).

$$f'(t) = \frac{d}{dt} f(t) = \lambda f(t)$$

we have,

$$\frac{f'(t)}{f(t)} = \lambda$$

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Thus,

$$\int \frac{f'(t)}{f(t)} dt = \int \lambda \, dt \tag{3}$$

$y = f(t)$, then $\frac{dy}{dt} = f'(t)$ and so

$$\int \frac{f'(t)}{f(t)} = \int \frac{f'(t)}{f(t)} \frac{dt}{dy} dy$$

$$= \int \frac{f'(t)}{y} \cdot \frac{1}{f'(t)} dy$$

$$= \int \frac{dy}{y}$$

$$= ln\, y$$

$$= ln\, f(t)$$

Hence, we obtain from (3) the following,

$ln\, f(t) = \lambda t + c$,

when t=0 , we have $c = ln\, f(0)$

Thus,

$$ln\, f(t) = \lambda t + ln\, f(0)$$

$$ln\, f(t) - ln\, f(0) = \lambda t$$

$$ln\, (\frac{f(t)}{f(0)}) = \lambda t$$

$$\frac{f(t)}{f(0)} = e^{\lambda t}$$

$$\Rightarrow f(t) = f(0)\, e^{\lambda t} \tag{4}$$

Which says that if the rate of change of a given quantity is proportional to the amount of the quantity at any given instance, then the amount $f(t)$ at time t is equal to the product of the original amount $f(0)$ and $e^{\lambda t}$, where λ is the constant of proportionality.

Derivation of the First Scheme:

Let us assume that the solution to equation (1)
Can be written in the form,

$$F(x) = be^{\beta x}, \quad \text{where } b \text{ is a constant} \tag{5}$$

At the point $x=x_n$, we have

$$F(x_n) = be^{\beta x_n} \tag{6}$$

while at the point $x = x_{n+1}$, equation (5) becomes,

$$F(x_{n+1}) = be^{\beta x_{n+1}} \tag{7}$$

Using (5), we have

$$F'(x) = b\beta e^{\beta x} \tag{8}$$

and

$$F'(x_n) = b\beta e^{\beta x_n} \tag{9}$$

Let us further assume that (9) coincides with (1) and hence we have,

$$b\beta e^{\beta x_n} = f(x_n, y_n)$$

$$\Rightarrow b = \frac{f(x_n, y_n)}{\beta e^{\beta x_n}} \tag{10}$$

To proceed further, let us assume that the theoretical and numerical solution initially coincides, hence

$$F(x_n) = y_n \tag{11}$$

$$F(x_{n+1}) = y_{n+1} \tag{12}$$

Using (6), (7), (11) and (12), we have,

$$y_{n+1} - y_n = b(e^{\beta x_{n+1}} - e^{\beta x_n}) \tag{13}$$

$$\Rightarrow y_{n+1} = y_n + \frac{(e^{\beta x_{n+1}} - e^{\beta x_n}) f(x_n, y_n)}{\beta e^{\beta x_n}} \tag{14}$$

Which is the required numerical scheme.

It is note worthy that this numerical method is strong enough to solve not only problems relating to radioactivity but also problems involving biological and other decay related problems in chemistry, biology and other physical sciences.

Equation (14) is a standard finite difference scheme and it is a one-step method.

Derivative of the Second Scheme:

Here, we shall derive an exact scheme which will be able to produce exact solution to problems in form of equation (1). The difference between the scheme developed here and that of (14) is that (14) produces approximate solution to equation (1) by exact difference scheme.

