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Application of HPEM to Investigate the Response and Stability of Nonlinear Problems in Vibration

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Abstract: In this work, a powerful analytical method, called He's Parameter Expanding Methods (HPEM) is used to obtain the exact solution of nonlinear problems in nonlinear vibration. In this work, the governing equation is obtained by using Lagrange method, then the nonlinear governing equation is solved analytically by He's Parameter Expanding Methods. It is shown that one term in series expansions is sufficient to obtain a highly accurate solution which is valid for the whole domain. Comparison of the obtained solutions with those obtained using numerical method shows that this method is effective and convenient for solving these problems.

Key words: Analytical solution, Nonlinear Dynamics; He's Parameter Expanding Method;

INTRODUCTION

Recently a large amount of research has been related to nonlinear systems having single or multi degrees of freedom, but hardly any of those problems have analytical solution. Also, most scientific problems in fluid mechanics and heat transfer are inherently nonlinear. Except a limited number of these problems, most of them do not have analytical solutions. Therefore, these nonlinear equations should be solved using other methods. Some of them are solved using numerical techniques and some are solved using perturbation method (Nayfeh, 1993). In the numerical method, stability and convergence should be considered so as to avoid divergence or inappropriate results. In the perturbation method, a small parameter is inserted in the equation. Therefore, finding the small parameter and exerting it into the equation are deficiencies of this method.

Recently, considerable attention has been directed towards analytical solutions for nonlinear equations without small parameters. Many new techniques have been appeared in the literature, for example, variational iteration method (Ganji and Sadighi, 2006; Odidat and Momani, 2006; Yusufoglu, 2007), the homotopy perturbation method (He, 2006; 2005a; 2005b; Ganji and Rajabi, 2006), homotopy analysis method (Kimiaeifar *et al.,* 2009a; 2009b; Fooladi *et al.,* 2009; Ghasempour *et al.,* 2009; D'Acunto, 2006), the, and the energy balance method (He, 2006; 2001).

He's Parameter expanding method (PEM) is the most effective and convenient method to analytical solve of nonlinear differential equations. HPEM has been shown to effectively, easily and accurately solve a large class of linear and nonlinear problems with components that converge rapidly to accurate solutions. HPEM was first proposed by He and was successfully applied to various engineering problems (He, 2002; Tao) J.H. He (2002) proposed modified Lindstedt–Poincare method for some strongly non-linear oscillations. Liu, (2005) studied approximate period of nonlinear oscillators with discontinuities by modified Lindstedt–Poincare method. Xu, (2007) suggested He's parameter-expanding methods for strongly nonlinear oscillators. Tao, propose frequency–amplitude relationship of nonlinear oscillators using He's Parameter expanding method.

In this study He's Parameter-Expanding Method is used to investigate the behaviors of nonlinear problems in vibration. To show the accuracy and application of this method an example is studied and compared with numerical methods. Some remarkable virtues of the methods are studied, and their applications to obtain the higher-order approximate periodic solutions are illustrated. By using simultaneously Lagrange and PEM method, it seems very easy to study the behavior of dynamical systems and also calculate the natural frequency and limit cycle.

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Lagrange Equation:

Lagrange equation is a differential equations of motion expressed in terms of generalized coordinates (Nayfeh, 1993). A brief development is presented here for the general form of these equations in terms of kinetic and potential energies.

Consider first a conservative system where the sum of kinetic and potential energies is a constant. The differential of the total energy is then zero.

$$d(T+U)=0 \tag{1}$$

The kinetic energy T is a function of the generalized coordinates, $\dot{q}$, and the generalized velocities $\dot{q}$ whereas the potential energy U is a function only of $\dot{q}$.

$$T=T(\dot{q}_1,\dot{q}_2,\ldots\dot{q}_n) \tag{2}$$

$$U=U(\dot{q}_1,\dot{q}_2,\ldots\dot{q}_n) \tag{3}$$

The differential of *T* is

$$dT=\sum_{i=1}^{N}\frac{\partial T}{\partial q_i}dq_i+\sum_{i=1}^{N}\frac{\partial T}{\partial \dot{q}_i}d\dot{q}_i \tag{4}$$

To eliminate the second term with $d\dot{q}$, we start with the equation for the kinetic energy:

$$T=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}m_{ij}\dot{q}_i\dot{q}_j$$

Differentiating this equation with respect to $\dot{q}$, multiplying by $\dot{q}$, and summing over *i* from *1* to *N*, the result is equal to:

$$2T=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}m_{ij}\dot{q}_i\dot{q}_j=\sum_{i=1}^{N}\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i \tag{6}$$

or

$$2T=\sum_{i=1}^{N}\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i \tag{7}$$

From the differential of *2T* from above equation by using the product rule in calculus:

$$2dT=\sum_{i=1}^{N}d\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i+\sum_{i=1}^{N}\frac{\partial T}{\partial \dot{q}_i}d\dot{q}_i \tag{8}$$

