

Strong Convergence of Iterative Selection for Common Fixed Point Problems

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Abstract: Many of classical iterative methods use the fixed point problems. Our convergence analysis with suitable assumption is shown in this paper to cover a variety of existing result, result as can be proposed to this consideration our use of selecting a particular common fixed point that match much that necessary and sufficient conditions for the strong convergence theorems of the modified iterative selection for common fixed point problems.

Key words: Fixed of point, Mann iteration; Uniformly smooth Banach space, Non-expansive mapping, Equilibrium problems.

INTRODUCTION

Let C be a closed convex subset of Banach space E with $\|\cdot\|$ and $T: C \rightarrow C$ a mapping is non-expansive with Domain D_T and Range R_T in Banach space. The fixed point set of mapping T is denoted by $F(T) = \{x \in C; T_x = x\}$, recall that if

$$\|T_x - T_y\| \leq \|x - y\| \text{ for all } x, y \in C$$

Such that $T_x = x \quad \forall x \in C \quad T_y = y \quad \forall y \in C$

Many iteration methods are used to approximate a fixed point of a non-expansive mapping, e.g Mann's iteration process (Mann, 1953), Ishikawa's iteration process (Ishikawa, 1974), and Halpern's iteration method [11] and so other iteration. Our interest or goal to achieve close convergence iteration to Halpern's iteration process, as $x \in C$ and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0 \tag{1.1}$$

Where $\{\alpha_n\}$ is a sequence in the interval $[0,1]$ with new assumption on the sequences $\{\alpha_n\}_{n \in \mathbb{N}}$ the convergence properties of (1.1).

For particular choices of the operator T are analyzed in (Kaczor, 2002; Chidume, *et al*), and some applications problems are considered in (Mann, 1953), Ishikawa's iteration process (Ishikawa, 1974), which define recursively by

$$y_n = \beta_n x_n + (1 - \beta_n) T x_n, \tag{1.2}$$

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n.$$

With initial guess x_0 is taken in C arbitrary, $\{\alpha_n\}_{n=1}^{+\infty}$ and $\{\beta_n\}_{n=1}^{+\infty}$ are sequences in the interval $[0,1]$.

Reich proved that if E is uniformly convex Banach space and if $\{\alpha_n\}$ is chosen such that

$$\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty, \tag{1.3}$$

the sequence $\{x_n\}$ in (1.2) is proved converges weakly to a fixed of point of T . on the other, process (1.1) is indeed more general than process (1.2). But research has been concentrated on the latter due probably to the reasons that the formulation of process (1.1) is simpler than that of (1.2) and that a convergence theorem of process (1.1) may lead to convergence theorem for process (1.2). In (Lions,1977), lion proved that

the sequence $\{x_n\}$ given by (1.1) converges strongly to the element of $F(T)$ nearest x_0 under the following assumptions,

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i- $\liminf_{n \rightarrow \infty} \alpha_n = 0$ C₁

ii- $\sum_{n=1}^{\infty} \alpha_n = \infty$ C₂

iii $\liminf_{n \rightarrow \infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0$ C₃

(Xu, 2002), proved this strong convergence theorem in a uniformly smooth Banach space under the conditions of C₁, C₂ and C₃. Moreover, (Cho, kang and Zhou), considered C₁,C₂ and

$$|\alpha_n - \alpha_{n-1}| = 0(\alpha_n) + \delta_n \text{ such that } \sum_{n=1}^{\infty} \delta_n < \infty \tag{1.4}$$

Under assumption on $(\alpha_n)_{n \in \mathbb{N}}$, is in [0,1], shown in (Mann, 1953), In (Senter, Dotson, Chidume, Ofedu and Zegeye), given operator $T : C \rightarrow C$ using different approach (Mondafi, 2000), extended (1.1) to fixed $x_0 \in C$ and set

$$x_{n+1} = \alpha_n Qx_n + (1 - \alpha_n)Tx_n \quad \forall n \in \mathbb{N} \tag{1.5}$$

Where $Q : C \rightarrow C$ Is a strict contraction, by taking good assumption of good convergence for the iteration,

$$x_{n+1} = \alpha_n Q(S_n x_n) + (1 - \alpha_n)T_n x_n \quad \forall n \in \mathbb{N} \tag{1.6}$$

