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Geometrical Probability and Lagrange Interpolation Higher-Order Polynomials

[1]Kamal Al-Dawoud and [2]Khomchenko A.N.

[1]Department of mathematics and Statistics, Mutah University – Jordan
[2]Department of applied mathematics, Kherson National Technical University – Ukraine

Abstract: In this paper, we study the possibility of using geometrical probability for designing Lagrange interpolational polynomials with quadratical and cubical interpolation by the example of functions of one and two arguments. We considered a one-dimensional element (3 nodes), a triangle (6 nodes), a square (9 nodes), and a triangle (10 nodes).

Key words: geometrical probability, Lagrange polynomials, quadratical and cubical interpolation, Finite-element method.

INTRODUCTION

The theory of interpolation functions of two arguments has received intensive development in the second half of the twentieth century in connection with the advent of a finite element method (Segerlind L.J. 1985, Strang G., Fix G.J. 1973, Zienkiewicz O.C., Taylor R.L. 2000). In (Kamal Al-Dawoud, Khomchenko A.N. 2007), the geometrical probability is applied on interpolational of a one-dimensional simplex, a two-dimensional simplex (triangle) and a two-dimensional multiplex (a square with bilinear basis). Algorithm design of polynomials of the higher order of one and two arguments using geometrical probability is considered. Polynomial interpolation on a finite element is considered as extension of the notion of expectation nodal values of function. A set of nodal values of function represents a sample. We will describe a radically new approach to the problem of interpolation. This excludes the need of compilation and solving systems of the linear algebraic equations. Regardless of the dimensions and configuration of finite element construction, Lagrange coefficients are reduced to a problem in geometrical probability. Actually, we calculate probability of hitting a random point in a domain in the "successful" outcomes on a finite element.

Formulation and Solution:

Quadratic Interpolation on One-dimensional Elements:

The one-dimensional finite element (FE) with any number of nodals can be considered as pooling of simplex (Fig. 1). A standard interval is considered $[-1; 1]$.

This is convenient for generalization on a standard square. Nodal are located regularly on points $x = -1$, $x = 0$, $x = 1$ (Fig. 1, *a*).

The procedure of Lagrange coefficients construction begin with node $x = -1$ (Fig. 1, *b*). The node $x = -1$ is common to the two simplexes $[-1; 1]$ and $[-1; 0]$, in each of the selected interval simplex with movable left end: $[x; 1]$ и $[x; 0]$. Now at each simplex we will define probability of hit of a random point respectively in $[x; 1]$ and $[x; 0]$. The details can be found in (Kamal Al-Dawoud, Khomchenko A.N. 2007).
Probability of hitting in $[x; 1]$ is equal

$$l_{-1}^{(1)} = \frac{1-x}{2}$$

Probability of hitting in $[x; 0]$ is:

$$l_{-1}^{(2)} = -x$$

Corresponding Author: Kamal Al-Dawoud, Department of mathematics and Statistics, Mutah University – Jordan
E-mail: aldawoud55@yahoo.com

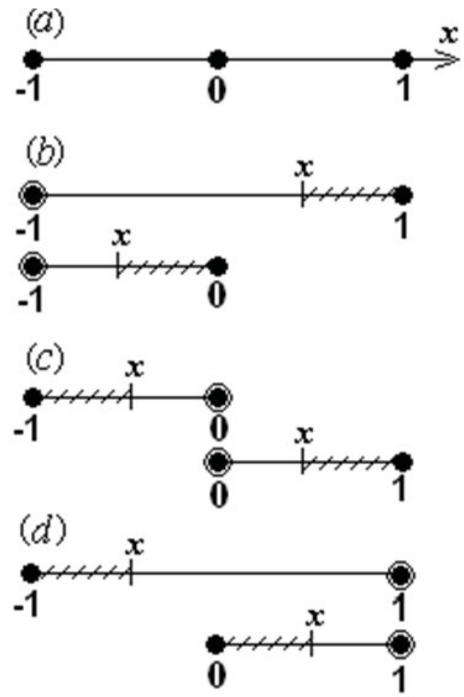

Fig. 1: Probabilistic determination of Lagrange coefficients.