Let us consider as ordinary differential equation of the form,

$$\frac{dy}{dx} = \beta y, \; y(b) = y_0, \; b \text{ is a constant} \tag{15}$$

with the theoretical solution given by

$$y(x) = be^{\beta x} \tag{16}$$

At the point $x = x_n$, (16) becomes

$$y(x_n) = be^{\beta x_n} \tag{17}$$

Therefore,

$$\begin{vmatrix} y_n & be^{\beta x} \\ y_{n+1} & be^{\beta x_{n+1}} \end{vmatrix} = 0 \qquad (18)$$

hence,

$$y_n(be^{\beta x_{n+1}}) - y_{n+1}(be^{\beta x_n}) = 0$$
$$\Rightarrow y_{n+1}b(e^{\beta x_n}) = y_n b(e^{\beta x_{n+1}})$$

Thus

$$\Rightarrow y_{n+1} = \frac{y_n(e^{\beta x_{n+1}})}{(e^{\beta x_n})} \qquad (19)$$

If we put $x_n = nh$ and $x_{n+1} = (n+1)h$, equation (19) can then be written as;

$$y_{n+1} = \frac{y_n(e^{\beta(n+1)h})}{(e^{\beta nh})} \qquad (20)$$

which is called the exact difference scheme that can be used to solve ordinary differential equations that involve decay.

One of the shortcomings of a standard finite difference scheme is that it is necessary that we must know the theoretical solution before we construct the method. However, the advantage of exact finite difference scheme is that it produces exact solution to the differential equations under consideration.

Numerical Experiments:

Now, we shall apply (14) and (20) to the radioactive decay problem below. Considerations will also be given to the performance of the schemes on some closely related problems.

Examples 1:

A certain radioactive substance is known to decay at the rate proportional to the amount present. A block of this substance having a mass of 100g originally is observed. After 40 hours, its mass reduces to 90g. Find an expression for the mass of the substance at anytime.

The problem above has a differential equation of the form;

$$\frac{dN}{dt} = -\lambda N, \; N(0) = 100 \qquad (21)$$

where N represents the mass of the substance at any time t and λ is a constant which specifies the rate at which these particular substance decay.

Thus, the theoretical solution to equation (21) is given by,

$$N(t) = 100e^{-0.0026t} \qquad (22)$$

Hence, (22) is the expression for the mass of the substance at any time t.

Note: To show that (22) is the theoretical solution of the problem above, we proceed, thus

$f(0) = 100g, \; t = 40 hours, \; f(40) = 90$ From equation (4), we have

$$f(t) = f(0)e^{\lambda t}$$

$$90 = 100e^{40\lambda}$$

$$e^{40\lambda} = ln(\frac{90}{10})$$

$$40\lambda = ln(\frac{90}{10})$$

$$\lambda = \frac{ln\ 9 - ln 10}{40}$$
$$= -0.0026$$

Hence, the required expression is

$$f(t) = 100e^{-0.0026t}$$

which can equally be written in the form (22) as $N(t) = 100e^{-0.0026t}$

Definitely, equation (21) is of the general form (1). With a uniform step length h=0.1, we obtain the following results.

Table 1: Scheme (14) with h= 0.1

S/N	*X*	y^n	EXACT	ERROR
0	0.0	100.00000000	100.00000000	0.00000000
1	0.1	99.97400338	99.97400338	0.00000000
2	0.2	99.94801352	99.94801352	0.00000000
3	0.3	99.92203041	99.92203041	0.00000000
4	0.4	99.89960541	99.89960541	0.00000000
5	0.5	99.87008446	99.87008446	0.00000000
6	0.6	99.84412162	99.84412162	0.00000000
7	0.7	99.81816552	99.81816552	0.00000000
8	0.8	99.79221617	99.79221617	0.00000000
9	0.9	99.76627357	99.76627357	0.00000000
10	1.0	99.74033771	99.74033771	0.00000000

Table 2: Scheme (20) with h=0.1

S/N	*X*	y^n	EXACT	ERROR
0	0.0	100.00000000	100.00000000	0.00000000
1	0.1	99.97400338	99.97400338	0.00000000
2	0.2	99.94801352	99.94801352	0.00000000
3	0.3	99.92203041	99.92203041	0.00000000
4	0.4	99.89960541	99.89960541	0.00000000
5	0.5	99.87008446	99.87008446	0.00000000
6	0.6	99.84412162	99.84412162	0.00000000
7	0.7	99.81816552	99.81816552	0.00000000
8	0.8	99.79221617	99.79221617	0.00000000
9	0.9	99.76627357	99.76627357	0.00000000
10	1.0	99.74033771	99.74033771	0.00000000

Schemes (14) and (20) are specially designed to solve the problem that represents the radioactive decay of substances. We may wish to apply the schemes to some selected problems which are also stated below.