Subtracting Eq. (3) from this equation, the second term with $d\dot{q}$ is eliminated. By shifting the scalar quantity *dt*, the term $d\left(\frac{\partial T}{\partial \dot{q}}\right)\dot{q}$ become $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right)d\dot{q}$ and the result is:

$$dT=\sum\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial \dot{q}_i}\right)d\dot{q}_i \tag{9}$$

From Eq. (2) the differential of *U* is:

$$dU=\sum_{i=1}^{N}\frac{\partial U}{\partial q_i}dq_i \tag{10}$$

Thus, Eq. (1) for the invariance of the total energy becomes

$$d(T+U)=\sum_{i=1}^{N}[\frac{d}{dT}(\frac{\partial T}{\partial \dot{q}_i})-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}]dq_i=0 \quad (11)$$

Since the N generalized coordinates are independent of one another, the dq_i may assume arbitrary value. Therefore, the above equation is satisfied only if:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i})-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}=0 \quad i=1,2,\ldots N \quad (12)$$

This Lagrange equation for the case in which all forces have a potential U. they can be somewhat modified by

introducing the Lagrange $L=(T-U)$. Since $\frac{\partial U}{\partial \dot{q}_i}=0$ Eq. (10) can be written in terms of L as:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})-\frac{\partial L}{\partial q_i}=0 \qquad i=1,2,\ldots N \quad (13)$$

When the system is also subjected to given forces that do not have a potential, we have instead of Eq. (1)

$$d(T+U)=dw \quad (14)$$

Where d_w is the work of the nonpotential forces when the system is subjected to an arbitrary infinitesimal

displacement. From Eq. (13) dw is expressed in term of the generalized coordinates q_i.

$$dw=\sum_{i=1}^{N}Q_i dq_i \quad (15)$$

Where the quantities Q_i are known as the generalized forces associated with the generalized coordinated q_i. Lagrange equation including nonconservative forces then becomes:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i})-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}=Q_i \qquad i=1,2,\ldots N \quad (16)$$

He's Parameter-Expanding Method:

In case no parameter exists in an equation, HPEM can be used. As a general example, it can be considered the following equation:

$$m x''+\omega_0^2 x+\eta f(x,x',x'')=0,\ x(0)=A, \qquad x'(0)=0 \quad (17)$$

According to the bookkeeping parameter method [19], the solution is expanded into a series of p in the form:

$$x(t)=u_0(t)+p\,x_1(t)+p^2x_2(t)+\ldots \quad (18)$$

Hereby the parameter p does not require being small .

The coefficients m and ω_0^2 are expanded in a similar way

$$m=1+p\,m_1+p^2m_2+\ldots$$

$$\omega_0^2=\omega^2+p\,\omega_1+p^2\omega_2+\ldots \quad (19)$$

$$\eta=p\,c_1+p^2c_2+\ldots \quad (20)$$

ω is assumed to be the frequency of the studied nonlinear oscillator, the values for m and ω_0^2 can be any real value positive, zero or negative.

Here, we are going to solve some problems by using He's parameter expanding method.

Example:

An example of a single-degree-of- freedom conservative system has been considered that is described by an equation as follows. A rigid rod is rigidly attached to the axle as shown in Fig (1). The wheels roll without slip as the pendulum swings back and forth. Only the ball on the end of the pendulum has appreciable mass and it may be considered a particle. The equation governing θ would be (Nayfeh, 1993):

$$\left(l^2+r^2-2rl\cos\theta\right)\ddot{\theta}+rl\,\dot{\theta}^2\sin\theta+gl\sin\theta=0 \tag{21}$$

Here, by using the Taylor's series expansion for $\cos(\theta(t))$ and $\sin(\theta(t))$ the above equation reduces to:

$$\ddot{\theta}+\frac{gl}{l^2+r^2-2rl}\theta-\frac{1}{l^2+r^2-2rl}\left(lr\ddot{\theta}\theta^2+2r\theta\dot{\theta}^2-\frac{1}{6}rl\,\dot{\theta}^2\theta^3-\frac{1}{6}gl\,\theta^3\right)=0 \tag{22}$$

With the boundary conditions:

$$\theta(0)=\lambda, \qquad \dot{\theta}(0)=0, \tag{23}$$

Application of HPEM:

According to the HPEM, Eq. (22) can be rewritten as:

$$1\ddot{\theta}+\omega_0^2\theta-\eta\left(lr\ddot{\theta}\theta^2+2r\theta\dot{\theta}^2-\frac{1}{6}rl\,\dot{\theta}^2\theta^3-\frac{1}{6}gl\,\theta^3\right)=0 \tag{24}$$