For the construction of Lagrange coefficients, we will find the product of these probabilities:

$$l_{-1}(x) = l_{-1}^{(1)} \cdot l_{-1}^{(2)} = \frac{1}{2}x(x-1) \tag{1}$$

Similarly, for that node x = 0 (Fig. 1, *c*)

$$l_0(x) = l_0^{(1)} \cdot l_0^{(2)} = (x+1)(1-x) = 1-x^2 \tag{2}$$

For node x = 1 (Fig. 1, *d*):

$$l_1(x) = l_1^{(1)} \cdot l_1^{(2)} = \frac{x+1}{2} \cdot x = \frac{1}{2}x(x+1) \tag{3}$$

It is useful to remark that Lagrange coefficients have distinct probability meaning.
Interpolation quadratic polynomials interpolation is given by:

$$P_2(x) = l_{-1}(x) \cdot f_{-1} + l_0(x) \cdot f_0 + l_1(x) \cdot f_1 \tag{4}$$

where f_{-1}, f_0, f_1 nodal values of function.

Note. It is always possible to move from any interval [*a,b*] to a standard interval [-1;1]. For this purpose we do replacement

$$x = \frac{2(t-a)}{b-a} - 1, a \le t \le b$$

Two-dimensional Elements (Quadratic Interpolation), Triangular Element, Quadratic Triangle:

It is anticipated that on the triangular element, interpolation nodes are located in vertexes and on the mid-side (Fig. 2, *a*).

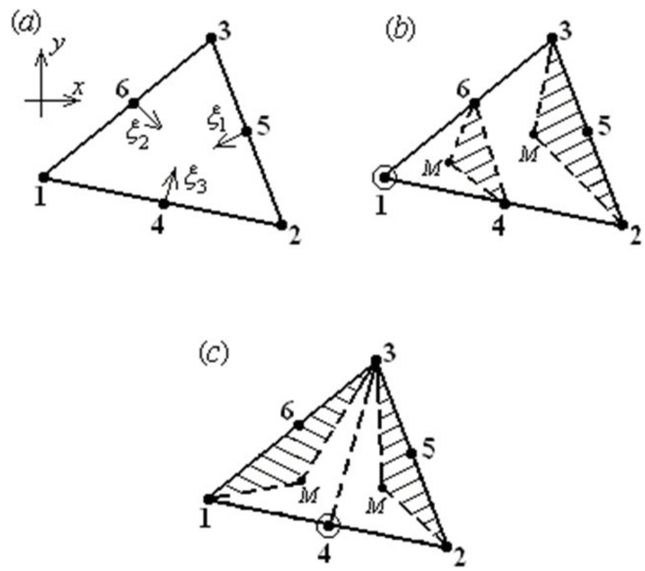

Fig. 2: Pooling of simplexes and modeling rule of quadratic triangle.

Theorem 1: At the finite element of any higher-order coefficient Lagrange (shape function) N_i is equal to the product of geometrical probabilities at the selected simplex, connected with general node *i*.

Proof: For the construction of the basic function N_i on an element of a second order it is necessary to present a quadratic triangle in the form of superposition of two simple (linear) triangles, connected by the general vertex "1" (Fig. 2, *b*). The rule of the construction of the basis of linear interpolation on a triangle is given in (Kamal Al-Dawoud, Khomchenko A.N. 2007) . Now in each triangle 146 and 123 A random point is thrown. The domain of "successful" outcomes of the test determines an arbitrary point of M in each triangle. We will find that the probability of hitting in the domain of M46 of a triangle 146 is (Kamal Al-Dawoud, Khomchenko A.N. 2007) :

$$N_1^{(1)} = \frac{\xi_1 - \frac{1}{2}}{\frac{1}{2}} = 2\xi_1 - 1$$

The probability of hitting in the domain of M23 of a triangle 123 is

$$N_1^{(2)} = \xi_1$$

Finally, N_i is defined as probability of joint event (Feller W. 1969):

$$N_1 = N_1^{(1)} \cdot N_1^{(2)} = (2\xi_1 - 1) \cdot \xi_1$$

Similarly, for other corner nodes, Lagrange coefficient is given by:

$$N_2 = (2\xi_2 - 1)\cdot\xi_2; \quad N_3 = (2\xi_3 - 1)\cdot\xi_3$$

Where ξ_1, ξ_2, ξ_3 - are barycentric coordinates of a triangular simplex (Segerlind L.J. 1985, Strang G., Fix G.J. 1973, Zienkiewicz O.C., Taylor R.L. 2000).In (Kamal Al-Dawoud, Khomchenko A.N. 2007) barycentric coordinates are denoted by l_i.