Example 2:

Consider the differential equation of the form;

$$y' = y, \quad y(0) = 1, \quad x \in [0,1]$$

The theoretical (exact) solution is given by

$$y(x) = e^x$$

We apply scheme (14) to solve the above problem and the result is as presented below in a graphical form.

S/N	X	y^n	EXACT	ERROR
0	0.0	1.000000000	1.000000000	0.000000000
1	0.1	1.105170918	1.105170918	0.000000000
2	0.2	1.221402758	1.221402758	0.000000000
3	0.3	1.349858807	1.349858807	0.000000000
4	0.4	1.491824697	1.491824697	0.000000000
5	0.5	1.648721270	1.648721270	0.000000000
6	0.6	1.822118800	1.822118800	0.000000000
7	0.7	2.013752707	2.013752707	0.000000000
8	0.8	2.225540928	2.225540928	0.000000000
9	0.9	2.459631110	2.459631110	0.000000000
10	1.0	2.718281828	2.718281828	0.000000000

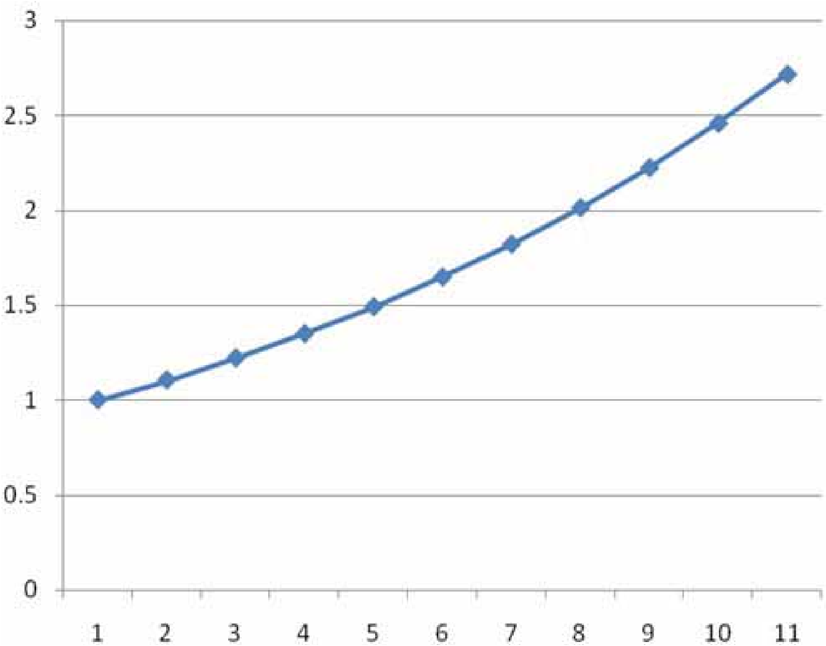

Fig. 1: Graphical Representation of Scheme (14) on Problem 2

Furthermore, we can also construct exact scheme for problem (2) once the exact solution is known. This is given as;

$$\begin{vmatrix} y_n & e^{x_n} \\ y_{n+1} & e^{x_{n+1}} \end{vmatrix} = 0 \quad (23)$$

$$y_n(e^{x_{n+1}}) - y_{n+1}(e^{x_n}) = 0$$

Thus,

$$y_{n+1}(e^{x_n}) = y_n(e^{x_{n+1}})$$

$$\Rightarrow y_{n+1} = \frac{y_n(e^{x_{n+}})}{(e^{x_n})} \quad (24)$$

The exact scheme (24) is implemented using $x_n = nh$, $x_{n+1} = (n+1)h$ and the result is as given below.