And the initial conditions are as follows:

$$\theta_0(0)=\lambda, \qquad \dot{\theta}_0(0)=0, \tag{25}$$

The form of solution and the constant one in Eq. (24) can be expanded as:

$$\theta(t)=\theta_0(t)+p\,\theta_1(t) \tag{26}$$

$$1=1+pa_1 \tag{27}$$

$$\omega_0^2=\omega^2+pb_1 \tag{28}$$

$$\eta=pc_1 \tag{29}$$

Substituting Eqs. (26), (27), (28) and (29) into Eq. (24), and processing as the standard perturbation method, it can be obtained as bellow:

$$\ddot{\theta}_0(t)+\omega^2\theta_0(t)=0, \qquad \theta_0(0)=\lambda, \qquad \dot{\theta}_0(0)=0 \tag{30}$$

$$\ddot{\theta}_1(t)+b_1\theta_0(t)-c_1lr\,\ddot{\theta}_0(t)\theta_0^2(t)+\frac{1}{6}c_1lr\,\dot{\theta}_0^2(t)\theta_0^3(t)-2c_1r\,\theta_0(t)\,\dot{\theta}_0^2(t)+\omega^2\theta_1(t)+\frac{1}{6}c_1gl\,\theta_0^3(t)=0, \tag{31}$$
$$\theta_1(0)=0,\ \dot{\theta}_1(0)=0$$

The solution of Eq. (30) is:

$$\theta_0(t)=\lambda\cos(\omega t) \tag{32}$$

Substituting $\theta_0(t)$ from the above equation, into Eq. (31) results in:

$$\ddot{\theta}_1(t)+b_1\theta_0(t)+c_1lr\lambda^3\omega^2\cos^3\omega t+\frac{1}{6}c_1lr\lambda^5\omega^2\cos^3\omega t\sin^2\omega t-2c_1r\lambda^3\omega^2\cos\omega t\sin^2\omega t$$
$$+\omega^2\theta_1(t)+\frac{1}{6}c_1gl\,\lambda^3\omega^2\cos^3\omega t=0, \tag{33}$$

But from Eq. (28) and (29):

$$b_1 = \frac{\omega_0^2 - \omega^2}{p} \tag{34}$$

And

$$c_1 = \frac{\eta}{p} \tag{35}$$

Based on trigonometric functions properties it is clear that:

$$\cos^3(\omega t) = 1/4\cos(3\omega t) + 3/4\cos(\omega t) \tag{36}$$

After $P=1$ and eliminating the secular term, ω has been obtained as follows:

$$\omega = \pm\frac{\sqrt{2}\sqrt{\left(4l^2+4r^2-8rl-3rl\lambda^2\right)gl\left(8+\lambda\hat{}\,2\right)}}{2\left(4l^2+4r^2-8rl-3rl\lambda^2\right)} \tag{37}$$

Replacing ω from Eq. (37) into Eq. (32) yields:

$$\theta(t) = \theta_0(t) = A\cos\left(\frac{\sqrt{2}\sqrt{\left(4l^2+4r^2-8rl-3rl\lambda^2\right)gl\left(8+\lambda\hat{}\,2\right)}}{2\left(4l^2+4r^2-8rl-3rl\lambda^2\right)}t\right) \tag{38}$$

Numerical Result:

In this paper, the usefulness of the presented parameter expanding method was investigated by considering a nonlinear vibration problem. To validate the HPEM results, convergence studies were carried out and the results were compared with those obtained using numerical results. The effects of constant parameters on position and velocity were studied in Fig. 2 and Fig. 3.

Based on figures 2 and 3, it can be concluded that only one term in series expansions is sufficient to obtain a highly accurate solution, which is valid for the whole solution domain. In addition, stability has been investigated in Fig. (4) and the phase plane has been shown.

Conclusions:

In this paper, two methods, Lagrange and HPEM, were simultaneously used to obtain an analytical solution for nonlinear problems in vibration. Also, a new method called He's parameter expanding method was studied. Some remarkable virtues of the methods have been illustrated and their applications to obtain the nonlinear problems approximate periodic solutions have been demonstrated. HPEM is simple to understand, it was shown that one term in series expansions is sufficient to obtain a highly accurate solution, which is valid for the whole solution domain. In addition, just by a simple formula the response and stability of system is predictable. The obtained results had a good agreement with those obtained using numerical method. The results showed that the method is promising tool to solve this type of problems and might find wide applications.

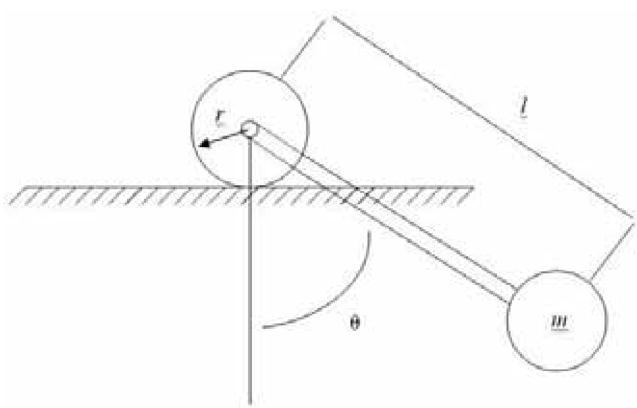

Fig. 1: The geometry of problem.