For mid-side nodes “4” the compiled composition of the two linear triangles are connected by the common vertexes “4” (Fig. 2, *c*).The domains of "successful" outcomes *M*23 and *M*13 are shaded. In the triangle 423 the probability of hitting in *M*23 is

$$N_4^{(1)} = \frac{\xi_1}{\frac{1}{2}} = 2\xi_1.$$

In the triangle 413 the probability of hitting in *M*13 is

$$N_4^{(2)} = \frac{\xi_2}{\frac{1}{2}} = 2\xi_2$$

Finally, N_4 is defined as probability combination of two events:

$$N_4 = N_4^{(1)}\cdot N_4^{(2)} = 4\xi_1\cdot\xi_2$$

Similarly, for other mid-side nodes, Lagrange coefficient is given by:

$$N_5 = 4\xi_1\cdot\xi_3; \quad N_6 = 4\xi_1\cdot\xi_3.$$

Note that any element of higher-order can be presented as a set-simplex element.

As an example, we will show how to extend the theorem 1 on cubic triangles (Fig.3). The repeated application of the geometrical procedure results in the following.

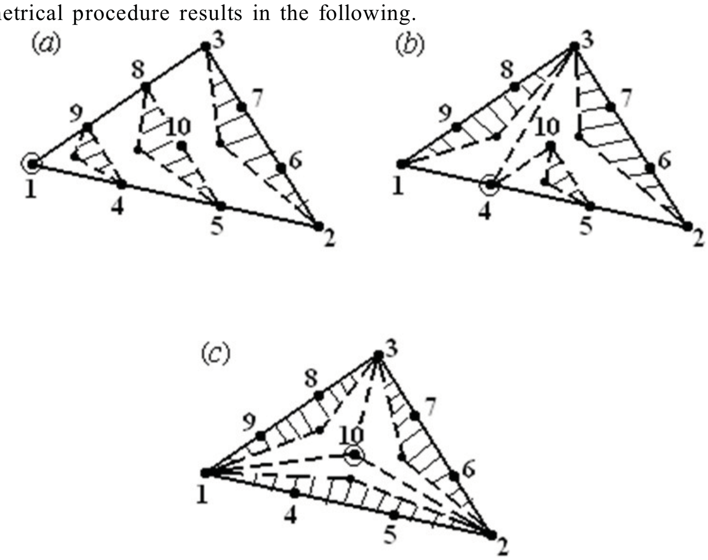

Fig. 3: Compositions of simplex elements on cubic triangle

Corner nodes (Fig. 3, *a*):

$$N_1 = \frac{\xi_1 - \frac{2}{3}}{\frac{1}{3}} \cdot \frac{\xi_1 - \frac{1}{3}}{\frac{2}{3}} \cdot \xi_1 = \frac{1}{2}(3\xi_1 - 1)(3\xi_1 - 2) \cdot \xi_1$$

$$N_2 = \frac{1}{2}(3\xi_2 - 1)(3\xi_2 - 2) \cdot \xi_2; \quad N_3 = \frac{1}{2}(3\xi_3 - 1)(3\xi_3 - 2) \cdot \xi_3$$

Mid-side nodes (Fig. 3, *b*):

$$N_4 = \frac{\xi_1}{\frac{2}{3}} \cdot \frac{\xi_2}{\frac{1}{3}} \cdot \frac{\xi_1 - \frac{1}{3}}{\frac{1}{3}} = \frac{9}{2}\xi_1\xi_2(3\xi_1 - 1); \quad N_5 = \frac{9}{2}\xi_1\xi_2(3\xi_2 - 1), \text{ etc.}$$

Internal nodes (Fig. 3, *c*):

$$N_{10} = \frac{\xi_1}{\frac{1}{3}} \cdot \frac{\xi_2}{\frac{1}{3}} \cdot \frac{\xi_3}{\frac{1}{3}} = 27\xi_1 \cdot \xi_2 \cdot \xi_3$$

This completes the proof of theorem 1.

As it is known (Segerlind L.J. 1985, Strang G., Fix G.J. 1973, Zienkiewicz O.C., Taylor R.L. 2000), Basis functions of Lagrange interpolation have the following properties:

$$\sum_{i=1}^{R} N_i = 1, \quad N_i(M_k) = \delta_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases} \tag{5}$$

Where R – represents the general number of nodes on a finite element, M_k- node with number k; i – number of the function.

Square Element (Quadratic Interpolation):

The standard element Lagrange family $(|x| \leq 1, |y| \leq 1)$ has 9 nodes (Fig. 4). An easy and systematic method of generating shape functions can be achieved by simple products of appropriate polynomials in the two coordinates.

Here we will show the application of geometrical probability for designing of shape functions on a square of a second order element. This model generalizes the described above quadratic interpolation on one-dimensional elements.