Where $(\alpha_n)_{n \in \mathbb{N}}$ and $(S_n)_{n \in \mathbb{N}}$ are sequence of quasi non-expansive operator from C to C such that

$$\phi \neq \bigcap_{n \in \mathbb{N}} F(T_n) \subset \bigcap_{n \in \mathbb{N}} F(S_n).$$

As $\{x_n\}$ is the sequence as defined in (1.2) and if $\phi \neq \bigcap_{n \in \mathbb{N}} F(T_n)$, then $\lim_{n \rightarrow \infty} \|x_n - u\|$ exist for $u \in \bigcap_{n \in \mathbb{N}} F(T_n)$, where $F(T_n)$ is denotes the fixed point set.

From which follows (1.2) we have

$$\|y_n - u\| \leq \|x_n - u\|$$

Attempts to modify the Mann iteration method, we prove strong convergence theorems of a common fixed point, (Nakajo and Takahashi), proposed the fallowing of the Mann iteration (1.6) for a single non expansive mapping T in Hilbert space

$$\begin{cases} x_0 \in C \text{ chosena arbitrary} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{u \in C : \|y_n - u\| \leq \|x_n - u\| \}, \\ Q_n = \{u \in C : \langle x_0 - x_n, x_n - u \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.7}$$

We shall prove that this iteration converge strongly to a common fixed point of non-expansive mapping S and T provide that $\{\alpha_n\}$ satisfy some appropriate conditions, and $\{x_n\}$ defined by (1.7) converges strongly

to $P_{F(T)}x$. (Yanes and Xu, 2001) has adapted (Nakajo and Takahashi's), idea to modify the process (1.2) for a single non-expansive mapping T .

$$\begin{cases} x_0 \in C \text{ chosen arbitrary} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \left\{ u \in C : \|y_n - u\|^2 \leq \|x_n - u\|^2 + \alpha_n (\|x_0\|^2 + 2 \langle x_n - x_0, u \rangle) \right\}, \\ Q_n = \left\{ u \in C : \langle x_0 - x_n, x_n - u \rangle \geq 0 \right\} \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \quad (1.8)$$

They proved that if $\{\alpha_n\} \subset (0, 1)$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$ then the sequence $\{x_n\}$ generated by (1.8) converges strongly to $P_{F(T)}x$

Preliminaries:

Let C be a closed convex subset of Banach space E with $\|\cdot\|$ and $T: C \rightarrow C$ a mapping is non-expansive. The fixed point set of mapping T is denoted by $F(T) = \{x \in C; T = x\}$

Lemma:

Let C be a closed convex subset of a smooth Banach space E and $T: C \rightarrow E$ Suppose that C is sunny non-expansive retract of E . If T satisfies the condition

$$T_x \in S_x^c \quad \forall x \in C \quad (C_4)$$

Where $S_x = \{y \in E : y \neq x, Qy = x\}$ and Q is a sunny nonexpansive retraction from E onto C , then $F(T) = F(QT)$

Proof:

Its clear that $F(T) \subset F(QT)$ to show the reverse inclusion, let $u \in F(QT)$. Since $T_u \in S_u^c$ and $QTu = u$, we have $Tu = u$.

Lemma:

let E be uniformly convex and smooth Banach space and let $\{x_n\}, \{y_n\}$ be two sequence of E . $\phi(y_n, x_n) \rightarrow 0$, If and either $\{x_n\}, \{y_n\}$ is bounded, then $y_n - x_n \rightarrow 0$.

Lemma:

(Nakajo, and Takahashi), Let E be Banach space and j be a normalized duality mapping. Then for $x, y \in E$ all ,

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle.$$

Theorem:

Let E be a uniformly differentiable. Let C be a nonempty closed subset of E and let T be a non-expansive mapping from C to E with $F(T)$ suppose that C in a sunny non-expansive retract of E . Let $\{\alpha_n\}$ be a sequence such that C_1, C_2 and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| = \infty$, let x and x_0 be an element of C and suppose that $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)QTx_n, \quad n \geq 0, \quad (2.1)$$

where Q is a sunny non-expansive retraction from E onto C . Then $\{x_n\}$ converges strongly

Proof:

Applying the theorem in [] we obtain that $\{x_n\}$ converges as $n \rightarrow \infty$ to a fixed point u of $Q(T)$. Since $F(T) \neq \emptyset$, using lemmas in [].