S/N	X	y^n	EXACT	ERROR
0	0.0	1.000000000	1.000000000	0.000000000
1	0.1	1.105170918	1.105170918	0.000000000
2	0.2	1.221402758	1.221402758	0.000000000
3	0.3	1.349858808	1.349858808	0.000000000
4	0.4	1.491824698	1.491824698	0.000000000
5	0.5	1.648721271	1.648721271	0.000000000
6	0.6	1.822118800	1.822118800	0.000000000
7	0.7	2.013752707	2.013752707	0.000000000
8	0.8	2.225540928	2.225540928	0.000000000
9	0.9	2.459803111	2.459803111	0.000000000
10	1.0	2.718281828	2.718281828	0.000000000

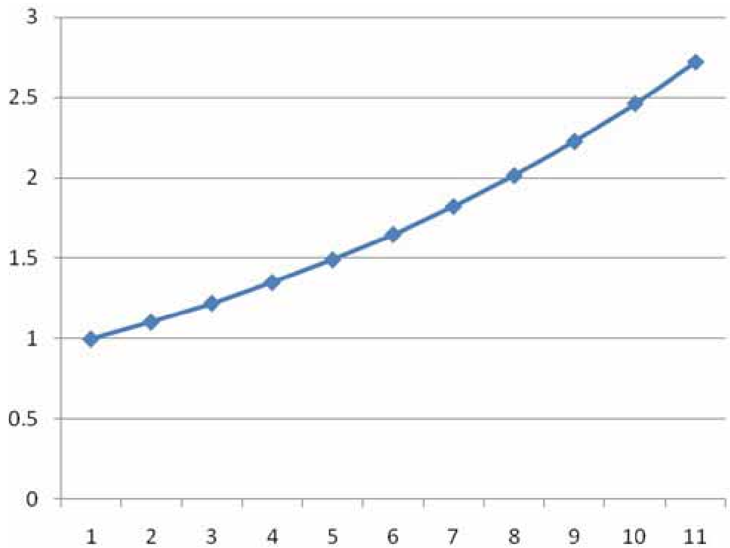

Fig. 2: Graphical Representation of Exact Scheme (24) on Problem (2)

Example 3:

Consider the differential equation of the form

$$y' = 4x - 2xy, \quad y(0) = 3, \quad x \in [0,1]$$

The theoretical (exact) solution is given by

$$y(x) = 2 + e^{-x^2}$$

We now apply scheme (14) to solve the above problem 3 and the result is as presented below in a graphical form.

S/N	X	y^n	EXACT	ERROR
0	0.0	3.000000000	3.000000000	0.000000000
1	0.1	3.000000000	2.990049834	0.009950166
2	0.2	2.978965816	2.960789439	0.018176377
3	0.3	2.937782323	2.913931185	0.023851138
4	0.4	2.878605866	2.852143789	0.026462077
5	0.5	2.804682838	2.778800783	0.025882055
6	0.6	2.720053605	2.697676326	0.020285973
7	0.7	2.629179167	2.612626394	0.016552773
8	0.8	2.536539276	2.527292424	0.009245852
9	0.9	2.446253951	2.444858066	0.001395885
10	1.0	2.377774663	2.367879441	0.009895222

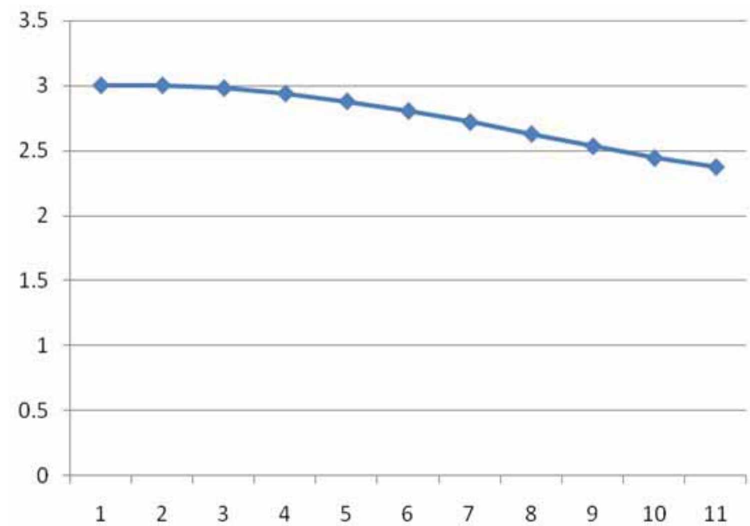

Fig. 3: Graphical Representation of Scheme (14) on Problem (3)