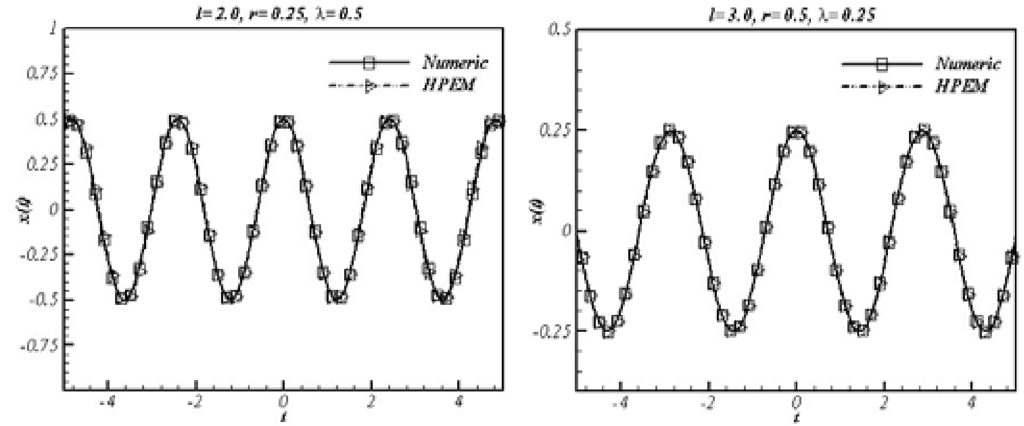

Fig. 2: The effects of constant parameters on position and comparison with numerical results

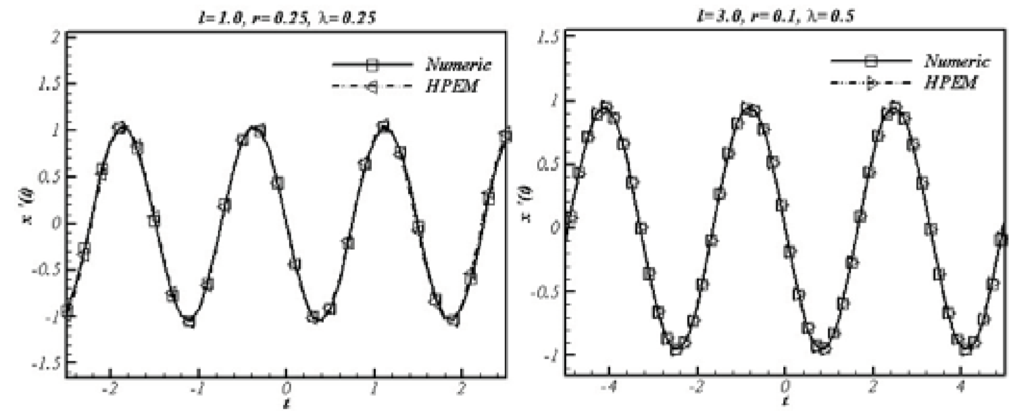

Fig. 3: The effects of constant parameters on velocity and comparison with numerical results

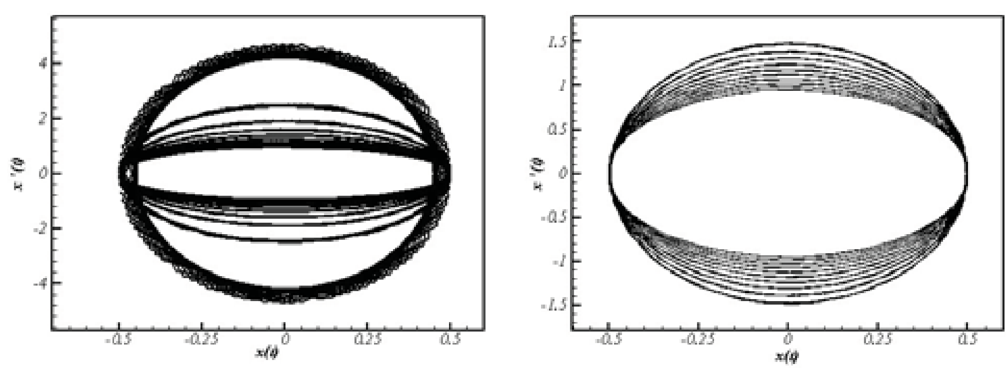

Fig. 4: The phase plane to show the stability

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