Theorem:

Let E be a uniformly differentiable. Let C be a nonempty closed subset of E and let T be a non-expansive mapping from C to E with $F(T)$ suppose that C in a sunny non-expansive retract of E . Let $\{\alpha_n\}$ be a sequence such that C_1, C_2 and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| = \infty$, let x and x_0 be an element of C and suppose that $\{x_n\}$ is given by

$$x_{n+1} = Q(\alpha_n x_n + (1 - \alpha_n)Tx_n), \quad n \geq 0, \tag{2.2}$$

where Q is a sunny non-expansive retraction from E onto C . Then $\{x_n\}$ converges strongly

Proof:

Since QT is a non-expansive mapping from $C \rightarrow C$ using lemma 2.1 we have that $\{x_n\}$ converges strongly to $u \in F(QT) = F(T)$

we show that $\{x_n\}$ and $\{Tx_n\}$ are bounded.

$$\begin{aligned} \|x_n - u\| &= \|Q(\alpha_n x_{n-1} + (1 - \alpha_n)Tx_{n-1}) - u\| \\ &\leq \|\alpha_n x_{n-1} + (1 - \alpha_n)Tx_{n-1} - u\| \\ &\leq \alpha_n \|x_{n-1} - u\| + (1 - \alpha_n) \|Tx_{n-1} - u\| \\ &\leq M, \end{aligned}$$

Where $M = \max\{\|x - u\|, \|x_1 - u\|, \|x_0 - u\|\}$. If $\|x_n - u\| \leq M$ for some $n \in \mathbb{N}$, then we can show that $\|x_{n+1} - u\| \leq M$ similarly. And hence $\{x_n\}$ and $\{Tx_n\}$ are bounded. Now we have to show that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0 \tag{2.3}$$

since

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|Q(\alpha_n x_n + (1 - \alpha_n)Tx_n) - Q(\alpha_{n-1} x_{n-1} + (1 - \alpha_{n-1})Tx_{n-1})\| \\ &= \|\alpha_n Qx_n + (1 - \alpha_n)QTx_n - \alpha_{n-1} Qx_{n-1} - (1 - \alpha_{n-1})QTx_{n-1}\| \\ &\leq \|\alpha_n Qx_n + (1 - \alpha_n)QTx_n - \alpha_{n-1} Qx_n - (1 - \alpha_{n-1})QTx_n\| \\ &\leq \|\alpha_n Qx_n + (1 - \alpha_n)QTx_n - \alpha_{n-1} Qx_n - (1 - \alpha_{n-1})QTx_n\| \\ &\leq (1 - \alpha_{n-1}) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| (M + M'), \end{aligned}$$

For each $n \in \mathbb{N}$, where $M' = \sup_{n \in \mathbb{N}} \|Tx_n\|$. By equation(1.3), C_2 and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| = \infty$ we can get $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Also from

$$\begin{aligned} \|x_n - QTx_n\| &= \|Q(\alpha_{n-1} x + (1 - \alpha_{n-1})Tx_{n-1}) - QTx_n\| \\ &\leq \|\alpha_{n-1} x + (1 - \alpha_{n-1})Tx_{n-1} - Tx_n\| \\ &\leq \alpha_{n-1} \|x - Tx_n\| + (1 - \alpha_{n-1}) \|x_{n-1} - x_n\|, \end{aligned}$$

Again using (2.4) and C_1 we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$$

Next we assume that E is a smooth Banach space and by lemma 2.2 and as defined in (Ya.1 and Alber, 1994; Ya.1 and Alber, 1996).

$$\phi(x, y) = \|x\|^2 - 2\langle x, j(y) \rangle + \|y\|^2 \quad \text{for all } x, y \in E$$

It is well known that if E is uniformly smooth, the j is uniformly norm-to-norm continuous on each bounded subset of E . A Banach space is said to have the Kadec-Klee property if a sequence $\{x_n\}$ converge weakly to $x \in E$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x$. It is known that if E is uniformly convex, then E has the

Kadec-Klee property. See [3,4].

Remark:

We show a strong convergence theorem for a new iteration process which introduced by theorem [2.2] under the conditions given in C_1, C_2 .

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