The geometrical procedure of the construction of quadratic shape functions on a square is simple and quite clear from Fig. 4.

Corner nodes (Fig. 4, *a*):

$$N_1 = \frac{1}{4}xy(1-x)(1-y) \text{ etc.}$$

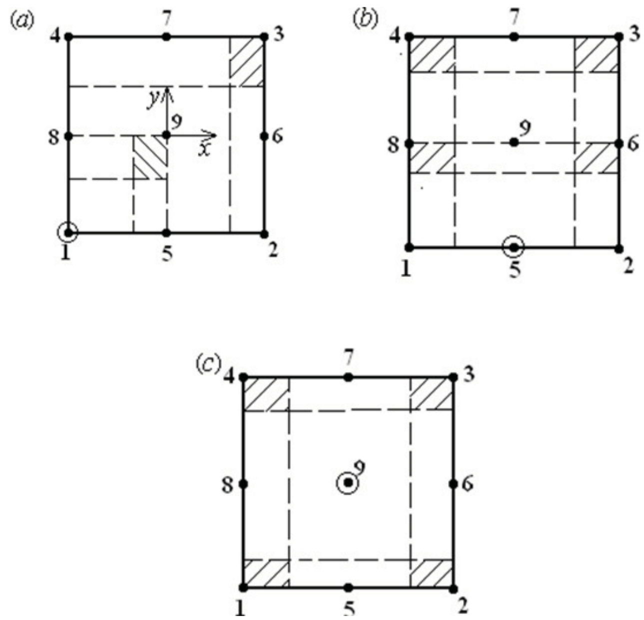

Fig. 4: The domain of "successful" outcomes of quadratic shape functions

Mid-side nodes (Fig. 4, *b*):

$$N_5 = \frac{1}{2}\left(1 - x^2\right)\left(y - 1\right)y \quad \text{, etc.}$$

Internal nodes (Fig. 4, *c*):

$$N_9 = \left(1 - x^2\right)\left(1 - y^2\right)$$

As mentioned above, these functions satisfy condition (5).
Interpolation of quadratic polynomial interpolation function $f(x,y)$ has the form:

$$P_2\left(x,y\right) = \sum_{i=1}^{9} N_i\left(x,y\right) f_i \tag{6}$$

Now it is possible to show, that the polynomial (6) has distinct probability meaning. In fact, the formula (6) realizes quick algorithm of Monte-Carlo method for the problem of recovery of the function on its discrete values f_i.

Theorem 2: In each point of a finite element Lagrange interpolation polynomial is defined as expectation of nodal values function.
Proof: Let us require to construct a field of temperatures $f(x,y)$ of a square plate (Fig. 4), if temperatures f_i in knots $i = \overline{1,9}$ are known.

The normal algorithm of Monte-Carlo method uses an orthogonal grid for routing of random walks and model of wanderings with the absorption of particles in the defined nodes *i*. Computer experiments provide a table:

F	f_1	f_2	…	f_8	f_9
$\frac{n_i}{n}$	$\frac{n_1}{n}$	$\frac{n_2}{n}$	…	$\frac{n_8}{n}$	$\frac{n_9}{n}$

$$\sum_{i=1}^{n} \frac{n_i}{n} = 1 .$$

Here n – is the general number of the particles starting from control point $P(x_p;y_p)$
n_i – number of the particles, i absorption node.
The temperature in point P is defined as the weighed average of nodal temperatures:

$$f(x_p;y_p)=\sum_{i=1}^{9}\frac{n_i}{n}\cdot f_i \tag{7}$$

To get a quick algorithm, enough posteriori probability $\frac{n_i}{n}$ in formula (7) and in the table replaces the a priori probabilities $N_i(x,y)$ Then the temperature at any point of a plate is defined as expectation

$$f(x,y)=\sum_{i=1}^{9}N_i(x,y)\cdot f_i$$

Remark: In the scheme of random walks on the lattice nodes, the internal node 9 plays an exclusive role. Therefore, adequate model walks with no conservative transition probability is difficult to create. In the formula (6) exclusive role of node 9 is considered automatically.

Conclusion:

Special places in the method of finite elements occupy square elements without internal nodes - ‘serendipity’ family (Zienkiewicz O.C., Taylor R.L. 2000). These elements resist formalization. It is interesting to study the possibility of application of geometric probability for the construction of bases of serendipity elements. Only by means of geometric probability can the element of serendipity families be obtained from alternative bases.

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