We proceed by constructing an exact scheme for problem (3) once the exact equation is known. Thus,

$$\begin{vmatrix} y_n & 2+e^{-x_n^2} \\ y_{n+1} & 2+e^{-x_{n+1}^2} \end{vmatrix} = 0 \qquad (25)$$

i.e $y_n(2+e^{-x_{n+1}^2}) - y_{n+1}(2+e^{-x_n^2}) = 0$

$$\Rightarrow y_{n+1}(2+e^{-x_n^2}) = y_n(2+e^{-x_{n+1}^2})$$

We have

$$\Rightarrow y_{n+1} = \frac{y_n(2+e^{-x_{n+1}^2})}{(2+e^{-x_n^2})} \qquad (26)$$

The exact scheme (26) is implemented using $x_n = nh$, $x_{n+1} = (n+1)h$ and the result is as given below.

S/N	X	y^n	EXACT	ERROR
0	0.0	3.000000000	3.000000000	0.000000000
1	0.1	2.990049834	2.990049834	0.000000000
2	0.2	2.960789439	2.960789439	0.000000000
3	0.3	2.913931185	2.913931185	0.000000000
4	0.4	2.852143789	2.852143789	0.000000000
5	0.5	2.778800783	2.778800783	0.000000000
6	0.6	2.697676326	2.697676326	0.000000000
7	0.7	2.612626394	2.612626394	0.000000000
8	0.8	2.527292424	2.527292424	0.000000000
9	0.9	2.444858066	2.444858066	0.000000000
10	1.0	2.367879441	2.367879441	0.000000000

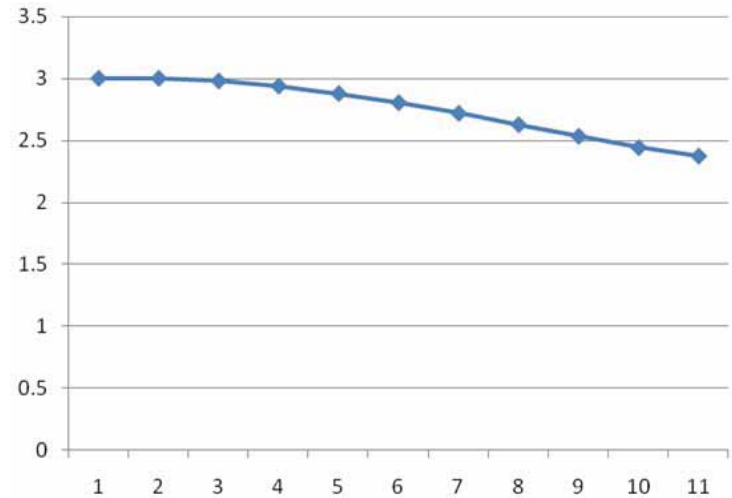

Fig. 4: Graphical representation of exact scheme (26) on problem (3)

Conclusion:

Two stable numerical schemes have been produced, namely the standard and exact finite difference schemes. We have also tested our methods on some problems which are common in radioactivity. The two methods performed well on the problems and the methods are seen to be numerically stable.

REFERENCES

Di Prima, C., 1969. Elementary Differential Equations and Boundary value problems, John Wiley and Sons Inc., New York.

Fatunla, S.O., 1998. Numerical Methods for Initial value Problems in Ordinary Differential Equations, Academic Press, Inc., New York.

Gakhov, F.D., 1966. Boundary Value Problems, Dover Publications, Inc., New York.

Gavurin, M.K., 1949. On a method of numerical integration of homogenous linear differential equations convenient for mechanization of the computation Trudy Mat. Ist. Steklov, 28: 152-156.

Henrici, P., 1962. Discrete Variable Methods in Ordinary Differential Equations, John Wiley and Sons, New York.

Kandasamy, P., 1997. Numerical Methods, S. Chand and Company Ltd, New Delhi.

Lambert, J.D., 1973. Computational Method in Ordinary Differential Equations, John Wiley and Sons, New York.

Lyons, L., 1995. Mathematics for Sciences Students, Cambridge University Press, United Kingdom.